Chapter 1

Arithmetic and algebraic expressions

CHALLENGE 1

EVALUATION 1

CHALLENGE 1

1. Under what conditions does $\sqrt[n]{a}$ not exist in $\mathbb{R}$? ***When $a < 0$ and n is even.***

2. Write the expression $\frac{8^2 \times \sqrt{32}}{4^4 \times \sqrt[5]{16}}$ as a power of 2.

$2^{-\frac{3}{10}}$

3. Rationalize the denominator of the expression $\frac{5}{\sqrt{2}}$.

$\frac{5\sqrt{2}}{2}$

4. Find an equivalent fraction to $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ with a rational denominator.

$\frac{7+2\sqrt{10}}{3}$

5. For which values of x is the expression $\sqrt{2x^2+x-6}$ defined?

$x \in]-\infty, -2] \cup \left[\frac{3}{2}, +\infty\right[$

6. Under what conditions is the trinomial $ax^2 + bx + c$ factorable in the set of real numbers?

$b^2 - 4ac \geqslant 0$

7. Factor the trinomial $x^2 + 2x - 2$. $(x+1+\sqrt{3})(x+1-\sqrt{3})$

8. For what values of m does the trinomial $mx^2 + (m-1)x + m$ factor as a product of two distinct first degree factors?

$m \in \left]-1, \frac{1}{3}\right[$

1.1 Powers

ACTIVITY 1 Powers with integer exponent

a) 1. What is the definition of a^n? ($n \in \mathbb{N}$; $n \geqslant 2$) $a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}}$

2. Complete: 1) $a^1 =$ a 2) $a^0 =$ 1 ($a \neq 0$)

b) 1. Under what conditions is the power a^n negative? $a < 0$ ***and n is odd.***

2. Calculate

1) 5^2 25 2) $(-5)^2$ 25 3) 5^3 125 4) $(-5)^3$ -125

c) Complete

1. $a^m \times a^n$ a^{m+n} 2. $a^m \div a^n$ a^{m-n} 3. $(a \times b)^n =$ $a^n \times b^n$

4. $\left(\frac{a}{b}\right)^n$ $\frac{a^n}{b^n}$ 5. $(a^m)^n$ a^{mn}

d) 1. What is the definition of a^{-n} when a is non-zero? $a^{-n} = \frac{1}{a^n}$

2. Calculate

1) 5^{-2} $\frac{1}{25}$ 2) $(-5)^{-2}$ $\frac{1}{25}$ 3) $\left(\frac{2}{3}\right)^{-2}$ $\frac{9}{4}$ 4) $\left(\frac{-2}{3}\right)^{-3}$ $\frac{-27}{8}$

POWERS WITH INTEGER EXPONENT

- **Definitions**

$\boxed{a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}} \quad (n \in \mathbb{N} \text{ and } n \geqslant 2)}$ $\boxed{a^1 = a}$ $\boxed{a^0 = 1 \ (a \neq 0)}$ $\boxed{a^{-n} = \frac{1}{a^n} \ (a \neq 0)}$

- **Laws of exponents**

1. Product of two powers with the same base: $a^m \times a^n = a^{m+n}$
2. Quotient of two powers with the same base: $\frac{a^m}{a^n} = a^{m-n}$
3. Power of a product: $(ab)^n = a^n b^n$
4. Power of a quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
5. Power of a power: $(a^m)^n = a^{mn}$

1. Calculate the following.

a) $(-2)^3$ -8 **b)** $(-3)^4$ 81 **c)** -5^2 -25 **d)** 10^{-2} $\frac{1}{100}$

e) $(-4)^{-2}$ $\frac{1}{16}$ **f)** -2^{-2} $\frac{-1}{4}$ **g)** $\left(\frac{2}{3}\right)^3$ $\frac{8}{27}$ **h)** $\left(\frac{-3}{4}\right)^{-2}$ $\frac{16}{9}$

2. Calculate the following.

a) $(-2)^2 \times (-2)^3 =$ **-32** b) $3^{-2} \times 3^4 =$ **9** c) $\frac{(-5)^6}{(-5)^4} =$ **25**

d) $[(-2)^3]^{-2} =$ **$\frac{1}{64}$** e) $\left(\frac{-2}{3}\right)^3\left(\frac{-2}{3}\right)^{-4} =$ **$\frac{-3}{2}$** f) $\left[\left(\frac{2}{3}\right)^{-2}\right]^2 =$ **$\frac{81}{16}$**

3. Simplify the following expressions.

a) $x^{-4} \cdot x^2$ **$\frac{1}{x^2}$** b) $(x^{-2})^{-3}$ **x^6** c) $\frac{x^6}{x^4}$ **x^2**

d) $(3x^2)^2$ **$9x^4$** e) $(5x^2y^3)^2$ **$25x^4y^6$** f) $(3x^2y)\,(2xy^3)$ **$6x^3y^4$**

g) $\frac{12x^4y^2}{6x^2y^3}$ **$\frac{2x^2}{y}$** h) $(3x^2)^2\,(2x^{-2})^3$ **$\frac{72}{x^2}$** i) $\left(\frac{-2x^3}{3y^2}\right)^2$ **$\frac{4x^6}{9y^4}$**

4. Write the following expressions as a power of 2.

a) $(2^3)^2 \times 4^3$ **2^{12}** b) $16^2 \times 8^2 \times 32$ **2^{19}**

c) $\frac{4^2 \times 16}{8^2 \times 2^3}$ **2^{-1}** d) $\frac{(2^3)^2 \times (4^2)^3}{8^4 \times (2^2)^3}$ **2^0**

5. Expand the following numbers as a sum of powers of 2 and write each number in the binary system.

a) $5 =$ **$1 \times 2^2 + 1 \times 2^0 = 101_2$**

b) $10 =$ **$1 \times 2^3 + 1 \times 2 = 1010_2$**

c) $13 =$ **$1 \times 2^3 + 1 \times 2^2 + 1 \times 2^0 = 1101_2$**

d) $29 =$ **$1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^0 = 11101_2$**

6. Simplify the following expressions using the laws of exponents. $(a \neq 0)$

a) $a^{2n} \times a^n$ **a^{3n}** b) $a^{2n-1} \times a^{3-n}$ **a^{n+2}** c) $a^{3n+1} \div a^{2n-1}$ **a^{n+2}**

d) $(a^{2n+1})^2$ **a^{4n+2}** e) $\frac{a^{3n-2}}{a^{n+2}}$ **a^{2n-4}** f) $(a^{n+1} \times a^{n-1})^2$ **a^{4n}**

7. Simplify ~~the~~ following expressions using the laws of exponents. $(a \neq 0)$

a) $(a^{n+1})^3 \times (a^{n-1})^2$ **a^{5n+1}** b) $\frac{(a^{2n+1})^3}{(a^{2n-1})^2}$ **a^{2n+5}**

c) $\frac{(a^{n-1})^3 \times (a^{2n+1})^2}{a^{5n+4}}$ **a^{2n-5}** d) $\frac{(a^{n-1})^2 \times (a^{n+1})^3}{a^{3n+1} \times a^{2n-1}}$ **a**

8. Write $(4^{2n+4} \times 2^{4n-5}) \div 16^{2n}$ as a power of 2.

2^3

9. If $x = a^3$ and $y = 3a^2$, write the following expressions in terms of a.

a) $3x^2y$ **$9a^8$** b) x^2y^3 **$27a^{12}$**

c) $(2x^3y)^2$ **$36a^{22}$** d) $4xy^2 \div 2xy$ **$6a^2$**

e) $3x^2y^4 \div (2xy)^2$ **$\frac{27a^4}{4}$** f) $\left(\frac{2xy^2}{3xy}\right)^2$ **$4a^4$**

g) $x^{-2}y^2$ **$9a^{-2}$** h) $9x^2y^{-2}$ **a^2**

ACTIVITY 2 n^{th} root of a real number

a) Complete.

1. $\sqrt[2]{64} =$ ***8*** because $8^2 = 64$
2. $\sqrt[2]{-64}$ does not exist in ℝ because $-64 < 0$
3. $\sqrt[3]{64} =$ ***4*** because $4^3 = 64$
4. $\sqrt[3]{-64} =$ ***-4*** because $(-4)^3 = -64$

b) Complete. $\sqrt[n]{a} = b \Leftrightarrow b^n = a$

c) Under what conditions does $\sqrt[n]{a}$ not exist in ℝ?

When n is even and a is negative.

N^{th} ROOT OF A REAL NUMBER

Given a natural number n and a real number a, the n^{th} root of the real number a, written $\sqrt[n]{a}$, is the unique real number b for which $b^n = a$.

$$\boxed{\sqrt[n]{a} = b \Leftrightarrow b^n = a}$$

Note that $\sqrt[n]{a}$ does not exist in ℝ when ***n*** **is even and** ***a*** **is negative.**

By convention, we do not write the index 2. Thus, $\sqrt[2]{a} = \sqrt{a}$.

Ex.: $\sqrt[4]{16} = 2$ because $2^4 = 16$; $\sqrt[3]{8} = 2$ because $2^3 = 8$; $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$

$\sqrt[4]{-16} \notin$ ℝ because the index 4 is even and the radicand −16 is negative.

n is called the index, a is the radicand, and $\sqrt{\ }$ is the radical.

10. Determine, if possible, the following roots.

a) $\sqrt{64}$ ***8*** **b)** $\sqrt[3]{64}$ ***4*** **c)** $\sqrt[2]{-16}$ $\notin$ ℝ **d)** $\sqrt{(-5)^2}$ ***5***

e) $\sqrt{\frac{16}{9}}$ $\frac{4}{3}$ **f)** $\sqrt[3]{\frac{8}{27}}$ $\frac{2}{3}$ **g)** $\sqrt[3]{\frac{-8}{125}}$ $\frac{-2}{5}$ **h)** $\sqrt[3]{0.001}$ ***0.1***

11. True or false?

a) The equation $x^2 = 25$ has 2 real solutions 5 and −5. ***True***

b) The equation $x^2 = -25$ has no real solutions. ***True***

c) The equation $x^3 = 8$ has the number 2 as its only solution. ***True***

d) The equation $x^3 = -8$ has the number −2 as its only solution. ***True***

12. Simplify $\sqrt{x^2}$ if

a) x is positive. $\sqrt{x^2} = x$ **b)** x is negative. $\sqrt{x^2} = -x$

ACTIVITY 3 Powers with rational exponent

a) For any real number a, and any non-zero natural number n, we have: $a^{\frac{1}{n}} = \sqrt[n]{a}$.
Evaluate, if possible.

1. $16^{\frac{1}{4}}$ **2**
2. $(-16)^{\frac{1}{4}}$ $\notin \mathbb{R}$
3. $8^{\frac{1}{3}}$ **2**
4. $(-8)^{\frac{1}{3}}$ **-2**

b) If $a^{\frac{1}{n}}$ exists in $\mathbb{R}$, then for any natural number m, we have: $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$.
Evaluate, if possible.

1. $16^{\frac{3}{2}}$ **64**
2. $(-8)^{\frac{2}{3}}$ **4**
3. $(-16)^{\frac{3}{4}}$ $\notin \mathbb{R}$
4. $64^{\frac{2}{3}}$ **16**

POWERS WITH RATIONAL EXPONENT

- **Definitions**
  - For any real number a, and any non-zero natural number n, we have:

$$\boxed{a^{\frac{1}{n}} = \sqrt[n]{a}}$$

Ex.: $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$; $(-64)^{\frac{1}{3}} = \sqrt[3]{-64} = -4$; $(-81)^{\frac{1}{4}} = \sqrt[4]{-81} \notin \mathbb{R}$

The power $a^{\frac{1}{n}}$ does not exist in $\mathbb{R}$ when n is even and a is negative.

  - If $a^{\frac{1}{n}}$ exists in $\mathbb{R}$, then for any natural number m, we have:

$$\boxed{a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m} \quad \text{or} \quad \boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m}}$$

Ex.: $16^{\frac{3}{2}} = \left(\sqrt[2]{16}\right)^3 = 4^3 = 64$; $(-8)^{\frac{2}{3}} = \left(\sqrt[3]{-8}\right)^2 = (-2)^2 = 4$

- **Laws of exponents**

The five laws of exponents also apply when the exponents m and n are **rational** numbers.

13. Calculate the following real numbers.

a) $4^{\frac{3}{2}}$ **8** **b)** $(-8)^{\frac{2}{3}}$ **4** **c)** $25^{\frac{-3}{2}}$ $\frac{1}{125}$ **d)** $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ $\frac{4}{9}$

14. Calculate the following.

a) $x^{\frac{2}{3}} \cdot x^{\frac{1}{2}}$ $x^{\frac{7}{6}}$ **b)** $\left(x^{\frac{2}{3}}\right)^{\frac{3}{4}}$ $x^{\frac{1}{2}}$ **c)** $\frac{x^{\frac{5}{6}}}{x^{\frac{2}{3}}}$ $x^{\frac{1}{6}}$ **d)** $\left(x^{\frac{2}{3}}y^{\frac{3}{2}}\right)^6$ x^4y^9

15. Write the expression $\frac{4^3 \times \sqrt{8}}{8^2 \times \sqrt[3]{16}}$ as a power of 2. $2^{\frac{1}{6}}$

16. Write the expression $\frac{\sqrt{5} \times \sqrt[3]{25}}{\sqrt[6]{625}}$ as a power of 5. $5^{\frac{1}{2}}$

17. Given that $a > 0$, simplify the expression $\frac{a^2 \cdot \sqrt{a^3}}{\sqrt[3]{a^2} \cdot \sqrt{a^4}}$. $a^{\frac{5}{6}}$

1.2 Square root of a real number

ACTIVITY 1 Properties of radicals

a) Consider two real numbers a and b and the equality $\sqrt{a} = b$.

1. What conditions must we put on the real numbers a and b? ***$a \geqslant 0$ and $b \geqslant 0$***
2. Complete: $\sqrt{a} = b \Leftrightarrow$ ***$b^2 = a$***

b)

1. Is the square root of a positive real number unique? ***Yes***
2. Does the square root of a negative real number exist in the set of real numbers? ***No***
3. Can the square root of a number be negative? ***No***

c) True or false?

1. $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$ ***False***
2. $\sqrt{a} - \sqrt{b} = \sqrt{a-b}$ ***False***
3. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ ***True***
4. $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ ***True***

d) Simplify.

1. $a\sqrt{b} \times c\sqrt{d} =$ ***$ac\sqrt{bd}$***
2. $a\sqrt{b} \div c\sqrt{d} =$ ***$\frac{a}{c}\sqrt{\frac{b}{d}}$***
3. $a\sqrt{b} + c\sqrt{b} =$ ***$(a+c)\sqrt{b}$***
4. $a\sqrt{b} - c\sqrt{b} =$ ***$(a-c)\sqrt{b}$***

PROPERTIES OF RADICALS

- The square root of a zero or positive real number a, written $\sqrt{a}$, is the zero or positive real number b such that $b^2 = a$.

  If $a \in \mathbb{R}_+$ and $b \in \mathbb{R}_+$, $\boxed{\sqrt{a} = b \Leftrightarrow b^2 = a}$ Ex.: $\sqrt{25} = 5$ since $5^2 = 25$

- If $a \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$, we have:

  $\boxed{\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}}$ Ex.: $\sqrt{16 \times 9} = \sqrt{16} \times \sqrt{9}$

  $\boxed{\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}}$ $(b \neq 0)$ Ex.: $\sqrt{\frac{16}{100}} = \frac{\sqrt{16}}{\sqrt{100}}$

  $\boxed{\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}}$ $(ab \neq 0)$ Ex.: $\sqrt{64 + 36} \neq \sqrt{64} + \sqrt{36}$

  $\boxed{\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}}$ $(ab \neq 0)$ Ex.: $\sqrt{100 - 64} \neq \sqrt{100} - \sqrt{64}$

ACTIVITY 2 Reducing the radicand

Justify the steps which enable you to reduce the radicand.

$\sqrt{75} = \sqrt{25 \times 3}$ $\boldsymbol{75 = 25 \times 3}$

$= \sqrt{25} \times \sqrt{3}$ $\boldsymbol{\sqrt{ab} = \sqrt{a} \times \sqrt{b}}$

$= 5\sqrt{3}$ ***Evaluation of*** $\boldsymbol{\sqrt{a}}$

REDUCING THE RADICAND

- Let us illustrate the procedure in the following example.

  Ex.: $\sqrt{80} = \sqrt{16 \times 5}$ The radicand is written as the product of 2 factors with one factor being a perfect square number.

  $= \sqrt{16} \times \sqrt{5}$ Apply the property $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.

  $= 4 \times \sqrt{5}$ Evaluate the square root of the square number factor.

  $= 4\sqrt{5}$
- The list of perfect square numbers less than 200 are:
  0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196.

1. Write each of the following numbers in the form $a\sqrt{b}$, where b is the smallest possible integer.

a) $\sqrt{50}$ $\boldsymbol{5\sqrt{2}}$

b) $\sqrt{48}$ $\boldsymbol{4\sqrt{3}}$

c) $\sqrt{80}$ $\boldsymbol{4\sqrt{5}}$

d) $\sqrt{72}$ $\boldsymbol{6\sqrt{2}}$

e) $\sqrt{1000}$ $\boldsymbol{10\sqrt{10}}$

f) $2\sqrt{63}$ $\boldsymbol{6\sqrt{7}}$

ACTIVITY 3 Rationalizing the denominator

a) Rationalize the denominator of the following expressions.

1. $\dfrac{3}{\sqrt{2}}$ $\boldsymbol{\dfrac{3\sqrt{2}}{2}}$

2. $\dfrac{a}{\sqrt{b}}$ $\boldsymbol{\dfrac{a\sqrt{b}}{b}}$

b) Rationalize the denominator of the following expressions.

1. $\dfrac{3}{\sqrt{5}+1}$ $\boldsymbol{\dfrac{3(\sqrt{5}-1)}{4}}$

2. $\dfrac{7}{\sqrt{5}-\sqrt{2}}$ $\boldsymbol{\dfrac{7(\sqrt{5}+\sqrt{2})}{3}}$

3. $\dfrac{a}{\sqrt{b}+\sqrt{c}}$ $\boldsymbol{\dfrac{a(\sqrt{b}-\sqrt{c})}{b-c}}$

4. $\dfrac{a}{\sqrt{b}-\sqrt{c}}$ $\boldsymbol{\dfrac{a(\sqrt{b}+\sqrt{c})}{b-c}}$

RATIONALIZING THE DENOMINATOR

Rationalizing the denominator of an irrational expression consists of determining an equivalent expression with a rational denominator.

There are 2 cases:

1. To rationalize the expression $\frac{a}{\sqrt{b}}$, we multiply the numerator and denominator by the denominator.

$$\boxed{\frac{a}{\sqrt{b}} = \frac{a \cdot \sqrt{b}}{\sqrt{b} \cdot \sqrt{b}} = \frac{a\sqrt{b}}{\left(\sqrt{b}\right)^2} = \frac{a\sqrt{b}}{b}}$$

Ex.: $\frac{2}{\sqrt{3}} = \frac{2 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{2\sqrt{3}}{\left(\sqrt{3}\right)^2} = \frac{2\sqrt{3}}{3}$

2. To rationalize the expression $\frac{a}{\sqrt{b} + \sqrt{c}}$, we multiply the numerator and denominator by the **conjugate** of the denominator.

$$\boxed{\frac{a}{\sqrt{b} + \sqrt{c}} = \frac{a\left(\sqrt{b} - \sqrt{c}\right)}{\left(\sqrt{b} + \sqrt{c}\right)\left(\sqrt{b} - \sqrt{c}\right)} = \frac{a\left(\sqrt{b} - \sqrt{c}\right)}{\left(\sqrt{b}\right)^2 - \left(\sqrt{c}\right)^2} = \frac{a\left(\sqrt{b} - \sqrt{c}\right)}{b - c}}$$

Ex.: $\frac{1}{\sqrt{5} + \sqrt{3}} = \frac{\left(\sqrt{5} - \sqrt{3}\right)}{\left(\sqrt{5} + \sqrt{3}\right)\left(\sqrt{5} - \sqrt{3}\right)} = \frac{\left(\sqrt{5} - \sqrt{3}\right)}{\left(\sqrt{5}\right)^2 - \left(\sqrt{3}\right)^2} = \frac{\sqrt{5} - \sqrt{3}}{5 - 3} = \frac{\sqrt{5} - \sqrt{3}}{2}$

Note that the **conjugate** of $(a + b)$ is $(a - b)$ and that $(a + b)(a - b) = a^2 - b^2$.

2. Rationalize the denominator of the following expressions.

a) $\frac{2}{\sqrt{6}}$ $\boldsymbol{\frac{\sqrt{6}}{3}}$ b) $\frac{-5}{\sqrt{5}}$ $\boldsymbol{-\sqrt{5}}$ c) $\frac{1}{\sqrt{3}}$ $\boldsymbol{\frac{\sqrt{3}}{3}}$ d) $\frac{5}{2\sqrt{10}}$ $\boldsymbol{\frac{\sqrt{10}}{4}}$

3. Rationalize the denominator of the following expressions.

a) $\frac{4}{\sqrt{2} + 1}$ $\boldsymbol{4\left(\sqrt{2} - 1\right)}$ b) $\frac{-2}{\sqrt{3} - \sqrt{2}}$ $\boldsymbol{-2\left(\sqrt{3} + \sqrt{2}\right)}$ c) $\frac{\sqrt{3}}{\sqrt{3} - 1}$ $\boldsymbol{\frac{3 + \sqrt{3}}{2}}$

4. Rationalize the numerator of the following expressions.

a) $\frac{\sqrt{3} + 1}{2}$ $\boldsymbol{\frac{1}{\sqrt{3} - 1}}$ b) $\frac{\sqrt{5} + \sqrt{2}}{6}$ $\boldsymbol{\frac{1}{2\left(\sqrt{5} - \sqrt{2}\right)}}$ c) $\frac{3\sqrt{2} - \sqrt{3}}{3\sqrt{2} + \sqrt{3}}$ $\boldsymbol{\frac{5}{7 + 2\sqrt{3}}}$

5. Consider the second degree equation $x^2 - 2x - 1 = 0$ in the form $ax^2 + bx + c = 0$.

a) Determine the solutions x_1 and x_2 of this equation. $\boldsymbol{x_1 = 1 - \sqrt{2};\ x_2 = 1 + \sqrt{2}}$

b) Verify the following properties.

1. $x_1 + x_2 = \frac{-b}{a}$. $\boldsymbol{x_1 + x_2 = 2 \text{ and } -\frac{b}{a} = 2}$

2. $x_1 \cdot x_2 = \frac{c}{a}$. $\boldsymbol{x_1 \cdot x_2 = (1 + \sqrt{2})(1 - \sqrt{2}) = -1 \text{ and } \frac{c}{a} = -1}$

6. For what values of x are the following expressions defined?

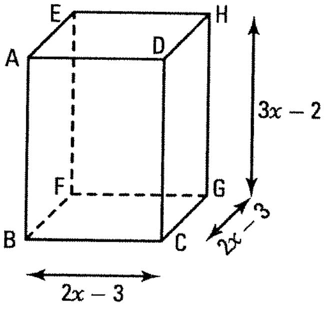
a) $\sqrt{x - 2}$ $\boldsymbol{x \in [2, +\infty[}$ b) $\sqrt[3]{x - 2}$ $\boldsymbol{x \in \mathbb{R}}$

c) $\sqrt{2x - 6}$ $\boldsymbol{x \in [3, +\infty[}$ d) $\sqrt{-3x + 6}$ $\boldsymbol{x \in]-\infty, 2]}$

e) $\frac{1}{\sqrt{x - 2}}$ $\boldsymbol{x \in]2, +\infty[}$ f) $\frac{x - 1}{\sqrt{x - 1}}$ $\boldsymbol{x \in]1, +\infty]}$

1.3 Polynomials

ACTIVITY 1 Polynomial operations



a) Consider the square base right prism on the right.

1. Determine the polynomial $P(x)$ that represents the total length of the prism's edges. $P(x) = 8(2x - 3) + 4(3x - 2) = 28x - 32$
2. Determine the polynomial $A(x)$ that represents the total area of the prism. $A(x) = 2(2x - 3)^2 + 4(2x - 3)(3x - 2) = 32x^2 - 76x + 42$
3. Determine the polynomial $V(x)$ that represents the volume of the prism. $V(x) = (2x - 3)^2\,(3x - 2) = 12x^3 - 44x^2 + 51x - 18$
4. Calculate the total length of the edges, the total area and the volume of the prism when $x = 2$ cm. $P(2) = 24$ cm; $A(2) = 18\text{ cm}^2$; $V(2) = 4\text{ cm}^3$

b) True or false?

1. The sum of 2 polynomials is a polynomial. True
2. The difference of 2 polynomials is a polynomial. True
3. The product of 2 polynomials is a polynomial. True

c) Prove the following remarkable identities.

1. $(a + b)^2 = a^2 + 2ab + b^2$

   $(a + b)^2 = (a + b)(a + b)$
   $= a^2 + ab + ab + b^2$
   $= a^2 + 2ab + b^2$

2. $(a - b)^2 = a^2 - 2ab + b^2$

   $(a - b)^2 = (a - b)(a - b)$
   $= a^2 - ab - ab - b^2$
   $= a^2 - 2ab + b^2$

3. $(a + b)\,(a - b) = a^2 - b^2$

   $(a + b)\,(a - b) = a^2 - ab + ab - b^2$
   $= a^2 - b^2$

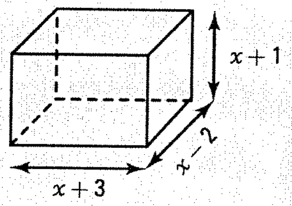
d) Use the remarkable identities to expand the following expressions.

1. $(2x + 3y)^2 = 4x^2 + 12xy + 9y^2$
2. $(3x^2 - 2x)^2 = 9x^4 - 12x^3 + 4x^2$
3. $(3x + 2y)(3x - 2y) = 9x^2 - 4y^2$

e) Use the remarkable identities to factor the following polynomials.

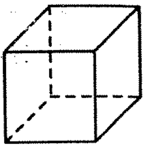
1. $9x^2 - 12xy + 4y^2 =$ $(3x - 2y)^2$
2. $9x^4 + 6x^2y + y^2 =$ $(3x^2 + y)^2$
3. $16x^2 - 9y^4 =$ $(4x + 3y^2)(4x^2 - 3y^2)$

POLYNOMIAL OPERATIONS



- The given right prism has the following dimensions: $(x + 3)$, $(x - 2)$ and $(x + 1)$.
  - The total length of the prism's edges $P(x)$ is equal to:
    $P(x) = 4(x + 3) + 4(x - 2) + 4\,(x + 1) = 12x + 8$
  - The total area $A(x)$ of the prism is equal to:
    $A(x) = 2(x + 3)(x - 2) + 2\,(x + 3)(x + 1) + 2(x - 2)(x + 1)$
    $= 2(x^2 + x - 6) + 2(x^2 + 4x + 3) + 2(x^2 - x - 2)$
    $= 6x^2 + 8x - 10$
  - The volume of the prism is equal to:
    $V(x) = (x + 3)(x - 2)(x + 1)$
    $= (x^2 + x - 6)\,(x + 1)$
    $= x^3 + 2x^2 - 5x - 6$

1. The cube on the right has edges of $(2x - 3)$. Use a polynomial to express



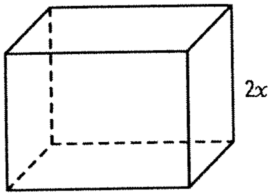
a) the total length of the edges. ***24x − 36*** b) the area of one face. ***$4x^2 - 12x + 9$***

c) the total area. ***$24x^2 - 72x + 54$*** d) the volume. ***$8x^3 - 36x^2 + 54x - 27$***

ACTIVITY 2 Dividing a polynomial by a monomial

a) The prism on the right has the volume $V(x) = 12x^3 + 10x^2 - 2x$. Determine the area of the base knowing that the height of the prism is equal to $2x$.

$6x^2 + 5x - 1$



b) Explain how to divide a polynomial by a monomial.

Divide each term of the polynomial by the monomial.

DIVISION BY A MONOMIAL

- To divide a polynomial by a monomial, we divide each term of the polynomial by the monomial.

  Ex.: The quotient $Q(x)$ of the polynomial $A(x) = 18x^4 - 9x^3 + 12x^2$ divided by the polynomial $B(x) = 6x^2$ is:

  $$\begin{aligned} Q(x) &= A(x) \div B(x) \\ &= (18x^4 - 9x^3 + 12x^2) \div 6x^2 \\ &= (18x^4 \div 6x^2) - (9x^3 \div 6x^2) + (12x^2 \div 6x^2) \\ &= 3x^2 - \frac{3}{2}x + 2. \end{aligned}$$

- The quotient of two polynomials is not always a polynomial.

  Ex.: $(12x^3 - 6x) \div (3x^2) = 4x - 2x^{-1}$ which is not a polynomial.

2. Perform the following divisions.

a) $(24x^4 + 12x^3 - 18x^2) \div 6x^2$ ***$4x^2 + 2x - 3$***

b) $(36x^2y^4 + 27x^3y^2 - 9x^2y^2) \div 9x^2y$ ***$4y^3 + 3xy - y$***

c) $(3x^2 + 2x)(4x + 6) \div 2x$ ***$6x^2 + 13x + 6$***

d) $(3x + 6)^2 \div 3x$ ***$3x + 12 + 12x^{-1}$***

e) $(4x^2 + 2x)(4x^2 - 2x) \div 4x^2$ ***$4x^2 - 1$***

EUCLIDEAN DIVISION

- Consider the polynomials $A(x) = 6x^2 + 5x - 4$ and $B(x) = 3x - 2$

  To divide $A(x)$ (the dividend) by $B(x)$ (the divisor), we proceed in the following manner to determine the quotient $Q(x)$ and the remainder $R(x)$:

  When writing the division, make sure $A(x)$ and $B(x)$ are in decreasing order of the exponents. →

  $$\begin{array}{r|l} A(x) & B(x) \\ \cline{2-2} R(x) & Q(x) \end{array}$$

1° **Divide** the term in the dividend with the highest degree $(6x^2)$ by the term in the divisor with the highest degree $(3x)$.
We get $(\mathbf{2x})$ which represents the term in the quotient $Q(x)$) with the highest degree.

$$\begin{array}{l|l} 6x^2+5x-4 & 3x-2 \\ \cline{2-2} & \mathbf{2x} \end{array}$$

2° Calculate the **product** between the divisor $(3x-2)$ and the 1st term $2x$ obtained in the first step. Align the resulting product $(6x^2 - \mathbf{4x})$ under the dividend.

$$\begin{array}{l|l} 6x^2+5x-4 & 3x-2 \\ \cline{2-2} 6x^2-\mathbf{4x} & 2x \end{array}$$

3° Calculate the first remainder by **subtracting** the product obtained in the 2nd step $(6x^2-4x)$ from the dividend. We then get $(\mathbf{9x}-4)$.

$$\begin{array}{r|l} 6x^2+5x-4 & 3x-2 \\ \cline{2-2} -\ 6x^2-4x & 2x \\ \cline{1-1} \mathbf{9x}-4 & \end{array}$$

Repeat the process...

Stop the division when the degree of the remainder is less than the degree of the divisor.

We therefore get the quotient $Q(x) = \mathbf{2x+3}$ and the remainder $R(x) = 2$.

$$\begin{array}{r|l} 6x^2+5x-4 & 3x-2 \\ \cline{2-2} -\ 6x^2-4x & \mathbf{2x+3} \\ \cline{1-1} -\ 9x-4 & \quad\uparrow \\ 9x-6 & \text{quotient} \\ \cline{1-1} \nearrow 2 & \\ \text{remainder} & \end{array}$$

- The dividend $A(x)$, the divisor $B(x)$, the quotient $Q(x)$ and the remainder $R(x)$ verify the following Euclidean relation:

$$\boxed{A(x) = B(x) \cdot Q(x) + R(x)} \quad \text{where} \quad \deg R(x) < \deg B(x).$$

Indeed, $6x^2 + 5x - 4 = (3x-2)(2x+3) + 2$.

3. Determine the quotient $Q(x)$ and the remainder $R(x)$ in the division of $A(x) = 2x^2 + 5x - 3$ by $B(x) = x - 1$. ***$Q(x) = 2x + 7$; $R(x) = 4$*** ***$(x - 1)(2x + 7) + 4 = 2x^2 + 5x - 3$***

4. In each of the following cases, determine the quotient $Q(x)$ and the remainder $R(x)$ in the division of $A(x)$ by $B(x)$.

a) $A(x) = 2x^2 - x - 6$; $B(x) = 2x + 3$ ***$Q(x) = x - 2$; $R(x) = 0$***
b) $A(x) = 3x^2 - 2x + 1$; $B(x) = x - 2$ ***$Q(x) = 3x + 4$; $R(x) = 9$***
c) $A(x) = 2x^3 + 3x^2 + 2x + 4$; $B(x) = x + 1$ ***$Q(x) = 2x^2 + x + 1$; $R(x) = 3$***
d) $A(x) = x^3 - 2x + 1$; $B(x) = x - 1$ ***$Q(x) = x^2 + x - 1$; $R(x) = 0$***
e) $A(x) = x^4 - 1$; $B(x) = x + 1$ ***$Q(x) = x^3 - x^2 + x - 1$; $R(x) = 0$***
f) $A(x) = x^3 + 27$; $B(x) = x + 3$ ***$Q(x) = x^2 - 3x + 9$; $R(x) = 0$***

ACTIVITY 3 Remainder in Euclidean division

a) Given $P(x) = 3x^2 - 5x + 1$.
1. Calculate $P(2)$. ***3***
2. Verify that the remainder of the division of $P(x)$ by $(x-2)$ is equal to $P(2)$.
3. Calculate $P(-2)$. ***23***
4. Verify that the remainder of the division of $P(x)$ by $(x+2)$ is equal to $P(-2)$.

b) Given $P(x) = x^2 + 2x - 15$.
A polynomial $A(x)$ is **divisible** by a polynomial $B(x)$ when the remainder of the Euclidean division of $A(x)$ by $B(x)$ is 0.
1. Show that $P(x)$ is divisible by $(x - 3)$.
2. Verify that $P(3) = 0$.
3. Show that $P(x)$ is divisible by $(x + 5)$.
4. Verify that $P(-5) = 0$.

REMAINDER THEOREM

- The remainder of a polynomial $P(x)$ divided by $(x - a)$ is equal to $P(a)$.
- A polynomial $P(x)$ is **divisible** by a polynomial $Q(x)$ if and only if the remainder of the division $P(x)$ by $Q(x)$ is equal to 0.

  Consequently,

$$\boxed{\begin{array}{c} P(x) \text{ is divisible by } (x - a) \\ \Leftrightarrow \\ P(a) = 0 \end{array}}$$

5. Given $P(x) = 2x^3 + 3x^2 - 4x - 1$.
Determine the remainder of the division of $P(x)$ by:
1. $(x - 2)$ ***19*** 2. $(x + 2)$ ***3*** 3. $(x - 1)$ ***0***

6. Given $P(x) = x^3 + 2x^2 - 5x - 6$.

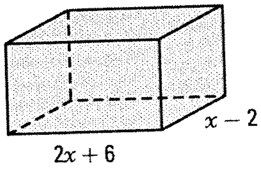
a) Show that $P(x)$ is divisible by:
1. $(x + 3)$ ***$P(-3) = 0$*** 2. $(x - 2)$ ***$P(2) = 0$*** 3. $(x + 1)$ ***$P(-1) = 0$***

b) Show that $P(x)$ is not divisible by:
1. $(x - 1)$ ***$P(1) \neq 0$*** 2. $(x + 2)$ ***$P(-2) \neq 0$*** 3. $(x - 3)$ ***$P(3) \neq 0$***

7. The area of a parallelogram is $A(x) = 10x^2 + 19x - 15$. The height is represented by the binomial $5x - 3$. Use a polynomial to express the parallelogram's base. ***$2x + 5$***

8. The area of a rectangle is given by the polynomial $A(x) = 6x^2 - 13x - 5$. The width is represented by the binomial $3x + 1$. Use a polynomial to express the perimeter of this rectangle.
$10x - 8$

9. The following right prism has the volume $V(x) = 2x^3 + 4x^2 - 10x - 12$. The dimensions of the prism's base are $(2x + 6)$ and $(x - 2)$.



a) Determine the height of the prism. ***$x + 1$***

b) Use a polynomial to express the total area of the prism. ***$10x^2 + 18x - 16$***

10. The area of a triangle is $A(x) = x^2 + x - 6$. The base is represented by the binomial $2x + 6$. Use a polynomial to express the height of this triangle.
$x - 2$

1.4 Factoring a polynomial

ACTIVITY 1 Factoring a polynomial

Factor the following polynomials.

a) $15x^4 + 20x^3 - 10x^2$ ***$5x^2(3x^2 + 4x - 2)$***

b) $6x^2 - 2xy + 9x - 3y$ ***$(2x + 3)(3x - y)$***

c) $16x^2 - 9$ ***$(4x + 3)(4x - 3)$***

d) $9x^2 + 12x + 4$ ***$(3x + 2)^2$***

e) $x^2 - 8x + 15$ ***$(x - 3)(x - 5)$***

f) $2x^2 + 7x + 3$ ***$(2x + 1)(x + 3)$***

FACTORING A POLYNOMIAL

- **Factoring** a polynomial means writing the polynomial as a product of factors.
- Removing a **common factor** is a method which can be used to factor a polynomial composed of monomials which all have a common factor. To factor, you need to apply the **distributive** property of multiplication over addition.

$$\boxed{ab + ac \underset{\text{expand}}{\overset{\text{factor}}{=}} a(b + c)}$$

Ex.: $P(x) = 6x^4 + 15x^3 - 18x^2$

$= 3x^2(2x^2 + 5x - 6)$

- Factoring by **grouping** is a method which enables you to factor polynomials by grouping the terms which contain a common factor. You then remove the common factor in each of the groupings.

Ex.: $\underbrace{9x^2 - 12xy^2}\ + \underbrace{6xy - 8y^3}$ ← Group the terms containing a common factor.

$= 3x(3x - 4y^2) + 2y(3x - 4y^2)$ ← Remove the common factor in each grouping.

$= (3x - 4y^2)(3x + 2y)$ ← Remove the common factor a 2nd time.

- A **difference of two squares** is factorable.

$$\boxed{a^2 - b^2 = (a + b)(a - b)}$$

Ex.: $9x^2 - 4y^2 = (3x + 2y)(3x - 2y)$

- The "**product and sum**" method enables you to factor a second degree trinomial. Let us illustrate this method by factoring $P(x) = 2x^2 + 7x + 6$.

1. **Identify** the coefficients a, b and c.	1. $a = 2; b = 7; c = 6$
2. **Find** two integers m and n such that $\begin{cases} m \cdot n = ac \leftarrow \text{product of the end coefficients} \\ m + n = b \leftarrow \text{middle coefficient} \end{cases}$	2. $\begin{cases} mn = 12 \\ m + n = 7 \end{cases}$ $m = 4, n = 3$
3. Write: $ax^2 + bx + c = ax^2 + mx + nx + c$ and **factor** by **grouping**.	3. $2x^2 + 7x + 6 = 2x^2 + 4x + 3x + 6$ $= 2x(x + 2) + 3(x + 2)$ $= (x + 2)(2x + 3)$

1. Factor the following polynomials.

a) $6x^3y^3 - 9x^3y + 6x^2y^2$ ***$3x^2y(2xy^2 - 3x + 2y)$***

b) $25x^2 - 9y^2$ ***$(5x + 3y)(5x - 3y)$***

Wrong Answer-

c) $2x^2 + 3xy - 10x - 15y$ $(2x + 3y)(x - 5)$

d) $x^3 + x^2 + x - 3$ $(x^2 + 1)(x - 3)$

e) $16x^2 - (x + 2)^2$ $(5x + 2)(3x - 2)$

f) $(x + 1)^2 - (2x + 3)^2$ $(3x + 4)(-x - 2)$

g) $4x^2 - 4x + 1$ $(2x - 1)^2$

h) $9x^2 + 12xy + 4y^2$ $(3x + 2y)^2$

i) $x^2 - 8x + 15$ $(x - 3)(x - 5)$

j) $2x^2 - 7x + 3$ $(2x - 1)(x - 3)$

k) $2x^2 - 9x + 4$ $(2x - 1)(x - 4)$

l) $3x^2 - 4x - 4$ $(3x + 2)(x - 2)$

2. Factor the following polynomials completely.

a) $2x^2 - 50x$ $2x(x + 5)(x - 5)$

b) $12x^3 - 12x^2 + 3x$ $3x(2x - 1)^2$

c) $4x^3 - 4x^2 - 8x$ $4x(x + 1)(x - 2)$

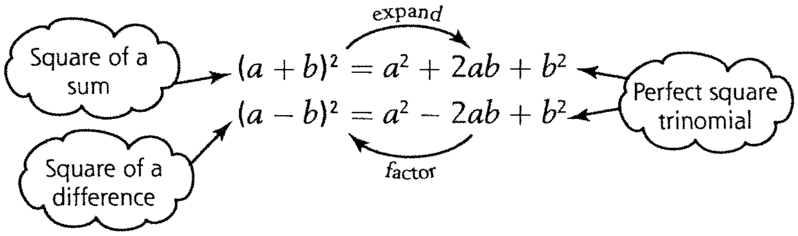
d) $x^4 - 81$ $(x^2 + 9)(x + 3)(x - 3)$

e) $(x^2 - 4) + (x - 2)^2$ $2x(x - 2)$

f) $x^4 - 8x^2 + 16$ $(x + 2)^2(x - 2)^2$

ACTIVITY 2 Perfect square trinomials

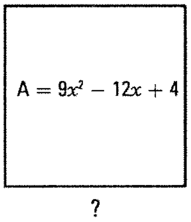
The following remarkable identities enable you to factor a perfect square trinomial.



a) Use the remarkable identities to factor the following perfect square trinomials.

1. $x^2 + 6x + 9$ $(x + 3)^2$
2. $x^2 - 8x + 16$ $(x - 4)^2$
3. $4x^2 + 20x + 25$ $(2x + 5)^2$
4. $9x^2 - 12xy + 4y^2$ $(3x - 2y)^2$

b) The square on the right has an area of $A = 9x^2 - 12x + 4$.
What is the measure of each side? $3x - 2$



c) 1. Explain why $4x^2 + 13x + 9$ is not a perfect square trinomial.
$4x^2 = (2x)^2$; $9 = (3)^2$ but $13x \neq 2(2x)(3)$

2. Explain why $9x^2 + 30x + 25$ is a perfect square trinomial and factor it.
$9x^2 = (3x)^2$; $25 = (5)^2$ and $30x = 2(3x)(5)$. $9x^2 + 30x + 25 = (3x + 5)^2$.

PERFECT SQUARE TRINOMIALS

- A **perfect square trinomial** is an algebraic expression of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$. A trinomial is a perfect square when the middle term is equal to twice the product of the square roots of the end terms.
- Every perfect square trinomial is factorable. You need only apply one of the remarkable identities below, depending on the sign of the middle term.

$\boxed{a^2 + 2ab + b^2 = (a + b)^2}$ or $\boxed{a^2 - 2ab + b^2 = (a - b)^2}$

Ex.: Factor:

- $4x^2 + 12x + 9 = (2x)^2 + 2(2x)(3) + 3^2$ ← Write in the form: $a^2 + 2ab + b^2$.
  $= (2x + 3)^2$ ← Apply the remarkable identity.
- $4x^2 - 12x + 9 = (2x)^2 - 2(2x)(3) + 3^2$ ← Write in the form: $a^2 - 2ab + b^2$.
  $= (2x - 3)^2$ ← Apply the remarkable identity.

3. Factor the following perfect square trinomials.

a) $x^2 + 10x + 25$ ***(x + 5)²***

b) $x^2 - 14x + 49$ ***(x − 7)²***

c) $4x^2 + 12xy + 9y^2$ ***(2x + 3y)²***

d) $25x^2 - 20xy + 4y^2$ ***(5x − 2y)²***

e) $9x^4 - 30x^2 + 25$ ***(3x² − 5)²***

f) $25x^4 + 30x^2y^3 + 9y^6$ ***(5x² + 3y³)²***

g) $x^2 - x + \frac{1}{4}$ $\left(x - \frac{1}{2}\right)^2$

h) $\frac{9}{16}x^2 + x + \frac{4}{9}$ $\left(\frac{3}{4}x + \frac{2}{3}\right)^2$

4. Explain why the following trinomials are not perfect squares.

a) $4x^2 + 6x + 9$ ***6x ≠ 2 × 2x × 3***

b) $4x^2 + 12x - 9$ ***The term −9 is negative.***

c) $-4x^2 + 12x + 9$ ***The term −4x² is negative.***

d) $9x^2 - 15x + 25$ ***15x ≠ 2 × 3x × 5***

5. Complete the trinomials to obtain a perfect square trinomial and then factor them.

a) $x^2 + \boxed{6x} + 9$ ***(x + 3)²***

b) $4x^2 - \boxed{12x} + 9$ ***(2x − 3)²***

c) $9x^2 + 30x + \boxed{25}$ ***(3x + 5)²***

d) $\boxed{25x^2} + 20x + 4$ ***(5x + 2)²***

e) $4x^2 - 28x + \boxed{49}$ ***(2x − 7)²***

f) $\boxed{9x^2} - 6x + 1$ ***(3x − 1)²***

g) $x^2 + \frac{2}{3}x + \boxed{\frac{1}{9}}$ $\left(x + \frac{1}{3}\right)^2$

h) $x^2 - \boxed{7x} + \frac{49}{4}$ $\left(x - \frac{7}{2}\right)^2$

ACTIVITY 3 Factoring a second degree trinomial: Method of completing the square.

a) Fill in the missing term to obtain a perfect square trinomial and then factor.

1. $x^2 + 6x + \underline{9} = (x + 3)^2$

2. $x^2 - 5x + \underline{\frac{25}{4}} = \left(x - \frac{5}{2}\right)^2$

3. $x^2 + \frac{5}{2}x + \underline{\frac{25}{16}} = \left(x + \frac{5}{4}\right)^2$

4. $x^2 - \frac{7}{3}x + \underline{\frac{49}{36}} = \left(x - \frac{7}{6}\right)^2$

Justify the steps which enable you to factor a trinomial of the form $ax^2 + bx + c$.

Steps	Justifications
$P(x) = 2x^2 + 5x + 2$	$a = 2, b = 5, c = 2$
$= 2\left(x^2 + \frac{5}{2}x + 1\right)$	***Factor out the coefficient a = 2.***
$= 2\left(x^2 + \frac{5}{2}x + \ldots + 1 - \ldots\right)$	
$= 2\left[x^2 + \frac{5}{2}x + \frac{25}{16} + 1 - \frac{25}{16}\right]$	***Complete the perfect square trinomial.***
$= 2\left[\left(x + \frac{5}{4}\right)^2 - \frac{9}{16}\right]$	***Factor the perfect square trinomial.***
$= 2\left(x + \frac{5}{4} + \frac{3}{4}\right)\left(x + \frac{5}{4} - \frac{3}{4}\right)$	***Factor the difference of two squares.***
$= 2(x + 2)\left(x + \frac{1}{2}\right)$	***Simplify the expression.***
$(x + 2)(2x + 1)$	***Write as a product of two factors.***

FACTORING A SECOND DEGREE TRINOMIAL: METHOD OF COMPLETING THE SQUARE

The method of completing the square is illustrated below by factoring $P(x) = 2x^2 + 7x + 6$.

$$P(x) = 2x^2 + 7x + 6$$

$$= 2\left(x^2 + \frac{7}{2}x + 3\right) \quad \leftarrow \text{Factor out the coefficient } a = 2.$$

$$= 2\left[\left(x^2 + \frac{7}{2}x + \frac{49}{16}\right) + 3 - \frac{49}{16}\right] \quad \leftarrow \text{Complete the perfect square trinomial.}$$

$\frac{49}{16}$ is the square of half of the coefficient $\frac{7}{2}$ of the middle term.

$$= 2\left[\left(x + \frac{7}{4}\right)^2 - \frac{1}{16}\right] \quad \leftarrow \text{Factor the perfect square trinomial.}$$

$$= 2\left[\left(x + \frac{7}{4} + \frac{1}{4}\right)\left(x + \frac{7}{4} - \frac{1}{4}\right)\right] \quad \leftarrow \text{Factor the difference of two squares.}$$

$$= 2(x + 2)\left(x + \frac{3}{2}\right) \quad \leftarrow \text{Simplify.}$$

$$= (x + 2)(2x + 3) \quad \leftarrow \text{Write as a product of two binomials.}$$

6. Factor the following second degree trinomials by completing the square.

a) $x^2 - 10x + 21$ ***(x − 7)(x − 3)***

b) $x^2 - 5x - 14$ ***(x − 7)(x + 2)***

c) $x^2 - 7x + 12$ ***(x − 3)(x − 4)***

d) $x^2 - 9x + 20$ ***(x − 5)(x − 4)***

e) $2x^2 + 7x + 3$ ***(2x + 1)(x + 3)***

f) $3x^2 + 5x - 2$ ***(3x − 1)(x + 2)***

g) $6x^2 + x - 2$ ***(3x + 2)(2x − 1)***

h) $10x^2 - 19x + 6$ ***(5x − 2)(2x − 3)***

ACTIVITY 4 Factoring a second degree trinomial: The roots method.

a) Justify the steps which enable you to write a second degree trinomial in standard form.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Factor out a.

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

Complete the perfect square trinomial.

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

Factor the perfect square trinomial.

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$$

$\Delta = \boldsymbol{b^2 - 4ac}$

b) From the standard form $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$,

1. show that when Δ is positive, the trinomial $ax^2 + bx + c$ is factorable into a product of two 1st degree factors, that is to say $ax^2 + bx + c = a(x - x_1)(x - x_2)$ where x_1 and x_2 are the roots of the trinomial.

$$\boldsymbol{a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] = a\left[\left(x + \frac{b}{2a} + \frac{\sqrt{\Delta}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{\Delta}}{2a}\right)\right]}$$

$$\boldsymbol{= a\left(x - \frac{-b - \sqrt{\Delta}}{2a}\right)\left(x - \frac{-b + \sqrt{\Delta}}{2a}\right)}$$

$$\boldsymbol{= a(x - x_1)(x - x_2) \text{ where } x_1 = \frac{-b - \sqrt{\Delta}}{2a} \text{ and } x_2 = \frac{-b + \sqrt{\Delta}}{2a}.}$$

2. Explain why the trinomial $ax^2 + bx + c$ is not factorable in ℝ when Δ is negative.

$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$ cannot be factored since the expression $\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}$ is a sum of two squares which cannot be factored in ℝ.

3. Factor the trinomial $ax^2 + bx + c$ when Δ is zero.

$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2$.

FACTORING A SECOND DEGREE TRINOMIAL: THE ROOTS METHOD

The second degree trinomial $ax^2 + bx + c$ is factorable in ℝ if and only if the discriminant Δ is positive or zero.

$$\boxed{\Delta \geqslant 0: ax^2 + bx + c = a(x - x_1)(x - x_2),\ \Delta = b^2 - 4ac}$$

where $x_1 = \frac{-b - \sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b + \sqrt{\Delta}}{2a}$ are the **roots** of the trinomial.

$$\boxed{\Delta < 0: ax^2 + bx + c \text{ is not factorable in } \mathbb{R}.}$$

Ex.: Factor $2x^2 + 7x + 6$.

$a = 2, b = 7, c = 6; \Delta = 1; x_1 = -2, x_2 = -\frac{3}{2}$.

$$2x^2 + 7x + 6 = 2(x + 2)\left(x + \frac{3}{2}\right)$$
$$= (x + 2)(2x + 3)$$

Ex.: Factor $4x^2 - 12x + 9$.

$a = 4, b = -12, c = 9; \Delta = 0; x_1 = x_2 = \frac{3}{2}$.

$$4x^2 - 12x + 9 = 4\left(x - \frac{3}{2}\right)^2$$

Ex.: $x^2 + x + 1$ is not factorable in ℝ since $\Delta < 0$.
Indeed, $a = 1, b = 1, c = 1$ and $\Delta = -3$.

7. For each of the following trinomials, indicate if the trinomial is factorable.

a) $2x^2 + 3x + 1$ ***Yes***

b) $2x^2 - x - 6$ ***Yes***

c) $x^2 - x + 1$ ***No***

d) $4x^2 - 28x + 49$ ***Yes***

e) $x^2 + 2x - 1$ ***Yes***

f) $4x^2 - 12x + 10$ ***No***

8. Factor the following trinomials using the roots method.

a) $2x^2 + 13x + 15$ $\boldsymbol{(2x + 3)(x + 5)}$

b) $4x^2 + 12x + 9$ $\boldsymbol{4\left(x + \frac{3}{2}\right)^2 = (2x + 3)^2}$

c) $-4x^2 + 4x - 1$ $\boldsymbol{-4\left(x - \frac{1}{2}\right)^2 = -(2x - 1)^2}$

d) $4x^2 + 4x - 3$ $\boldsymbol{4\left(x + \frac{3}{2}\right)\left(x - \frac{1}{2}\right) = (2x + 3)(2x - 1)}$

e) $x^2 + 2x - 1$ $\boldsymbol{\left(x + 1 + \sqrt{2}\right)\left(x + 1 - \sqrt{2}\right)}$

f) $2x^2 + 4x - 4$ $\boldsymbol{2\left(x + 1 + \sqrt{3}\right)\left(x + 1 - \sqrt{3}\right)}$

g) $6x^2 - 7x + 2$ $\boldsymbol{6\left(x - \frac{1}{2}\right)\left(x - \frac{2}{3}\right) = (2x - 1)(3x - 2)}$

h) $8x^2 + 2x - 1$ $\boldsymbol{8\left(x - \frac{1}{4}\right)\left(x + \frac{1}{2}\right) = (4x - 1)(2x + 1)}$

9. Simplify the following rational expressions after indicating the restrictions.

a) $\dfrac{x^2 - 3x + 2}{(x-1)^2} = \boldsymbol{\dfrac{(x - 1)(x - 2)}{(x - 1)^2}}$

$\boldsymbol{= \dfrac{(x - 2)}{(x - 1)}\ (x \neq 1)}$

b) $\dfrac{x^2 - 9}{x^2 - 8x + 15} = \boldsymbol{\dfrac{(x + 3)(x - 3)}{(x - 3)(x - 5)}}$

$\boldsymbol{= \dfrac{(x + 3)}{(x - 5)}\ (x \neq 3,\ x \neq 5)}$

c) $\dfrac{2x^2 - 3x - 2}{2x^2 - 5x - 3} = \boldsymbol{\dfrac{(2x + 1)(x - 2)}{(2x + 1)(x - 3)}}$

$\boldsymbol{= \dfrac{x - 2}{x - 3}\left(x \neq -\frac{1}{2} \text{ and } x \neq 3\right)}$

d) $\dfrac{2x^2 - 12x^2 + 18x}{2x^3 - 18x} = \boldsymbol{\dfrac{2x(x - 3)^2}{2x(x + 3)(x - 3)}}$

$\boldsymbol{= \dfrac{x - 3}{x + 3}(x \neq 0,\ x \neq -3,\ x \neq 3)}$

10. Perform the following operations, knowing that the variable does not take values that would make the denominator zero.

a) $\dfrac{x^2 - 4x}{x^2 - 16}$ $\boldsymbol{\dfrac{x}{x + 4}\ (x \neq -4 \text{ and } x \neq 4)}$

b) $\dfrac{x^3 - x}{5x^2 - 5x}$ $\boldsymbol{\dfrac{x}{5}\ (x \neq -1 \text{ and } x \neq 1)}$

11. Perform the following operations, knowing that the variable does not take values that would make the denominator zero.

a) $\dfrac{x^2 - 1}{x + 1} \times \dfrac{2x}{x^2 - x}$ $\boldsymbol{2}$

b) $\dfrac{x^2 - x - 2}{x^2 - 9} \times \dfrac{x + 3}{x - 2}$ $\boldsymbol{\dfrac{x + 1}{x - 3}}$

c) $\dfrac{4x^2 - 9}{4x^2 - 1} \div \dfrac{2x - 3}{2x + 1}$ $\boldsymbol{\dfrac{2x + 3}{2x - 1}}$

d) $\dfrac{2x^2 - 5x - 3}{x^2 - 6x + 5} \div \dfrac{2x^2 + 5x + 2}{x^2 - 1}$ $\boldsymbol{\dfrac{x^2 - 2x - 3}{x^2 - 3x - 10}}$

12. Perform the following operations, knowing that the variable does not take values that would make the denominator zero.

a) $\dfrac{2}{x + 3} + \dfrac{5}{2x + 6}$ $\boldsymbol{\dfrac{9}{2x + 6}}$

b) $\dfrac{5}{3x - 6} - \dfrac{2}{2x - 4}$ $\boldsymbol{\dfrac{2}{3x - 6}}$

c) $\dfrac{3}{x + 1} + \dfrac{2}{x - 2}$ $\boldsymbol{\dfrac{5x - 4}{(x + 1)(x - 2)}}$

d) $\dfrac{x}{x + 1} - \dfrac{x}{x - 1}$ $\boldsymbol{\dfrac{-2x}{x^2 - 1}}$

e) $\dfrac{x + 1}{x - 2} - \dfrac{x + 2}{x - 1}$ $\boldsymbol{\dfrac{3}{x^2 - 3x + 2}}$

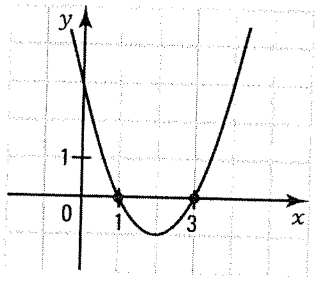
f) $\dfrac{x + 1}{x - 1} - \dfrac{x - 1}{x + 1}$ $\boldsymbol{\dfrac{4x}{x^2 - 1}}$

1.5 Second degree inequalities

ACTIVITY 1 Sign of a second degree trinomial

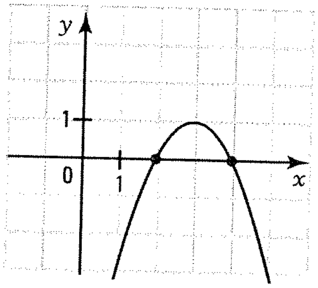
a) The quadratic function $f(x) = x^2 - 4x + 3$ is represented on the right.

1. What are the zeros of the function? ***1 and 3***
2. Using the graph of the function, determine the sign of $f(x) = x^2 - 4x + 3$.
   $f(x) \geqslant 0$ if $x \in$]−∞, 1] ∪ [3, +∞[; $f(x) \leqslant 0$ if $x \in$ [1, 3]



b) The quadratic function $f(x) = -x^2 + 6x - 8$ is represented on the right.

1. What are the zeros of the function? ***2 and 4***
2. Using the graph of the function, determine the sign of $f(x) = -x^2 + 6x - 8$.
   $f(x) \geqslant 0$ if $x \in$ [2, 4]; $f(x) \leqslant 0$ if $x \in$]−∞, 2] ∪ [4, +∞[

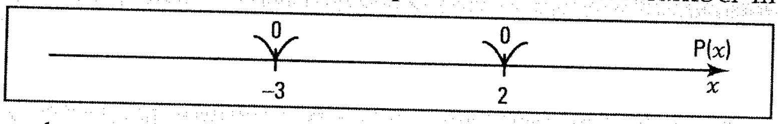


The preceding situations show that the sign of a polynomial does not change over each of the intervals limited by the zeros of the polynomial. This observation makes it possible to present the following axis method for determining the sign of a polynomial.

SIGN OF A SECOND DEGREE TRINOMIAL – NUMBER LINE METHOD

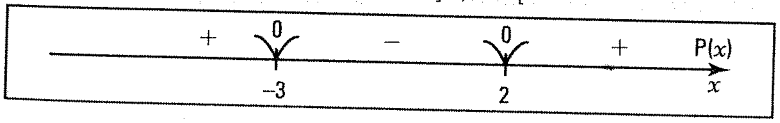
To determine the sign of the trinomial $P(x) = x^2 + x - 6$.

1. Determine the zeros of the polynomial to be placed on the real number line.



The zeros of the polynomial then define 3 intervals:]−∞, −3[,]−3, 2[and]2, +∞[.

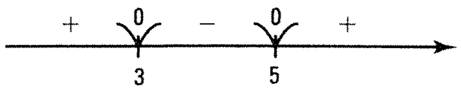
2. Since the sign of a trinomial does not change (see activity 1) over each of the intervals, we determine the sign of the trinomial **in each interval** by evaluating the trinomial for a randomly chosen x in each interval.
   - Since $P(-4) = 6$, we deduce that $P(x) > 0$ over]−∞ , −3[.
   - Since $P(0) = -6$, we deduce that $P(x) < 0$ over]−3, 2[.
   - Since $P(3) = 6$, we deduce that $P(x) > 0$ over]2, +∞[.



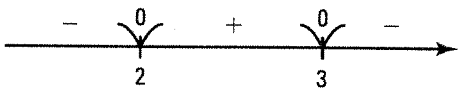
[illegible]) $\geqslant 0$ if $x \in$]−∞, −3] ∪ [2, +∞[and $P(x) \leqslant 0$ if $x \in$ [−3, 2].

1. Use the number line method to determine the sign of the following trinomials.

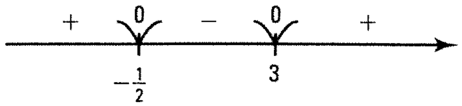
a) $P(x) = x^2 - 8x + 15$



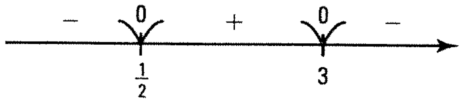
b) $P(x) = -x^2 + 5x - 6$



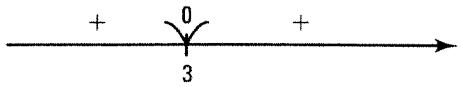
c) $P(x) = 2x^2 - 5x - 3$



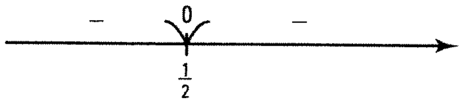
d) $P(x) = -2x^2 + 7x - 3$



e) $P(x) = x^2 - 6x + 9$



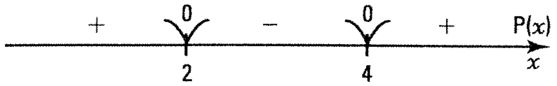
f) $P(x) = -4x^2 + 4x - 1$



ACTIVITY 2 Second degree inequality with one variable

a) Solving the second degree inequality $x^2 - 6x + 8 < 0$ consists of finding the values of x for which the trinomial $P(x) = x^2 - 6x + 8$ is strictly negative.

1. Study the sign of the trinomial $P(x) = x^2 - 6x + 8$.



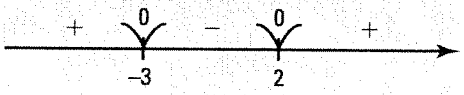
2. Deduce the solution set S. ***S =]2, 4[***

b) Refer to the sign of the trinomial $P(x) = x^2 - 6x + 8$ obtained in question a) to solve

1. $x^2 - 6x + 8 \leqslant 0$. ***S = [2, 4]***
2. $x^2 - 6x + 8 > 0$. ***S =]–∞, 2[∪]4, +∞[***
3. $x^2 - 6x + 8 \geqslant 0$. ***S =]–∞, 2] ∪ [4, +∞[***

SOLVING A SECOND DEGREE INEQUALITY – NUMBER LINE METHOD

- Solving the second degree inequality $x^2 + x - 6 < 0$ finding where $x^2 + x - 6$ is strictly negative.
  1. We study the sign of the trinomial using a number line.



  2. We dèduce the solution set to the inequality. S =]–3,2[
- Note that
  $x^2 + x - 6 \leqslant 0$ has the solution set S = [–3, 2].
  $x^2 + x - 6 > 0$ has the solution set S =]–∞, –3[∪]2, +∞[.
  $x^2 + x - 6 \geqslant 0$ has the solution set S =]–∞, –3] ∪ [2, +∞[.

2. Solve the following inequalities.

a) $2x^2 - x - 6 > 0$ $\left]-\infty, \frac{-3}{2}\right[\cup \left]2, +\infty\right[$

b) $-6x^2 + 11x \geqslant 4$ $\left[\frac{1}{2}, \frac{4}{3}\right]$

c) $x^2 + x > 3$ ℝ

d) $x^2 < 5x$ ***]0, 5[***

3. From the top of a seaside cliff, a stone is thrown upwards. The polynomial $h(t) = -2t^2 + 12t + 10$ gives the height h (in m) of the stone relative to sea level as a function of the elapsed time t in seconds. Between which times is the stone at a height of 26 m above sea level?

Between the times t = 2 s and t = 4 s

Evaluation 1

1. Write, as a power of 2, the expression $\frac{8^2 \times 2^{-3} \times \sqrt{12}}{\sqrt{3} \times 4^{\frac{3}{2}}}$. 2^1

2. Simplify the following expressions.

a) $(3x^2)^3$ $27x^6$

b) $(2x^3)^2 \cdot (3x^2)^2$ $36x^{10}$

c) $\left(\frac{2x^3}{3y^2}\right)^2$ $\frac{4x^6}{9y^4}$

d) $(3x^{-2}y^2)^2 \cdot (2xy^{-1})^3$ $\frac{72y}{x}$

3. If $x = 3a^2$ and $y = 2a^3$, write the following expressions in terms of a.

a) $3x^2y$ $54a^7$

b) $2xy^3$ $48a^{11}$

c) $(2x^2y)^2$ $1296a^{14}$

d) $12x^2y \div 3xy^2$ $\frac{6}{a}$

e) $x^{-2}y^2$ $\frac{4a^2}{9}$

f) $\left(\frac{2x^2}{3y}\right)^2$ $9a^2$

4. Express $(2^{3n+2} \times 4^{n-1}) \div 8^{n-1}$ as a power of 2. 2^{2n+3}

5. Given that $a > 0$, simplify the expression $\frac{a^3 \cdot \sqrt[3]{a^2}}{\sqrt[2]{a^3} \cdot \sqrt{a^6}}$. $a^{\frac{-5}{6}}$

6. Rationalize the denominator.

a) $\frac{12}{\sqrt{6}}$ $2\sqrt{6}$

b) $\frac{5}{2\sqrt{3} + \sqrt{2}}$ $\frac{2\sqrt{3} - \sqrt{2}}{2}$

c) $\frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}}$ $\frac{7 + 2\sqrt{10}}{3}$

7. Factor the trinomial $2x^2 + 7x + 6$ using the method of completing the square.

$$\begin{aligned}2x^2 + 7x + 6 &= 2\left(x^2 + \frac{7}{2}x + \frac{49}{16} + 3 - \frac{49}{16}\right)\\ &= 2\left[\left(x + \frac{7}{4}\right)^2 - \frac{1}{16}\right]\\ &= 2\left[\left(x + \frac{7}{4} + \frac{1}{4}\right)\left(x + \frac{7}{4} - \frac{1}{4}\right)\right]\\ &= 2(x + 2)\left(x + \frac{3}{2}\right)\\ &= (x + 2)(2x + 3)\end{aligned}$$

8. Factor the trinomial $x^2 + 2x - 1$.

$\Delta = 8; x_1 = -1 - \sqrt{2}; x_2 = -1 + \sqrt{2}$

$x^2 + 2x - 1 = (x + 1 + \sqrt{2})(x + 1 - \sqrt{2})$

9. Factor the following polynomials completely.

a) $x^4 - 16$ $(x^2 + 4)(x + 2)(x - 2)$

b) $x^4 - 18x^2 + 81$ $(x + 3)^2(x - 3)^2$

c) $(x^2 + 1)^2 - 4x^2$ $(x + 1)^2(x - 1)^2$

d) $2x^5 - 4x^3 + 2x$ $2x(x + 1)^2(x - 1)^2$

10. Simplify the following rational expressions after indicating the restrictions.

a) $\frac{x^2 - 9}{x^2 + 8x + 15}$ $\frac{x - 3}{x + 5}$, $x \neq -3$ and $x \neq -5$

b) $\frac{2x^2 - 3x - 2}{x^2 - 7x + 10}$ $\frac{2x + 1}{x - 5}$, $x \neq 2$ and $x \neq 5$

11. Explain why the trinomial $x^2 + x + 1$ is not factorable in the set $\mathbb{R}$.

$\Delta = -3$ is negative.

12. Solve the following second degree equations.

a) $2x^2 - 9x - 5 = 0$ $S = \left\{-\frac{1}{2}, 5\right\}$

b) $9x^2 - 12x + 4 = 0$ $S = \left\{\frac{2}{3}\right\}$

c) $x^2 + 2x + 3 = 0$ $S = \varnothing$

d) $x^2 + 4x + 1 = 0$ $S = \left\{-2 - \sqrt{3}, -2 + \sqrt{3}\right\}$

13. Solve the following second degree inequalities.

a) $x^2 + 2x \geqslant 8$

$S =]-\infty, -4] \cup [2, +\infty[$

b) $-2x^2 + 5x - 3 \geqslant 0$

$S = \left[1, \frac{3}{2}\right]$

c) $4x^2 \geqslant 9$

$S = \left]-\infty, \frac{-3}{2}\right] \cup \left[\frac{3}{2}, +\infty\right[$

d) $x^2 - 2x + 1 \geqslant 0$

$S = \mathbb{R}$

14. The profit $P(x)$, in millions of dollars, of a company depends on the number of units x sold in the month. This profit is given by $P(x) = -0.5x^2 + 40x - 750$. Determine over which interval the number of units sold must be for the company's profit to be strictly positive.

$x \in]30, 50[$

Chapter 2

Optimization