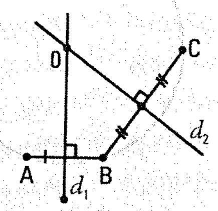
CHALLENGE 10

1. Determine the centre 0 of the circle represented on the right. Explain your procedure.

1. We draw any two chords AB and BC.

2. We draw the respective right bisectors of these two chords.

3. The common point of the perpendicular bisectors is the centre of the circle.



2. From point P exterior to the circle, draw two lines PA and PB such that A is the only common point of the circle and the line PA and B is the only common point of the circle and the line PB.

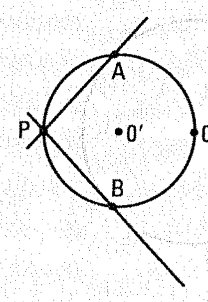
Explain your procedure.

1. We locate the point 0′, midpoint of $\overline{0P}$.

2. We draw the circle centred at 0′ passing through 0.

3. We locate the intersection points A and B of the two circles.

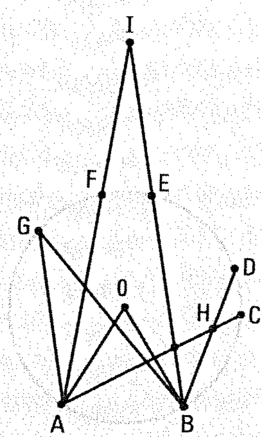
4. Lines PA and PB are the lines we seek.



3. Arcs AB, CD and EF on the right measure, in degrees, respectively a, b and c. Deduce

a) m ∠A0B a **b)** m ∠AGB $\frac{a}{2}$

c) m ∠CHD $\frac{a+b}{2}$ **d)** m ∠AIB $\frac{a-c}{2}$



4. The circle of radius 4 cm centred at 0 is represented on the right. From a point P the line segments PA and PB tangent to the circle are drawn.

Knowing that m $\overline{0P}$ = 5 cm, determine the area of the quadrilateral PA0B.

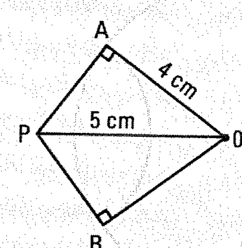
1. m$\overline{PA}$ = m$\overline{PB}$, theorem on tangent line segments.

2. m$\overline{PA}$ = 3 cm, Pythagoras' theorem.

3. Area △0AP = $\frac{3 \times 4}{2}$ = 6 cm².

4. △0AP ≅ △0BA, congruency case SSS.

5. Area of quadrilateral PA0B = 2 × Area △0AP = 12 cm².



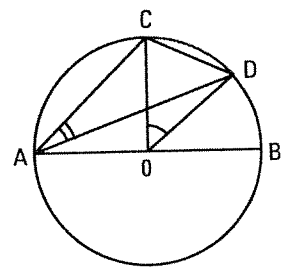
10.1 Geometric elements of the circle

ACTIVITY 1 Geometric elements of the circle

a) 1. Give the definition of a circle. ***A circle is the set of all points in the plane located at the same distance from a point called centre.***

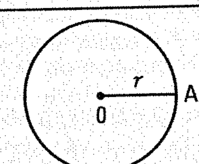
2. Give the definition of the radius of a circle. ***The distance between each point on a circle and the centre is called radius. We also call radius any line segment connecting the centre of the circle and a point on the circle.***

b) 1. Draw a circle of radius 3 cm centred at 0.
2. Draw a diameter AB.
3. Draw a chord CD.
4. The chord CD subtends two arcs. Colour the minor arc $\overset{\frown}{CD}$ in green.
5. What do we call angle ∠COD? ***Central angle***
6. What do we call angle ∠CAD? ***Inscribed angle***



DEFINITIONS

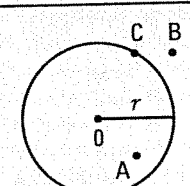
- A **circle** of radius r centred at 0 is the set of all points in the plane located at a distance r from the centre.
- A **radius** is a line segment connecting the centre and any point on the circle.



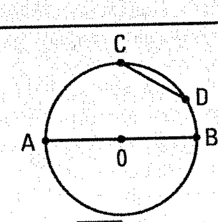
$\overline{0A}$ is a radius; $m\overline{0A} = r$

- A circle divides the plane into three regions,
  - the region containing points inside the circle.
  - the region containing points outside the circle.
  - the circle itself.

  Ex.: On the figure on the right, A is an interior point, B an exterior point and C a point on the circle.
  We have: $d(0, A) < r$, $d(0, B) > r$ and $d(0, C) = r$.



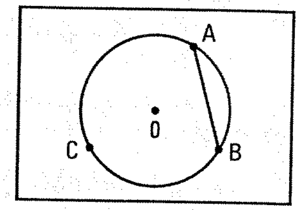
- A **chord** is a line segment connecting any two points on the circle.
- A **diameter** is a chord passing through the centre of the circle.



$\overline{AB}$ and $\overline{CD}$ are chords.
$\overline{AB}$ is a diameter.

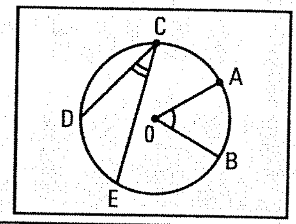
- An arc of circle is a portion of a circle bounded by two points on a circle.

  Ex.: On the figure on the right, we distinguish the chord AB, denoted $\overline{AB}$, the minor arc AB, denoted $\overset{\frown}{AB}$, and the major arc ACB, denoted $\overset{\frown}{ACB}$. We say that the chord AB **subtends** the arc AB or that the arc AB is **subtended** by the chord AB.



- A **central angle** is an angle whose vertex is the centre of the circle.
- An **inscribed angle** is an angle whose vertex is a point on the circle and whose sides intersect the circle.

  Ex.: On the figure on the right, central angle A0B **intercepts** arc AB and the inscribed angle DCE intercepts arc DE.



1. Complete using the appropriate term.

a) $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ are ***cords.*** b) $\overline{0A}$, $\overline{0B}$ and $\overline{0C}$ are ***radii.***

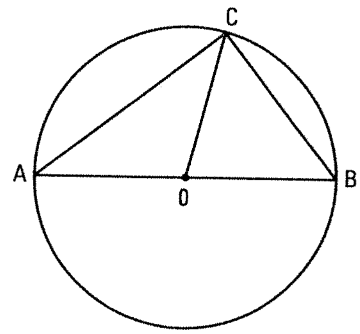
c) Chord AB is ***a diameter.*** d) $\overset{\frown}{BC}$ is ***an arc.***

e) ∠B0C is ***a central*** angle f) ∠CAB is ***an inscribed*** angle.

g) ∠A0C ***intercepts*** arc AC.

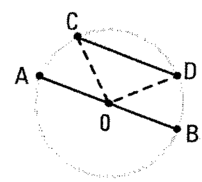
h) Chord AC subtends ***arc AC.***

i) Points A and C bound two arcs denoted $\overset{\frown}{AC}$ and $\overset{\frown}{ABC}$.



ACTIVITY 2 The largest chord

Justify the steps showing that, in a circle, the diameter is the largest chord.



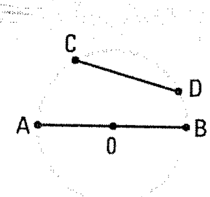
Consider a diameter $\overline{AB}$ and any chord $\overline{CD}$ that is not a diameter.

Steps	Justifications
1. In triangle C0D, we have: $m\overline{C0} + m\overline{0D} > m\overline{CD}$	***In any triangle, the sum of the measures of 2 sides is greater than the measure of the 3rd side.***
2. $m\overline{C0} = m\overline{0A}$ et $m\overline{0D} = m\overline{0B}$	***$\overline{C0}$, $\overline{0A}$, $\overline{0D}$ and $\overline{0B}$ are radii.***
3. We deduce that: $m\overline{0A} + m\overline{0B} > m\overline{CD}$	***We substitute $\overline{C0}$ by $\overline{0A}$ and $\overline{0D}$ by $\overline{0B}$ in the inequality established in 1.***
4. $m\overline{AB} > m\overline{CD}$	***$\overline{AB}$ being a diameter, we have: $m\overline{0A} + m\overline{0B} = m\overline{AB}$.***

THEOREM OF THE LARGEST CHORD

In a a circle, the **diameter** is the largest chord.

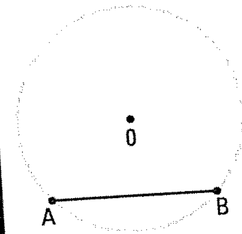
Ex.: On the figure on the right, we have: $m\overline{AB} > m\overline{CD}$.



ACTIVITY 3 Perpendicular bisector of a chord

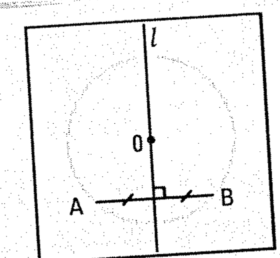
Justify the steps showing that the perpendicular bisector of a chord passes through the centre of the circle. Let AB, be a chord of a circle centred at 0.

	Steps	Justifications
1.	A and B are 2 points on the circle.	***$\overline{AB}$ is a chord.***
2.	$m\overline{0A} = m\overline{0B}$	***$\overline{0A}$ and $\overline{0B}$ are radii.***
3.	0 belongs to the perpendicular bisector of AB.	***Characteristic property of the perpendicular bisector: Any point on the perpendicular bisector of a line segment lies at the same distance from the endpoints of the segment.***



PROPERTY OF THE PERPENDICULAR BISECTOR OF A CHORD

Given a chord of a circle, the perpendicular bisector of a chord passes through the centre of the circle.

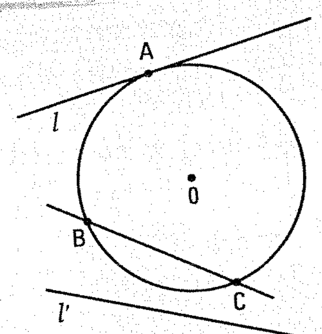


ACTIVITY 4 Line tangent to a circle

RELATIVE POSITIONS OF A LINE AND A CIRCLE

- A **secant** to a circle is a line having two distinct points in common with the circle.
- A **tangent** to a circle is a line having only one point in common with the circle.
- A line and a circle having no common point are called **disjoint**.

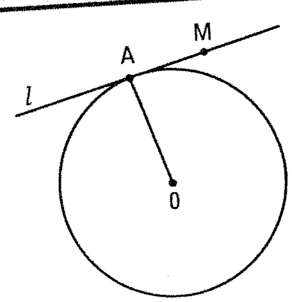
Ex.: On the figure on the right, line BC is a secant to the circle, line l' and the circle are disjoint and line l is tangent to the circle at point A. Point A is called **tangent point** or **point of contact**.



a) Consider on the right the circle centred at 0 and the line l tangent to the circle at point A.

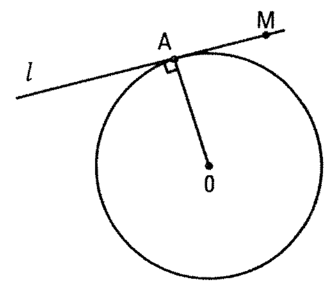
Justify the steps showing that line l is perpendicular to the radius 0A.

Hypotheses: – l is tangent to the circle at point A.
– M is any point on l distinct from A.



	Steps	Justifications
1.	M is an exterior point of the circle.	***If M were an interior point, line l would be secant to the circle (contradicting the hypothesis).***
2.	$m\overline{0M} > m\overline{0A}$.	***M being an exterior point, the length of $\overline{0M}$ is greater than the radius of the circle.***
3.	$m\overline{0A} = d(0, l)$.	***Since $m\overline{0M} > m\overline{0A}$, for any point M on line l, then $m\overline{0A} = d(0,l)$ (Property of the distance from a point to a line).***
4.	$\overline{0A} \perp l$.	***Definition of the distance from a point to a line.***

b) Consider on the right the circle centred at 0 and the radius 0A. Justify the steps showing that any line perpendicular to the endpoint of the radius is tangent to the circle.



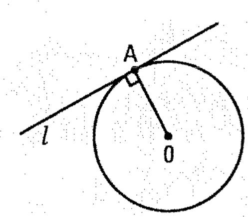
Hypotheses: – $l \perp \overline{0A}$.
– M is any point on line l, distinct from A.

	Steps	Justifications
1.	Triangle 0AM is right at A.	***$\overline{0A} \perp l$ (hypothesis).***
2.	$m\overline{0M} > m\overline{0A}$.	***$\overline{0M}$ is the hypotenuse.***
3.	M is an exterior point of the circle.	***d(0, M) > radius.***
4.	l is tangent to the circle.	***Any point of l other than A is an exterior point of the circle.***

THEOREM ON THE RADIUS AT THE TANGENT POINT

- In a circle, any radius ending at the tangent point is perpendicular to the tangent at this point.
- Conversely, any line perpendicular to the endpoint of a radius is tangent to the circle.

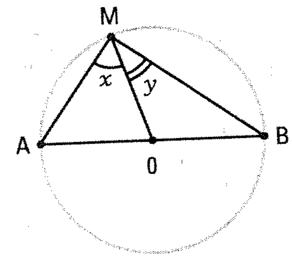
Thus, l is tangent to the circle at A $\Leftrightarrow l \perp \overline{0A}$



ACTIVITY 5 Right inscribed angle

Consider on the right the circle centred at 0 having diameter $\overline{AB}$. Let M be any point on the circle. Justify the steps showing that the inscribed angle AMB is right.

Let: m $\angle$0MA $= x$ and m $\angle$0MB $= y$ and let us show that $x + y = 90°$.

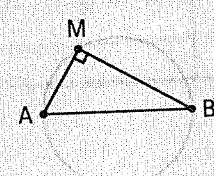


	Steps	Justifications
1.	Triangle A0M is isosceles.	*Sides 0A and 0M are congruent (radii).*
2.	Triangle B0M is isosceles.	*Sides 0B and 0M are congruent (radii).*
3.	m $\angle$0AM = m $\angle$0MA.	*The base angles of the isosceles $\triangle$A0M are congruent.*
4.	m $\angle$0MB = m $\angle$0BM.	*The base angles of the isosceles $\triangle$B0M are congruent.*
5.	m $\angle$A0M + $2x = 180°$.	*The sum of the angles of $\triangle$A0M is equal to 180°.*
6.	m $\angle$M0B + $2y = 180°$.	*The sum of the angles of $\triangle$B0M is equal to 180°.*
7.	m $\angle$A0M + m $\angle$M0B = 180°.	*Angle A0B is straight since AB is a diameter (hypothesis).*
8.	$x + y = 90°$	*We add the corresponding sides of the equalities established in steps 5 and 6 and we reduce using the equality established in step 7.*

THEOREM ON THE INSCRIBED RIGHT ANGLE

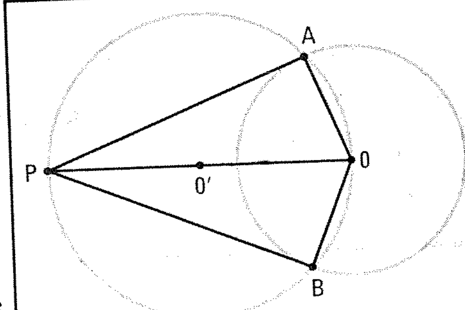
If M is any point on a circle having diameter $\overline{AB}$ then the inscribed angle AMB is right.

In other words, any inscribed angle intercepting a half-circle is right.



ACTIVITY 6 Construction of a tangent to a circle from a given point

Consider on the right the circle centred at 0 and an exterior point P. From point 0', midpoint of the line segment P0, we have drawn the circle centred at 0' with radius 0'0. We denote by A and B the intersection points of the two circles.



a) Explain why triangles 0AP and 0BP are right.

Angle 0AP is an inscribed angle intercepting the half-circle centred at 0'.

Similarly, angle 0BP is an inscribed angle intercepting the half-circle centred at 0'.

b) What can be said of lines PA and PB?

They are the two tangents to the circle centred at 0 passing through point P.

c) Find a procedure to draw the two tangents to a circle centred at 0 passing through a given exterior point P.

1. We locate the midpoint 0' of the line segment P0.

2. We draw the circle centred at 0' passing through P.

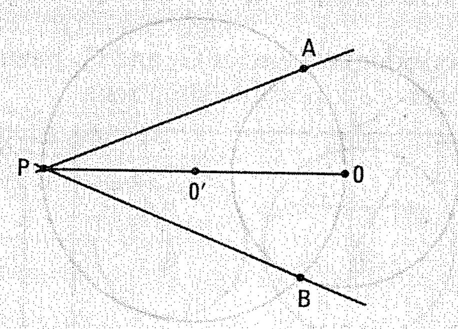
3. We locate the intersection points A and B of the two circles.

4. We draw the lines PA and PB tangent to the circle centred at 0.

CONSTRUCTION OF A TANGENT TO A CIRCLE FROM A GIVEN POINT

To draw the two tangents to a circle from a given exterior point P, we proceed in the following way.

1. We locate point 0', midpoint of $\overline{0P}$.
2. We draw the circle centred at 0' passing through P.
3. We locate the intersection points A and B of the two circles, the tangent points that we seek.
4. We draw the two tangents PA and PB.



d) What can be said of the line segments PA and PB?

Consider the right triangles 0AP and 0BP.

We have $m\overline{0A} = m\overline{0B}$ (radii), $m\overline{P0} = m\overline{P0}$ (common side).

Pythagoras' relation applied to each triangle implies that $m\overline{PA} = m\overline{PB}$.

e) What does the half-line P0 represent for angle APB?

Triangles 0AP and 0BP are congruent (case SSS).

$\angle AP0 \cong \angle BA0$ (corresponding elements).

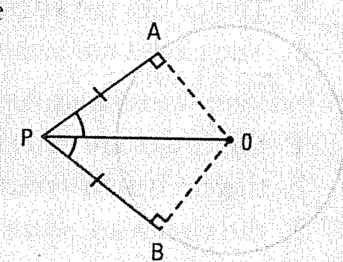
P0 is therefore the bisector of angle APB.

THEOREM ON TANGENT LINE SEGMENTS

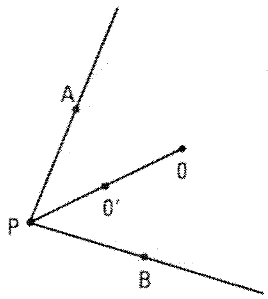
If from a point P, exterior to a circle, we draw the tangents to the circle PA and PB, then

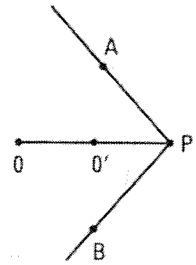
- the tangent line segments PA and PB are congruent.
- P0 is the bisector of angle APB.

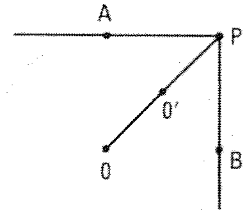
$\overline{PA} \cong \overline{PB}$ and $\angle 0PA \cong \angle 0PB$



2. Construct, in each case, the two tangents to the circle passing through P.

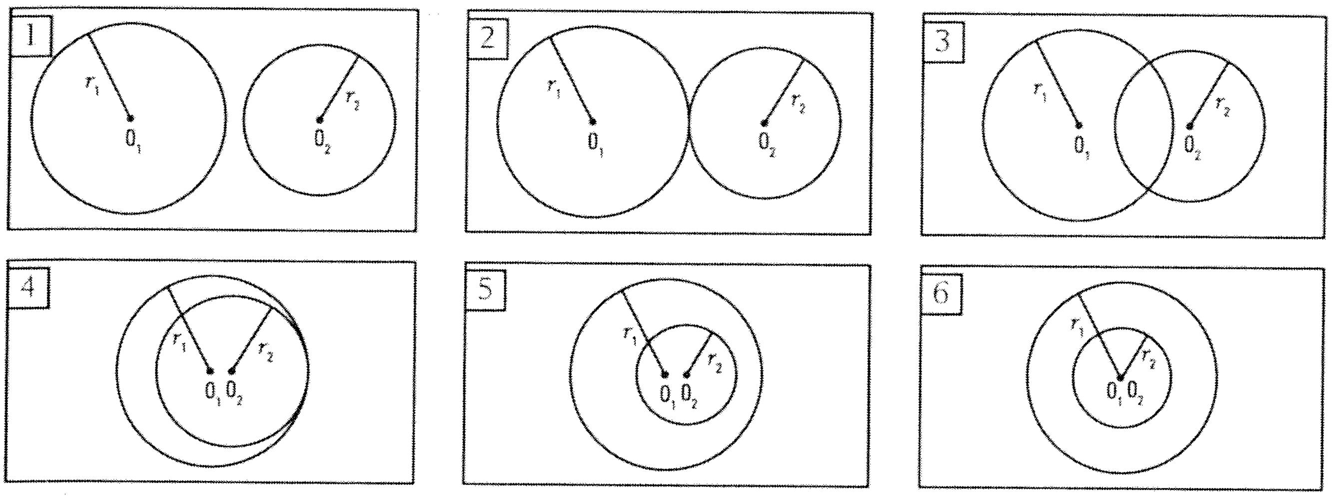
a) 

b) 

c) 

ACTIVITY 7 Relative position of two circles

Consider two circles C_1 and C_2 having respectively centres 0_1 and 0_2 and radii r_1 and r_2 ($r_1 > r_2$). We observe six possible situations.



a) Match each of the following descriptions with one of the situations above.

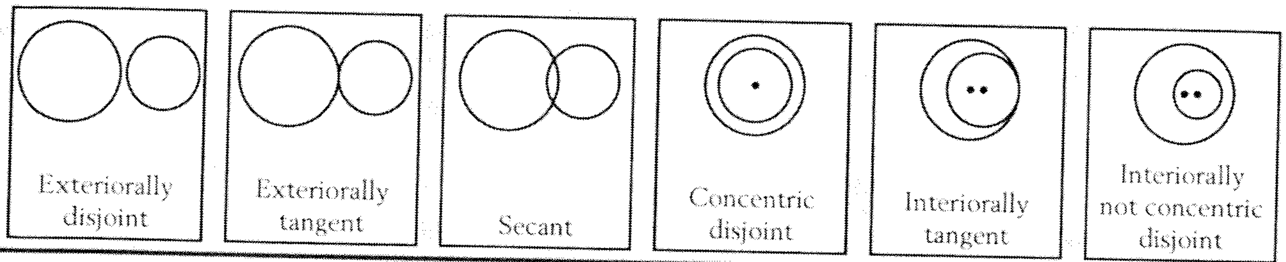
1. The circles are exteriorly disjoint. *1*
2. The circles are interiorly disjoint and are concentric. *6*
3. The circles are interiorly disjoint and are not concentric. *5*
4. The circles are exteriorly tangent. *2*
5. The circles are interiorly tangent. *4*
6. The circles are secant. *3*

b) Indicate the relation between $d(0_1, 0_2)$ and the radii r_1 and r_2 when

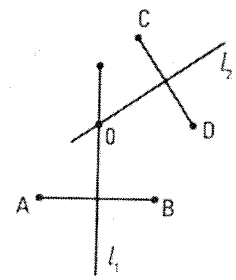
1. the circles are exteriorly disjoint. .... $d(0_1, 0_2) > r_1 + r_2$
2. the circles are exteriorly tangent. .... $d(0_1, 0_2) = r_1 + r_2$
3. the circles are secant. .... $d(0_1, 0_2) < r_1 + r_2$
4. the circles are interiorly tangent. .... $d(0_1, 0_2) = r_1 - r_2$
5. the circles are interiorly disjoint and are not concentric. .... $d(0_1, 0_2) < r_1 - r_2$
6. the circles are interiorly disjoint and are concentric. .... $d(0_1, 0_2) = 0$

RELATIVE POSITION OF TWO CIRCLES

Depending on the position of the centres and the length of the radii, we have the following positions for two circles.



3. Explain how to determine the centre of the circle on the right and then determine the centre.



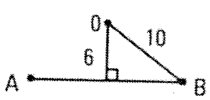
1. We draw any two chords $\overline{AB}$ and $\overline{CD}$.

2. We draw the perpendicular bisectors l_1 and l_2 of chords $\overline{AB}$ and $\overline{CD}$.

3. The intersection point 0 of the perpendicular bisectors is the centre of the circle.

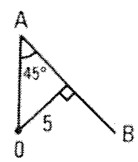
4. In each case, find the length of the chord AB.

a)



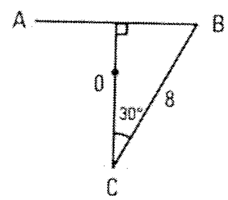
$m\overline{AB}$ = 16

b)



$m\overline{AB}$ = 10

c)



$m\overline{AB}$ = 8

5. A 16 cm chord lies 6 cm from the centre of a circle. What is the radius of the circle?

10 cm

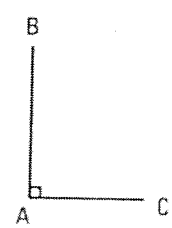
6. On a circle of radius 5 cm, at what distance from the centre is a 6 cm chord located?

4 cm

7. What is the length of a chord located 5 cm from the centre of a circle of radius 13 cm?

24 cm

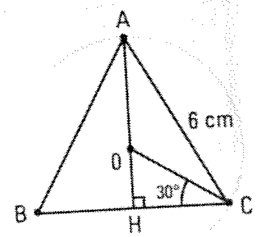
8. The two chords AB and AC on the right are perpendicular and measure respectively 6 cm and 8 cm. What is the area of the disk?



$m\overline{BC}$ = 10 cm; $\overline{BC}$ is a diameter; radius = 5 cm.

Area of the disk 25π cm^2.

9. The equilateral triangle ABC with side length 6 cm is inscribed in a circle centred at 0.



a) What is the radius of the circle?

C0 is the bisector of $\angle ACB$ and the perpendicular bisector of $\overline{AB}$.

(Property of the equilateral triangle) $\Rightarrow m\angle 0CH = 30°$

A0 is the perpendicular bisector of $\overline{BC} \Rightarrow \Delta 0HC$ is a rectangle and

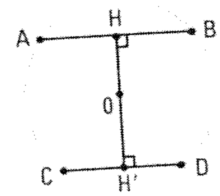
$m\overline{HC} = 3$ cm

$r = m\overline{0C} = \frac{m\overline{HC}}{\cos 30°} = \frac{3}{\cos 30°} = 2\sqrt{3}$ cm.

b) At what distance from the centre is each side of the triangle?

$m\overline{0H} = m\overline{HC} \cdot \tan 30° = \sqrt{3}$ cm.

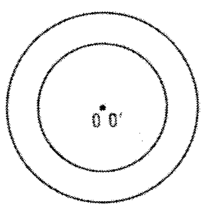
10. Two chords AB and CD, parallel and at a distance of 7 cm from each other, are located in a circle of radius 5 cm. What is the length of the chord CD if chord AB measures 8 cm?



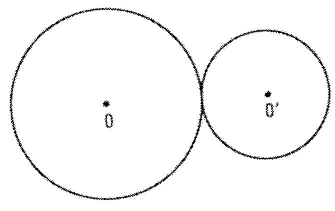
$m\overline{0H} = 3$ cm; $m\overline{0H'} = 4$ cm; $m\overline{H'C} = 3$ cm; $m\overline{CD} = 6$ cm

11. Draw two circles centred at 0 and 0′, having respective radii 3 cm and 2 cm and which are

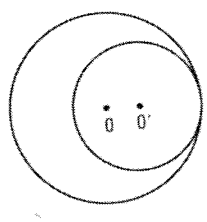
a) concentric disjoint.



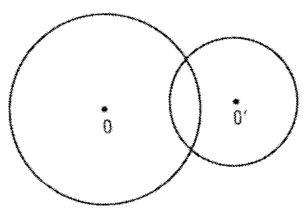
b) exteriorally tangent.



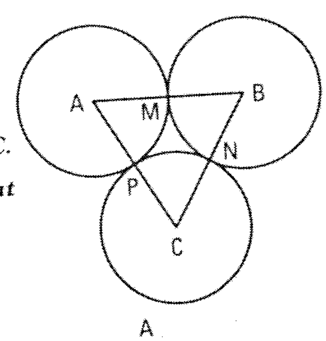
c) interiorally tangent.



d) secant and whose centres are at a distance of 4 cm from each other.



12. Starting with the equilateral triangle ABC on the right, draw three circles which are pairwise tangent. Explain your procedure.

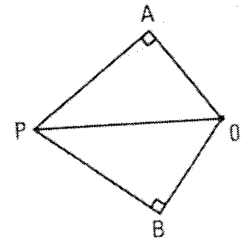


We locate points M, N and P, respective midpoints of sides AB, BC and AC.

We draw the circle centred at A with radius AM, the circle centred at

B with radius BN and the circle centred at C with radius CP.

13. Consider the circle of radius 3 cm centred at 0. Starting from point P located 5 cm from 0, we draw the tangent line segments PA and PB. Calculate the area of the shaded region.

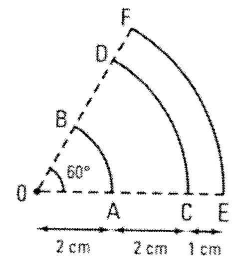


$m\overline{PA} = 4$ cm, Area $\Delta PA0 = 6$ cm^2, $\overline{PA} \cong \overline{PB}$ (theorem)

$\Delta PA0 \cong \Delta PB0$ (CCC); (SSS); Area of shaded region $= 12$ cm^2.

10.2 Angles, arcs and chords

ACTIVITY 1 Measure of an arc of circle

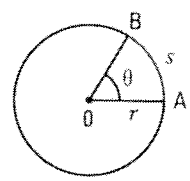


a) In terms of angle measure, the measure of an arc is equal to the measure of the central angle which intercepts this arc.

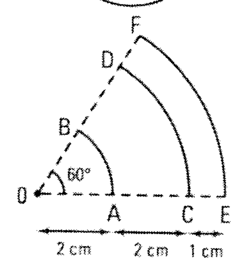
1. What is the measure, in degrees, of an arc corresponding to
   1) a half-circle? *180°* 2) a quarter-circle? *45°*
2. What is the measure, in degrees, of arcs AB, CD and EF represented on the right?
   $m\overset{\frown}{AB} = 60°, m\overset{\frown}{CD} = 60°, m\overset{\frown}{EF} = 60°.$

b) In terms of length measure, the measure of an arc is expressed using the radius of the circle on which this arc lies.

1. A circle with radius r has a circumference equal to $2\pi r$.
   What is the length of an arc corresponding to
   1) a half-circle? πr 2) a quarter-circle? $\frac{\pi r}{2}$



2. On the circle of radius r on the right, arc AB was drawn.
   We denote by θ, the measure of $\overset{\frown}{AB}$ in terms of angle measure and by s the measure of $\overset{\frown}{AB}$ in terms of length measure. Complete the ratio:
   $\frac{s}{2\pi r} = \frac{\theta}{360°}$.



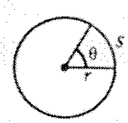
3. Using the ratio you just established, calculate the measure, in cm, of arcs AB, CD and EF on the right.
   1) $m\overset{\frown}{AB} = 2\pi r \cdot \frac{\theta}{360°} = 2\pi \cdot 2 \cdot \frac{60°}{360°} = \frac{2\pi}{3}$ cm
   2) $m\overset{\frown}{CD} = \frac{4\pi}{3}$ cm
   3) $m\overset{\frown}{EF} = \frac{5\pi}{3}$ cm

MEASURE OF AN ARC OF CIRCLE

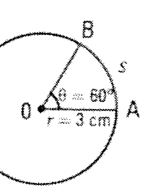
There exist two ways of measuring an arc, in terms of angle measure or in terms of length.

- In terms of angle measure, the measure θ of an arc is equal to the measure of the central angle intercepting this arc.
- In terms of length measure, the measure s of an arc is calculated using the ratio:

$$\boxed{\frac{s}{2\pi r} = \frac{\theta}{360°}}$$



Ex.: The circle on the right has radius $r = 3$ cm.
The central angle A0B measures 60° and intercepts arc AB.
In terms of angle measure, we have: $m\overset{\frown}{AB} = 60°$.
In terms of length measure, we have: $m\overset{\frown}{AB} = \pi$ cm.
Indeed, $s = 2\pi r \cdot \frac{\theta}{360°} = 2\pi \cdot 3 \cdot \frac{60°}{360°} = \pi \text{ cm} \approx 3.14$ cm.



1. A chord divides a circle into 2 arcs such that the measure of one is equal to 4 times the measure of the other. What is the measure, in degrees, of each arc?

The minor arc measures 72°, the major arc 288°.

2. In a circle of radius 6 cm, a central angle A0B measuring 40° intercepts arc AB. Determine the measure of arc AB

a) in degrees. ***$m\overset{\frown}{AB} = 40°$***   b) in cm. ***$m\overset{\frown}{AB} = \frac{4\pi}{3}$ cm***

3. In a circle of radius 10 cm, an arc measures $\frac{10\pi}{3}$ cm. What is the measure of this arc in degrees?

60°

4. What is the measure of a central angle which intercepts an arc of length 5 cm in a circle of radius 10 cm (π 3,14)? ***28.7°***

5. What is the measure of a central angle which intercepts an arc of length equal to the radius?

57.3°

6. Consider a circle of radius *r* and circumference C. θ denotes the measure, in degrees, of an arc and *s* its length, in cm. Complete the table on the right.

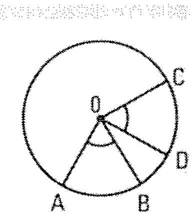
r	C	θ	s
5 cm	***10π cm***	60°	***$\frac{5\pi}{3}$ cm***
10 cm	***20π cm***	60°	***$\frac{10\pi}{3}$ cm***
2 cm	***4π cm***	45°	0.5π cm
4 cm	***8π cm***	***80°***	$\frac{16\pi}{9}$ cm
20 cm	40π cm	***120°***	$\frac{40\pi}{3}$ cm
12 cm	24π cm	180°	***12π cm***

7. Consider two concentric circles with respective radii 9 cm and 36 cm. A central angle intercepts an arc on each circle. What is the measure, in cm, of the arc intercepted on the larger circle if the arc intercepted on the smaller circle measures 4π cm?

The arc intercepted on the large circle measures16π cm.

ACTIVITY 2 Congruent central angles and congruent arcs

Explain why two congruent central angles intercept two congruent arcs and conversely, i.e. show the following equivalence:



$$AOB \quad COD \Leftrightarrow \overset{\frown}{AB} \cong \overset{\frown}{CD}$$

We know that the measure of an arc is equal to the measure of the angle intercepting this arc. Thus,

AOB   COD   ⇔ m   AOB = m   COD

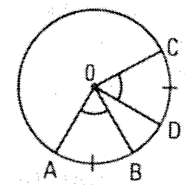
⇔ $m\overset{\frown}{AB} = m\overset{\frown}{CD}$

⇔ $\overset{\frown}{AB} \cong \overset{\frown}{CD}$

CONGRUENT CENTRAL ANGLES AND CONGRUENT ARCS

- Two congruent central angles intercept two congruent arcs.
- Two congruent arcs are intercepted by two congruent central angles.
  Thus, we have the following property:

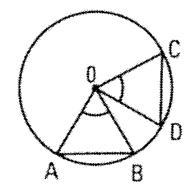
$$\angle A0B \cong \angle C0D \Leftrightarrow \overset{\frown}{AB} \cong \overset{\frown}{CD}$$



ACTIVITY 3 Congruent central angles and congruent chords

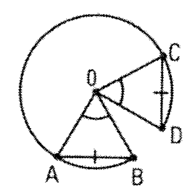
a) Justify the steps showing that two congruent central angles determine two congruent chords, i.e.: $\angle A0B \cong \angle C0D \Leftrightarrow \overline{AB} \cong \overline{CD}$.

Hypothesis $\angle A0B \cong \angle C0D$.



	Steps	Justifications
1.	$\overline{0A} \cong \overline{0C}$	*$\overline{0A}$ and $\overline{0C}$ are radii therefore they are congruent.*
2.	$\overline{0B} \cong \overline{0D}$	*$\overline{0B}$ and $\overline{0D}$ are radii therefore they are congruent.*
3.	$\angle A0B \cong \angle C0D$	*Hypothesis.*
4.	$\Delta A0B \cong \Delta C0D$	*Congruence case SAS.*
5.	$\overline{AB} \cong \overline{CD}$	*Corresponding elements of two congruent triangles are congruent.*

b) Justify he steps showing that two congruent chords determine two congruent central angles, i.e.: $\overline{AB} \cong \overline{CD} \Rightarrow \angle A0B \cong \angle C0D$.

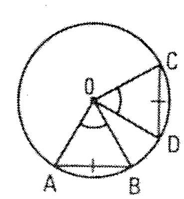


	Steps	Justifications
1.	$\overline{0A} \cong \overline{0C}$	*$\overline{0A}$ and $\overline{0C}$ are radii therefore they are congruent.*
2.	$\overline{0B} \cong \overline{0D}$	*$\overline{0B}$ and $\overline{0D}$ are radii therefore they are congruent.*
3.	$\overline{AB} \cong \overline{CD}$	*Hypothesis.*
4.	$\Delta A0B \cong \Delta C0D$	*Congruence case SSS.*
5.	$\angle A0B \cong \angle C0D$	*Corresponding elements of two congruent triangles are congruent.*

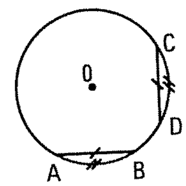
CONGRUENT CENTRAL ANGLES AND CONGRUENT CHORDS

- Two congruent central angles determine two congruent chords.
- Two congruent chords are determined by two congruent central angles.
  Thus, we have the following property:

$$\angle A0B \cong \angle C0D \Leftrightarrow \overline{AB} \Leftrightarrow \overline{CD}$$



ACTIVITY 4 Congruent chords and congruent arcs



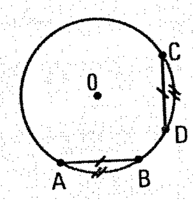
Explain why two congruent chords subtend two congruent arcs and conversely. In other words, show the following equivalence:

$\overline{AB} \cong \overline{CD} \Leftrightarrow \overset{\frown}{AB} \cong \overset{\frown}{CD}$.

We know that $\overline{AB} \cong \overline{CD} \Leftrightarrow \angle A0B \cong \angle C0D$ (property) and that $\angle A0B \cong \angle C0D \Leftrightarrow \overset{\frown}{AB} \cong \overset{\frown}{CD}$ (property).

We deduce, by transitivity, that $\overline{AB} \cong \overline{CD} \Leftrightarrow \overset{\frown}{AB} \cong \overset{\frown}{CD}$.

CONGRUENT CHORDS AND CONGRUENT ARCS

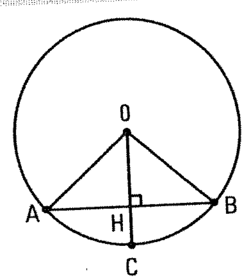


- Two congruent chords subtend two congruent arcs.
- Two congruent arcs are subtended by two congruent chords.

Thus, we have the following property:

$$\overline{AB} \cong \overline{CD} \Leftrightarrow \overset{\frown}{AB} \cong \overset{\frown}{CD}$$

ACTIVITY 5 Radius perpendicular to a chord



Justify the steps showing that any radius perpendicular to a chord

- divides the chord into two congruent line segments.
- divides the subtended arc into two congruent arcs.

Hypothesis: The radius 0C is perpendicular to chord AB at H.

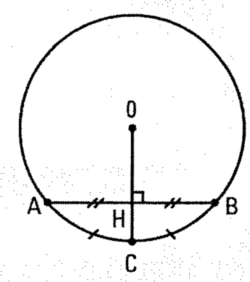
	Steps	Justifications
1.	Triangles 0HA and 0HB are right at H.	*Radius 0C is perpendicular to AB at H.*
2.	$\overline{0A} \cong \overline{0B}$	*$\overline{0A} \cong \overline{0B}$ are radii and therefore they are congruent.*
3.	$\overline{0H} \cong \overline{0H}$	*$\overline{0H}$ is a common side of triangles 0HA and 0HB.*
4.	$\overline{HA} \cong \overline{HB}$	*Pythagoras' relation applied to two right triangles having congruent hypotenuses $(\overline{0A} \cong \overline{0B})$ and side 0H in common.*
5.	$\Delta 0HA \cong \Delta 0HB$	*Congruence case SSS.*
6.	$\angle A0H \cong \angle B0H$	*Corresponding elements of 2 congruent triangles are congruent.*
7.	$\overset{\frown}{AC} \cong \overset{\frown}{BC}$	*Congruent central angles A0H and B0H intercept congruent arcs AC and BC (property).*

THEOREM ON THE RADIUS PERPENDICULAR TO A CHORD

- Any radius perpendicular to a chord:
  - divides the chord into two congruent line segments.
  - divides the subtended arc into two congruent arcs.

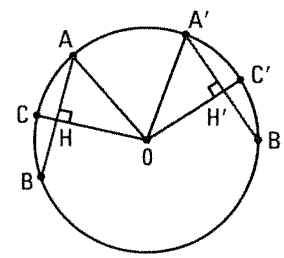
Thus, $\overline{0C} \perp \overline{AB} \Rightarrow \overline{HA} \cong \overline{HB}$

$\overline{0C} \perp \overline{AB} \Rightarrow \overset{\frown}{AC} \cong \overset{\frown}{CB}$



ACTIVITY 6 Congruent chords

a) Justify the steps showing that two congruent chords are located at the same distance from the centre.



Hypothesis: Chords $\overline{AB}$ and $\overline{A'B'}$ are congruent.

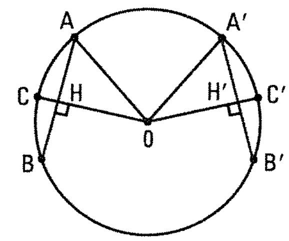
Construction: $\overline{0A}, \overline{0B}, \overline{0A'}, \overline{0B'}$ are radii.

$\overline{0C}$ is a radius perpendicular to chord $\overline{AB}$ at H.

$\overline{0C'}$ is a radius perpendicular to chord $\overline{A'B'}$ at H'.

	Steps	Justifications
1.	$\Delta 0HA$ and $\Delta 0H'A'$ are right.	***$\overline{0C} \perp \overline{AB}$ (construction) and $\overline{0C'} \perp \overline{A'B'}$ (construction).***
2.	$\overline{HA} \cong \overline{HB}$	***The radius perpendicular to chord AB (construction) divides chord AB into two congruent line segments HA and HB (theorem).***
3.	$\overline{H'A'} \cong \overline{H'B'}$	***The radius perpendicular to chord A'B' (construction) divides chord A'B' into two congruent line segments H'A' and H'B' (theorem).***
4.	$\overline{HA} \cong \overline{H'A'}$	***Line segments AB and A'B' being congruent, their halves $\overline{HA}$ and $\overline{H'A'}$ are congruent.***
5.	$\overline{0A} \cong \overline{0A'}$	***$\overline{0A}$ and $\overline{0A'}$ are radii and therefore they are congruent.***
6.	$\overline{0H} \cong \overline{0H'}$	***Pythagoras' relation applied to 2 right triangles having congruent hypotenuses $(\overline{0A} \cong \overline{0A'})$ and congruent sides $(\overline{HA} \cong \overline{H'A'})$.***
7.	0 is at the same distance from chords AB and A'B'.	***$\overline{0H'} \perp \overline{A'B'}$ (construction), and $\overline{0H} \perp \overline{AB}$ (construction) $m\overline{0H} \cong m\overline{0H'}$ (see 6.).***

b) Justify the steps showing that two chords located at the same distance from the centre are congruent.



Hypothesis: – $\overline{0H} \perp \overline{AB}$ and $\overline{0H'} \perp \overline{A'B'}$.

– $\overline{0H} \cong \overline{0H'}$.

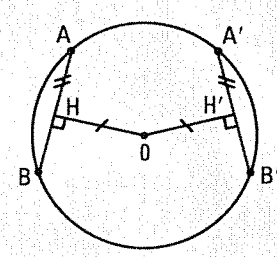
Construction: $\overline{0A}, \overline{0C}, \overline{0A'}$ and $\overline{0C'}$ are radii.

	Steps	Justifications
1.	$\Delta 0HA$ and $\Delta 0H'A'$ are right	***$\Delta 0HA$ is right, since $\overline{0H} \perp \overline{AB}$ (hypothesis)*** ***$\Delta 0H'A'$ is right, since $\overline{0H'} \perp \overline{A'B'}$ (hypothesis)***
2.	$\overline{0H} \cong \overline{0H'}$	***Hypothesis***
3.	$\overline{0A} \cong \overline{0A'}$	***$\overline{0A}$ and $\overline{0A'}$ are radii and therefore they are congruent.***
4.	$\overline{HA} \cong \overline{H'A'}$	***Pythagoras' relation applied to 2 right triangles having congruent hypotenuses $(\overline{0A} \cong \overline{0A'})$ and congruent sides $(\overline{0H} \cong \overline{0H'})$.***
5.	$\overline{HA} \cong \overline{HB}$ and $\overline{H'A'} \cong \overline{H'B'}$	***Any radius perpendicular to a chord divides the chord into 2 congruent chords.***
6.	$\overline{AB} \cong \overline{A'B'}$	***Consequence of steps 4 and 5.***

THEOREM ON CONGRUENT CHORDS

- Two congruent chords are located at the same distance from the centre.
- Two chords located at the same distance from the centre are congruent.

Thus, $\overline{AB} \cong \overline{A'B'} \Leftrightarrow \overline{0H} \cong \overline{0H'}$

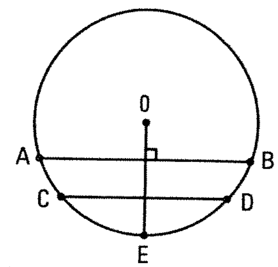


ACTIVITY 7 Parallel chords

Justify the steps showing that the arcs between two parallel chords are congruent.

Hypothesis: Chords AB and CD are parallel.

Construction: Radius 0E is perpendicular to chord AB.

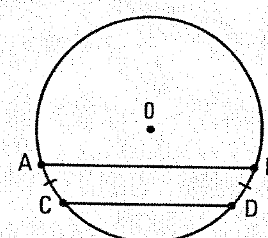


	Steps	Justifications
1.	$\overset{\frown}{AE} \cong \overset{\frown}{EB}$	*Radius $\overline{0E}$ perpendicular to $\overline{AB}$ (construction) divides arc AB into two congruent arcs AE and EB (theorem).*
2.	$\overline{0E} \perp \overline{CD}$	*When two lines are parallel, any line perpendicular to one of them is perpendicular to the other one (theorem). $\overline{AB}$ // $\overline{CD}$ (hypothesis) and $\overline{0E} \perp \overline{AB}$ (construction).*
3.	$\overset{\frown}{CE} \cong \overset{\frown}{ED}$	*Radius 0E perpendicular to $\overline{CD}$ (see 2) divides arc CD into two congruent arcs CE and ED (theorem).*
4.	$\overset{\frown}{AC} \cong \overset{\frown}{BD}$	$\begin{cases} \overset{\frown}{AE} \cong \overset{\frown}{EB} \\ \overset{\frown}{CE} \cong \overset{\frown}{ED} \end{cases} \Rightarrow m\overset{\frown}{AE} - m\overset{\frown}{CE} = m\overset{\frown}{EB} - m\overset{\frown}{ED} \Leftrightarrow m\overset{\frown}{AC} = m\overset{\frown}{BD} \Leftrightarrow \overset{\frown}{AC} \cong \overset{\frown}{BD}$.

THEOREM ON PARALLEL CHORDS

The arcs between two parallel chords are congruent.

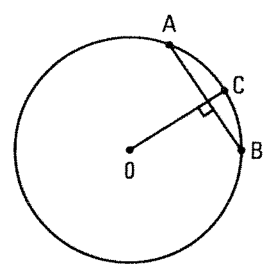
Thus, $\overline{AB} // \overline{CD} \Rightarrow \overset{\frown}{AC} \cong \overset{\frown}{BD}$



8. Consider the circle centred at 0 and arc AB. Explain how to determine the midpoint C of arc AB without using a protractor and then locate point C.

1. We draw chord AB.

2. We draw radius 0C perpendicular to chord AB. Radius 0C divides arc AB, according to the theorem on the radius perpendicular to a chord, into two congruent arcs.

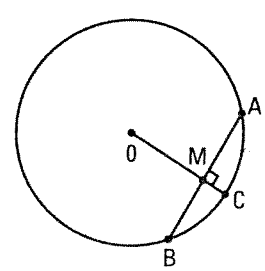


9. Consider the circle centred at 0 and the interior point M. Explain how to draw chord AB whose midpoint is point M and then draw chord AB.

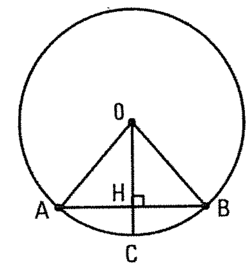
1. We draw radius 0C passing through M.

2. We draw chord AB perpendicular to radius 0C at M.

Radius 0C, perpendicular to chord AB, divides this chord into two congruent line segments according to the theorem on the radius perpendicular to a chord.



10. In a circle of radius 5 cm, a chord AB has length 6 cm. Calculate the area of triangle A0B.

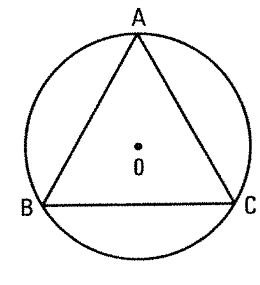


Let us draw radius $\overline{0C}$ perpendicular to $\overline{AB}$ at H. We have:

$m\overline{0A}$ = 5 cm (radius); $m\overline{HA}$ = 3 cm cm (theorem on the radius perpendicular to a chord); $m\overline{0H}$ = 4 cm (by Pythagoras).

Area $\triangle A0B = \frac{1}{2} \cdot 6 \cdot 4 = 12\ cm^2$

11. Let A, B And C be three points on a circle centred at 0 such that chords AB, AC and BC are located at the same distance from the centre.
Explain why triangle ABC is equilateral.



$\overline{AB} \cong \overline{AC}$ since chords AB and AC are located at the same distance from the centre of the circle (theorem on congruent chords).

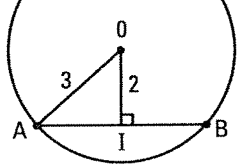
$\overline{AC} \cong \overline{BC}$ since chords AC and BC are located at the same distance from the centre (same).

We deduce that $\overline{AB} \cong \overline{AC} \cong \overline{BC}$. Congruence is transitive.

Therefore, triangle ABC is equilateral since its three sides are congruent.

12. Find, in each case, the length of chord AB.

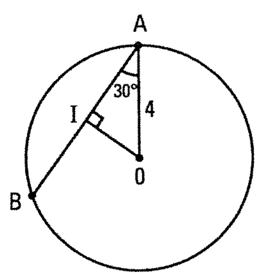
a)



$(mAI)^2 = 3^2 - 2^2 = 13$

$m\overline{AB} = 2\sqrt{13}$ u

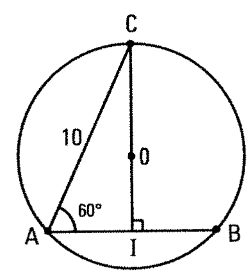
b)



$m\overline{AI} = 4 \cos 30° = 2\sqrt{3}$

$m\overline{AB} = 4\sqrt{3}$ u

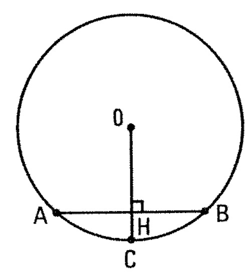
c)



$m\overline{IA} = 10 \cos 60° = 5$ u

$m\overline{AB} = 10$ u

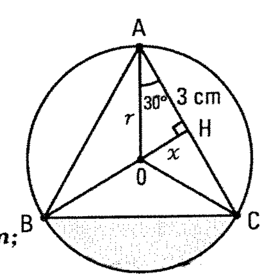
13. In a circle, a chord of length 16 cm is located 6 cm from the centre. Determine the circumference of this circle.



Hypotheses: $m\overline{AB}$ = 16 cm; $\overline{0C} \perp \overline{AB}$; $m\overline{0H}$ = 6 cm.
$m\overline{HA}$ = 8 cm (theorem on the radius perpendicular to a chord)
$m\overline{0H}$ = 6 cm; $m\overline{0A}$ = 10 cm (Pythagoras' theorem).

Circumference = $2\pi r = 20\pi$ cm

14. Equilateral triangle ABC on the right with a side length of 6 cm is inscribed in a circle centred at 0.



a) What is the distance between the centre 0 and each of the sides of the triangle? Justify your reasoning.

We draw the bisectors A0, B0 and C0 that are simultaneously altitudes and medians (property of the equilateral △). We deduce that: $m\overline{AH}$ = 3 cm; $m\angle 0AH = 30°$.

We have: $\tan 30° = \frac{x}{3} = \frac{\sqrt{3}}{3} \Rightarrow x = \sqrt{3}$

The distance between the centre 0 and each side of the triangle is equal to $\sqrt{3}$ cm.

b) What is the radius of the circle?

$(\sqrt{3})^2 + 3^2 = r^2 \Rightarrow r^2 = 12 \Rightarrow r = 2\sqrt{3}$ cm.

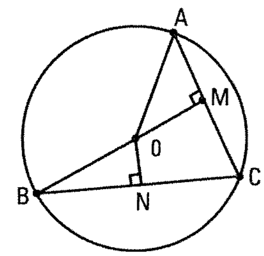
c) What is the area of the region bounded by arc BC and chord BC?

Height of triangle ABC = $6 \sin 60° = 3\sqrt{3}$ cm.

Area $\triangle ABC = \frac{6 \cdot 3\sqrt{3}}{2} = 9\sqrt{3}$ cm²; Area of disk = $\pi(\sqrt{12})^2 = 12\pi$ cm².

Area we seek = $\frac{1}{3}(12\pi - 9\sqrt{3}) = (4\pi - 3\sqrt{3})$ cm².

15. On the right, chord AC of length 12 cm is located 8 cm from the centre. What is the length, rounded to the nearest tenth, of chord BC located 4 cm from the centre?



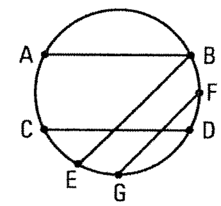
Let M be the midpoint of $\overline{AC}$ and N the midpoint of $\overline{BC}$.

$\triangle 0MA$ is right. We have: $r^2 = 6^2 + 8^2$; $r = 10$ cm.

$\triangle 0BN$ is right. We have: $10^2 = 4^2 + (m\overline{BN})^2$; $m\overline{BN} = \sqrt{84}$; $m\overline{BC} \approx 18.3$ cm

16. In the circle on the right, we have: $\overline{AB} \parallel \overline{CD}$ and $\overline{BE} \parallel \overline{FG}$.

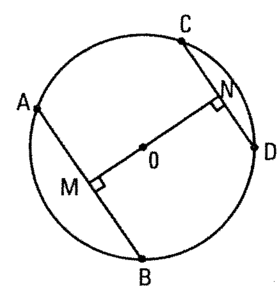
If $m\overset{\frown}{AC} = 60°$ and $m\overset{\frown}{EG} = 40°$, find the measure of $\overset{\frown}{DF}$.



$m\overset{\frown}{DF} = 20°$

17. In the circle of radius 10 cm on the right, two parallel chords AB and CD were drawn.

If chord AB measures 16 cm and the distance between the two chords is equal to 14 cm, calculate the length of chord CD.

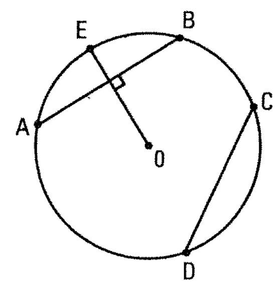


Consider the right $\triangle 0MA$. We have: $10^2 = 8^2 + (m\overline{0M})^2$

$m\overline{0M} = 6$ cm. Consider the right $\triangle 0NC$. We have: $10^2 = 8^2 + (m\overline{CN})^2$

$m\overline{CN} = 6$ cm; $m\overline{CD} = 12$ cm.

18. Chords AB and CD in the circle on the right are at the same distance from the centre 0. Let 0E be a radius perpendicular to chord AB. Show that $m\overset{\frown}{AE} = \frac{1}{2} m\overset{\frown}{CD}$.



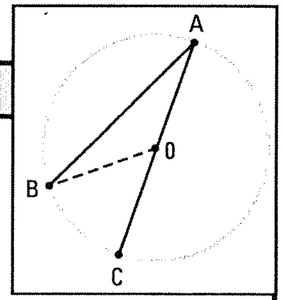
Steps	Justifications
1. $\overline{AB} \cong \overline{CD}$	*Two chords located at the same distance from the centre are congruent (theorem).*
2. $\overset{\frown}{AB} \cong \overset{\frown}{CD}$	*Two congruent chords subtend two congruent arcs (theorem).*
3. $m\overset{\frown}{AE} = \frac{1}{2} m\overset{\frown}{AB}$	*Any radius perpendicular to a chord divides the subtended arc into 2 congruent arcs (theorem).*
4. $m\overset{\frown}{AE} = \frac{1}{2} m\overset{\frown}{CD}$	*We substitute, in the equality established in 3, $m\overset{\frown}{AB}$ by $m\overset{\frown}{CD}$ since $m\overset{\frown}{AB} = m\overset{\frown}{CD}$ (see 2).*

10.3 Inscribed angle - Interior angle - Exterior angle

Activity 1 Measure of an inscribed angle

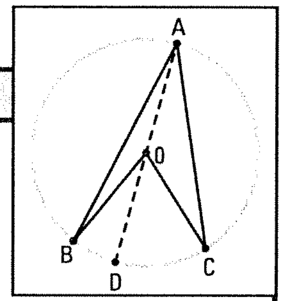
a) Justify, in each of the following cases, the steps showing that an inscribed angle measures half the measure of the central angle when these two angles intercept the same arc.

1st case: one of the sides of the inscribed angle is a diameter.



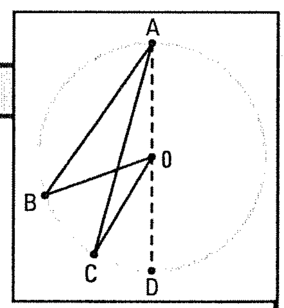
	Steps	Justifications
1.	Let us draw the radius 0B.	***Construction.***
2.	$\triangle A0B$ is isosceles.	$\overline{0A} \cong \overline{0B}$ ***(radii of the circle).***
3.	$\angle 0AB \cong \angle 0BA$	***The base angles of an isosceles triangle are congruent.***
4.	$m\ \angle B0C = m\ \angle 0AB + m\ \angle 0BA$	***The measure of an exterior angle of a triangle equals half the sum of the measures of the interior angles not adjacent to it.***
5.	$m\ \angle B0C = 2 \times m\ \angle 0AB$	***m*** $\angle 0BA$ ***= m*** $\angle 0AB$***, substitution in equality 4.***
6.	$m\ \angle B0C = 2 \times m\ \angle BAC$ ou $m\ \angle BAC = \frac{1}{2} \times m\ \angle B0C$	$\angle 0AB \cong \angle BAC$

2nd case: the centre of the circle is located in the interior of the inscribed angle.



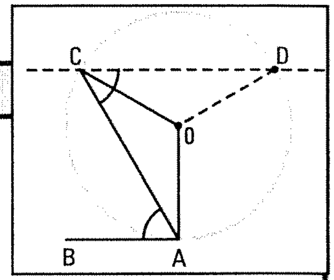
	Steps	Justifications
1.	Let us draw the diameter AD.	***Construction.***
2.	$m\ \angle BAC = m\ \angle BAD + m\ \angle DAC$	***See figure.***
3.	Thus, $m\ \angle BAC = \frac{1}{2} \times m\ \angle B0D + \frac{1}{2} \times m\ \angle D0C$	$m\ \angle BAD = \frac{1}{2} \cdot m\ \angle B0D$ ***and*** $m\ \angle DAC = \frac{1}{2} \cdot m\ \angle D0C$ ***(see 1st case)***
4.	Thus, $m\ \angle BAC = \frac{1}{2} \times (m\ \angle B0D + m\ \angle D0C) = \frac{1}{2} \times m\ \angle B0C$	***See figure.***

3rd case: the centre of the circle is outside the inscribed angle.



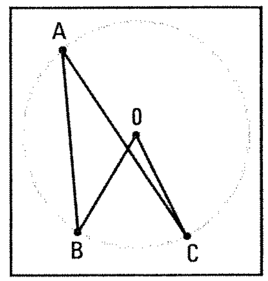
	Steps	Justifications
1.	Let us draw the diameter AD.	***Construction.***
2.	$m\ \angle BAC = m\ \angle BAD - m\ \angle CAD$	***See figure.***
3.	$m\ \angle BAC = \frac{1}{2} \times m\ \angle B0D - \frac{1}{2} \times m\ \angle C0D$	$m\ \angle BAD = \frac{1}{2} \cdot m\ \angle B0D$ ***and*** $m\ \angle CAD = \frac{1}{2} \cdot m\ \angle C0D$ ***(see 1st case)***
4.	Thus, $m\ \angle BAC = \frac{1}{2}(m\ \angle B0D - m\ \angle C0D) = \frac{1}{2} \times m\ \angle B0C$	***See figure.***

4th case: the inscribed angle has a side tangent to the circle.



	Steps	Justifications
1.	Through point C, we draw the line CD parallel to AB.	***Construction.***
2.	$\angle BAC \cong \angle ACD$	***Alternate interior angles.***
3.	$\overset{\frown}{AC} \cong \overset{\frown}{AD}$	***2 parallel lines secant to a circle intercept congruent angles (theorem).***
4.	$\angle A0C \cong \angle A0D$	***Central angles intercepting 2 congruent arcs are congruent.***
5.	$m\ \angle ACD = \frac{1}{2} \times m\ \angle A0D$	***See 2nd case. The inscribed angle ACD and the central angle A0D (0 being interior to the inscribed angle ACD) intercept the same arc AD.***
6.	Thus, $m\ \angle BAC = \frac{1}{2} \times m\ \angle A0C$	***$m\ \angle A0D = m\ \angle A0C$ and $m\ \angle ACD = m\ \angle BAC$ We substitute in equality 5.***

b) Explain why the measure of an inscribed angle is equal to half the measure of the intercepted arc.



We know that $m\ \angle BAC = \frac{1}{2} \cdot m\ \angle B0C$ (see a)) and that $m\ \angle B0C = m\overset{\frown}{BC}$ (the measure of a central angle is equal to the measure of the intercepted arc).

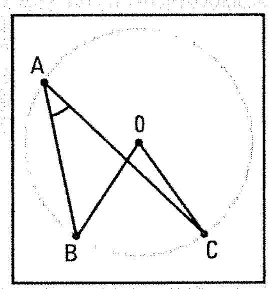
We deduce, by transitivity, that:

$m\ \angle BAC = \frac{1}{2} \cdot m\overset{\frown}{BC}$.

c) What can be said of two inscribed angles which intercept the same arc?

These inscribed angles are congruent.

THEOREM ON THE MEASURE OF AN INSCRIBED ANGLE



- If an **inscribed** angle and a **central** angle intercept the same arc, the measure of the inscribed angle is equal to **half the measure** of the central angle.

$$m\ \angle BAC = \frac{1}{2} m\ \angle B0C$$

- Thus, the measure of an inscribed angle is equal to **half the measure** of the **intercepted arc.**

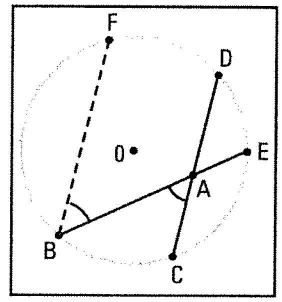
$$m\ \angle BAC = \frac{1}{2} m\overset{\frown}{BC}$$

- Two **inscribed angles** which intercept the same arc are **congruent.**

ACTIVITY 2 Measure of an interior angle

Justify the steps showing that an interior angle measures half the sum of the measures of the arcs intercepted by the sides of the angle and by their extensions.

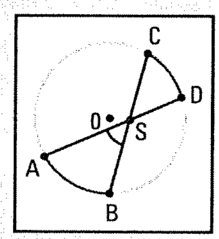
	Steps	Justifications
1.	We draw the line BF parallel to line CD.	*Construction.*
2.	$\angle BAC \cong \angle FBE$	*Alternate interior angles.*
3.	$m\angle FBE = \frac{1}{2} \cdot m\overset{\frown}{FE}$	*The measure of an inscribed angle is equal to half the measure of the intercepted arc.*
4.	$m\angle FBE = \frac{1}{2}\left(m\overset{\frown}{FD} + m\overset{\frown}{DE}\right)$	*See figure.* $m\overset{\frown}{FE} = m\overset{\frown}{FD} + m\overset{\frown}{DE}$
5.	$m\overset{\frown}{FD} = m\overset{\frown}{BC}$	*Two parallel secants to a circle determine 2 congruent arcs.*
6.	$m\angle FBE = \frac{1}{2}\left(m\overset{\frown}{BC} + m\overset{\frown}{DE}\right)$	$m\overset{\frown}{FD} = m\overset{\frown}{BC}$*, substitution in equality 4.*
7.	$m\angle BAC = \frac{1}{2}\left(m\overset{\frown}{BC} + m\overset{\frown}{DE}\right)$	$m\angle BAC = m\angle FBE$*, substitution in equality 6.*



THEOREM ON THE MEASURE OF AN INTERIOR ANGLE

The measure of an angle whose vertex is located in the interior of a circle is equal to **half the sum** of the measures of the arcs intercepted by the sides of the angle and by their extensions.

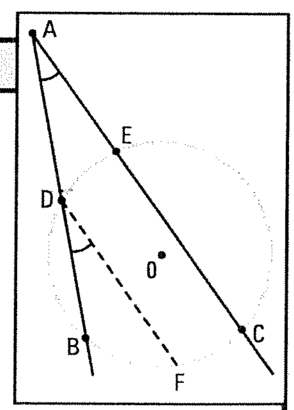
Thus, $m\angle ASB = \frac{1}{2}\left(m\overset{\frown}{AB} + m\overset{\frown}{CD}\right)$



ACTIVITY 3 Measure of an exterior angle

Justify the steps showing that an exterior angle measures half the difference of the arcs intercepted by the sides of the angle.

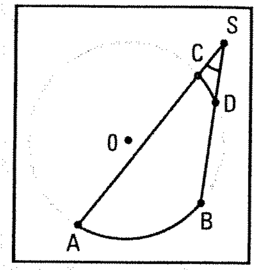
	Steps	Justifications
1.	We draw the line DF parallel to line AC.	*Construction.*
2.	$\angle BAC \cong \angle BDF$	*Corresponding angles.*
3.	$m\angle BDF = \frac{1}{2} \cdot m\overset{\frown}{BF}$	*The measure of an inscribed angle is equal to half the measure of the intercepted arc.*
4.	$m\angle BDF = \frac{1}{2}\left(m\overset{\frown}{BC} - m\overset{\frown}{FC}\right)$	*See figure.* $m\overset{\frown}{BF} = m\overset{\frown}{BC} - m\overset{\frown}{FC}$
5.	$m\overset{\frown}{DE} = m\overset{\frown}{FC}$	*2 parallel lines secant to a circle determine 2 congruent arcs.*
6.	$m\angle BDF = \frac{1}{2}\left(m\overset{\frown}{BC} - m\overset{\frown}{DE}\right)$	$m\overset{\frown}{DE} = m\overset{\frown}{FC}$*, substitution in equality 4.*
7.	$m\angle BAC = \frac{1}{2}\left(m\overset{\frown}{BC} - m\overset{\frown}{DE}\right)$	$m\angle BAC = m\angle BDF$*, substitution in equality 6.*



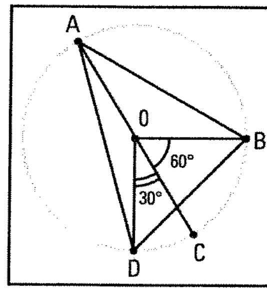
THEOREM ON THE MEASURE OF AN EXTERIOR ANGLE

The measure of an angle whose vertex is located outside a circle is equal to **half the difference** of the measures of the **arcs intercepted** by the sides of the angle..

Thus, $m\angle ASB = \frac{1}{2}\left(m\overset{\frown}{AB} - m\overset{\frown}{CD}\right)$



1. Using the figure on the right, determine the measures of the following angles. Justify each answer.



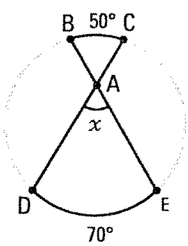
a) m ∠BAC = ***30°***
Angle BAC is inscribed and intercepts arc BC measuring 60°.

b) m ∠DAC = ***15°***
Angle DAC is inscribed and intercepts arc CD measuring 30°.

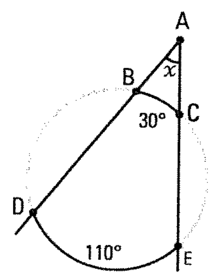
c) m ∠AB0 = ***30°***
Triangle 0AB is isosceles ($\overline{0A}$ and $\overline{0B}$ are radii). The base angles 0AB and 0BA are congruent.

d) $m\overset{\frown}{AB}$ = ***120°***
We have: $m\overset{\frown}{AC} = 180°$ and $m\overset{\frown}{AB} + m\overset{\frown}{BC} = m\overset{\frown}{AC}$

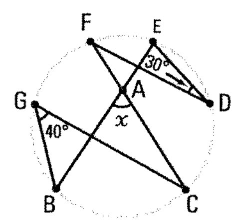
2. Find the value of x in each of the following cases.

a)


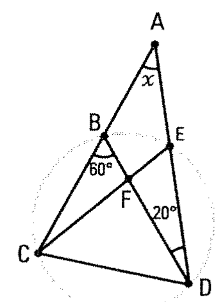
x = 60°

b)


x = 40°

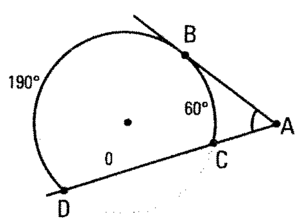
c)


x = 70°

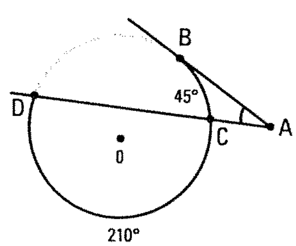
d)


x = 40°

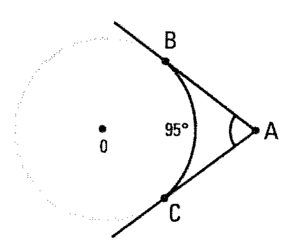
3. Find the measure of angle A in each of the following cases.

a)


65°

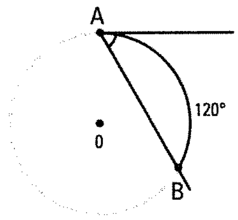
b)


30°

c)


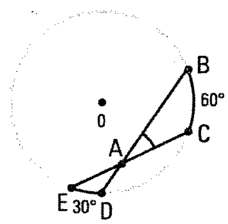
85°

d)



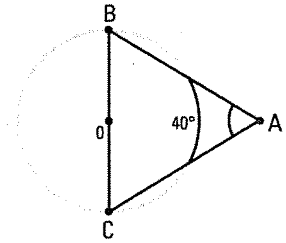
60°

e)



45°

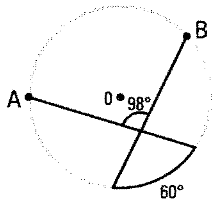
f)



70°

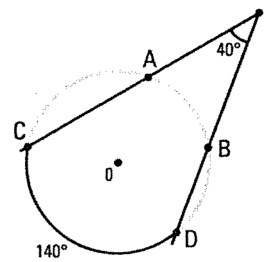
4. In each of the following cases, give the measure, in degrees, of arc AB.

a)



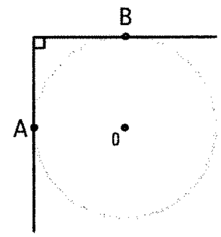
$m\widehat{AB} = 136°$

b)



$m\widehat{AB} = 60°$

c)



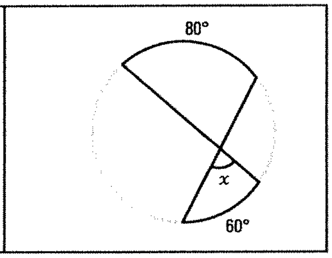
$m\widehat{AB} = 90°$

5. Find the value of x in each of the following cases.

a)

$x = \dfrac{80° + 60°}{2}$

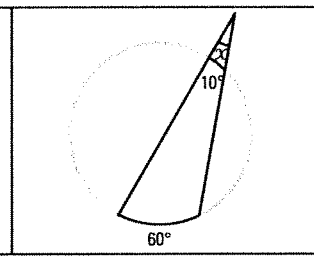
$x = 70°$



b)

$x = \dfrac{60° - 10°}{2}$

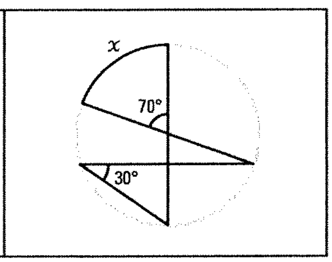
$x = 25°$



c)

$70° = \dfrac{60° + x}{2}$

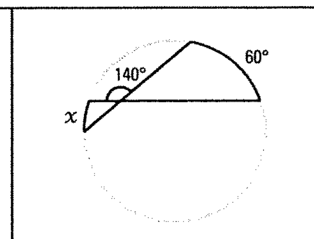
$x = 80°$



d)

$40° = \dfrac{x + 60°}{2}$

$x = 20°$



e)

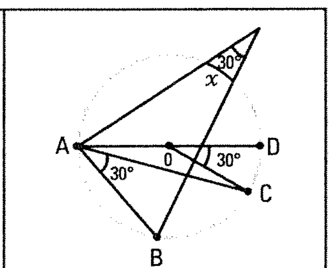
$m\widehat{CD} = 30°$

$m\widehat{BC} = 60°$

$m\widehat{AB} = 90°$

$30° = \dfrac{90 - x}{2}$

$x = 30°$



f)

$m\angle AOB = 48°$

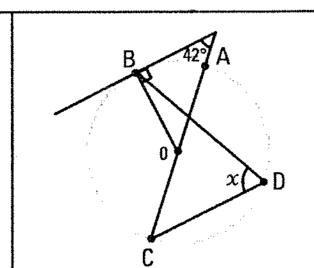
$m\widehat{AB} = 48°$

$m\widehat{AC} = 180°$

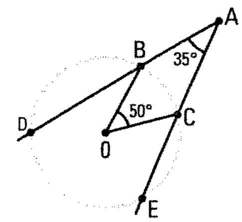
$m\widehat{BC} = 132°$

$m\angle BDC = 66°$

$x = 66°$



6. On the figure on the right, the central angle B0C measures 50° and the exterior angle BAC measures 35°. What is the measure of arc DE?



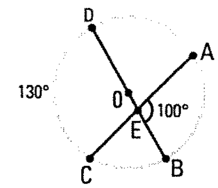
$m\overset{\frown}{BC} = 50°$; $m\overset{\frown}{DE} = x$; $35° = \frac{x - 50}{2}$; $x =$ ~~125°~~. 120°

70 = x − 50

7. In the circle on the right, $\overline{BD}$ is a diameter.

We have: $m\overset{\frown}{CD} = 130°$ and $m\angle AEB = 100°$.

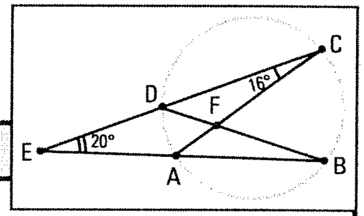
What is the measure of arc AD?



$m\overset{\frown}{CB} = 50°$, $m\angle CEB = 80°$, $m\overset{\frown}{AD} = x$

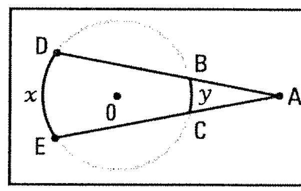
$80° = \frac{x + 50°}{2}$; $x = 110°$.

8. On the figure on the right, angle ACD measures 16° and angle AEC measures 20°. Determine the measure of angle AFB. Justify your reasoning.



	Steps	Justifications
1.	$m\angle ABD = 16°$	$\angle ABD$ and $\angle ACD$ are congruent since they intercept the same arc AD.
2.	$m\overset{\frown}{BC} = 72°$	$20° = \frac{m\overset{\frown}{BC} - 32°}{2}$ (theorem)
3.	$m\angle CAB = 36°$	The measure of the inscribed angle is equal to half the measure of the intercepted arc.
4.	$m\angle AFB = 128°$	The sum of the measures of the angles of triangle AFB equals 180°.

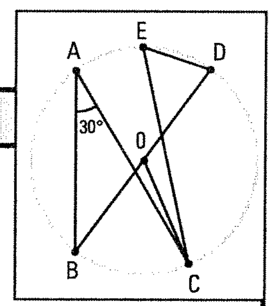
9. On the figure on the right, angle BAC measures 15°. If the central angles B0C and D0E are complementary, determine the measure of the intercepted arcs BC and DE.



Let $x = m\overset{\frown}{DE}$ and $y = m\overset{\frown}{BC}$. We have:

$$\begin{cases} \frac{x - y}{2} = 15° \\ x + y = 90° \end{cases} \Rightarrow x = 60° \text{ and } y = 30°$$

10. Consider the circle centred at 0 represented on the right. Determine the measure of angle CED. Justify your reasoning.

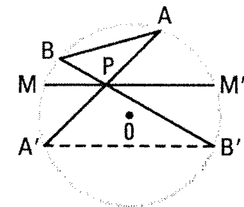


	Steps	Justifications
1.	$m\overset{\frown}{BC} = 60°$	The measure of an inscribed angle is equal to half the measure of the intercepted arc (theorem).
2.	$m\overset{\frown}{BD} = 180°$	$\overline{BD}$ is a diameter, since $0 \in \overline{BD}$. $m\angle B0D = 180°$.
3.	$m\overset{\frown}{CD} = 120°$	$m\overset{\frown}{CD} = m\overset{\frown}{BD} - m\overset{\frown}{BC}$.
4.	$m\angle CED = 60°$	The measure of an inscribed angle is equal to half the measure of the intercepted arc.

10.4 Power of a point with respect to a circle

ACTIVITY 1 Invariant product

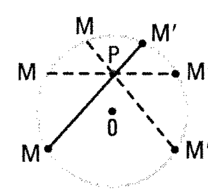
a) Consider a circle centred at 0 and any two chords AA′ and BB′ that intersect at a point P, exterior to the circle. Justify the steps showing that $m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$.
Consider triangles APB and B′PA′.



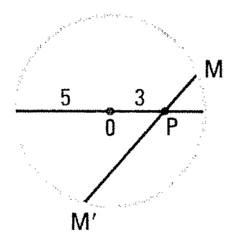
	Steps	Justifications
1.	$\angle APB \cong \angle A'PB'$	***Opposite angles.***
2.	$\angle BAA' \cong \angle BB'A'$	***Inscribed angles intercepting the same arc A'B, therefore they are congruent.***
3.	$\Delta APB \sim \Delta B'PA'$	***Similarity case AA.***
4.	$\frac{m\overline{PA}}{m\overline{PB'}} = \frac{m\overline{PB}}{m\overline{PA'}}$	***The corresponding sides of 2 similar triangles are proportional.***
5.	$m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$	***In a ratio, the cross-products are equal.***

b) Draw any chord MM′ passing through point P.
Does the product $m\overline{PM} \times m\overline{PM'}$ remain equal to the product $m\overline{PA} \times m\overline{PA'}$? ***Yes***

c) Given a point P inside a circle and a secant MM′ always passing through point P, does the product $m\overline{PM} \times m\overline{PM'}$ remain constant whatever the position of the secant?
Yes



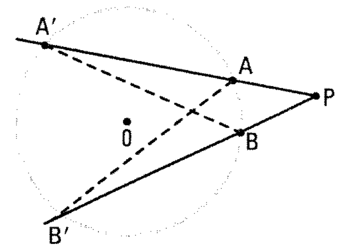
d) In the circle of radius 5 cm on the right, point P is located 3 cm from the centre 0. Let $\overline{MM'}$ be a chord passing through P. Complete the table below giving the measure of $\overline{PM'}$ according to the measure of $\overline{PM}$.



$m\overline{PM}$	2	2.5	3.2	4	5	6	6.4	8
$m\overline{PM'}$	8	***6.4***	***5***	***4***	***3.2***	***2.$\bar{6}$***	***2.5***	***2***

...ITY 2 Another invariant product

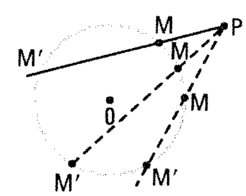
...sider a circle centred at O and a point P exterior to the circle. ...point P, we draw two secants to the circle intersecting the ...at points A and A′ and at points B and B′ respectively. Justify ...eps showing that $m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$.
...der triangles PA′B and PB′A.



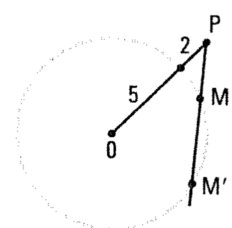
Steps	Justifications
...′PB ≅ ∠B′PA	*Common angle of the two triangles.*
...A′P ≅ ∠AB′P	*Inscribed angle intercepting the same arc AB, therefore they are congruent.*
...B ~ △PB′A	*Similarity case AA.*
... $= \frac{m\overline{PB}}{m\overline{PA}}$	*The corresponding sides of 2 similar triangles are proportional.*
... $\times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$	*In a ratio, the cross-products are equal.*

...point P, a secant to the circle intersecting the circle successively at point M then at ...oes the product $m\overline{PM} \times m\overline{PM'}$ remain equal to the product $m\overline{PA} \times m\overline{PA'}$? *Yes*

...t P outside the circle and a secant always passing through point ...cting the circle at point M then at point M′, does the product ... remain constant whatever the position of the secant?



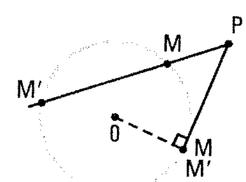
...radius 5 cm on the right, point P is located 7 cm from the ...der any secant passing through P intersecting the circle at ...point M′. Complete the table below giving the measure of ... the measure of PM.



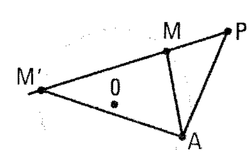
...2.4	2.5	3	3.2	3.6	4	$\sqrt{24}$
...*10*	*9.6*	*8*	*7.5*	*6.$\bar{6}$*	*6*	*$\sqrt{24}$*

...n of line PM when $m\overline{PM} = \sqrt{24}$?
...*to the circle at point M.*

...ide the circle and a secant always passing through point ...circle at point M then at point M′, does the product ...constant when, in the limit, the secant assumes the ..., in other words when M and M′ are merged?



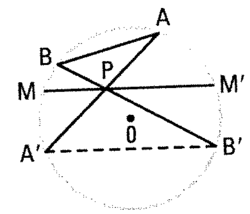
...ht, we have drawn from a point P a secant to the ...tangent at point A. Justify the steps showing that



10.4 Power of a point with respect to a circle

ACTIVITY 1 Invariant product

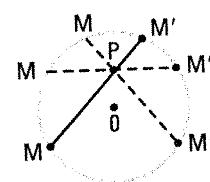
a) Consider a circle centred at 0 and any two chords AA′ and BB′ that intersect at a point P, exterior to the circle. Justify the steps showing that $m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$.
Consider triangles APB and B′PA′.



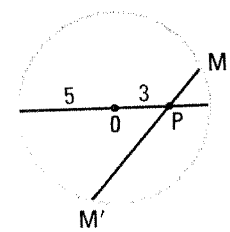
	Steps	Justifications
1.	$\angle APB \cong \angle A'PB'$	*Opposite angles.*
2.	$\angle BAA' \cong \angle BB'A'$	*Inscribed angles intercepting the same arc A'B, therefore they are congruent.*
3.	$\Delta APB \sim \Delta B'PA'$	*Similarity case AA.*
4.	$\frac{m\overline{PA}}{m\overline{PB'}} = \frac{m\overline{PB}}{m\overline{PA'}}$	*The corresponding sides of 2 similar triangles are proportional.*
5.	$m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$	*In a ratio, the cross-products are equal.*

b) Draw any chord MM′ passing through point P.
Does the product $m\overline{PM} \times m\overline{PM'}$ remain equal to the product $m\overline{PA} \times m\overline{PA'}$? *Yes*

c) Given a point P inside a circle and a secant MM′ always passing through point P, does the product $m\overline{PM} \times m\overline{PM'}$ remain constant whatever the position of the secant?
Yes



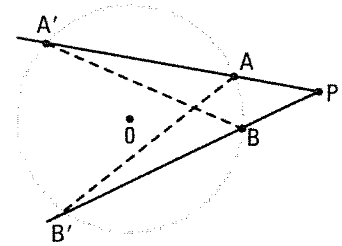
d) In the circle of radius 5 cm on the right, point P is located 3 cm from the centre 0. Let $\overline{MM'}$ be a chord passing through P. Complete the table below giving the measure of $\overline{PM'}$ according to the measure of $\overline{PM}$.



$m\overline{PM}$	2	2.5	3.2	4	5	6	6.4	8
$m\overline{PM'}$	8	***6.4***	***5***	***4***	***3.2***	***2.$\bar{6}$***	***2.5***	***2***

...er invariant product

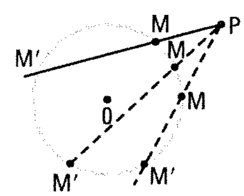
...ntred at O and a point P exterior to the circle. ...raw, two secants to the circle intersecting the ...d A', and at points B and B' respectively. Justify ...hat $m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$. ...A'B and PB'A.



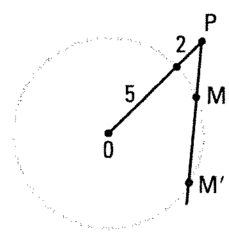
	Justifications
	Common angle of the two triangles.
	Inscribed angle intercepting the same arc AB, therefore they are congruent.
	Similarity case AA.
	The corresponding sides of 2 similar triangles are proportional.
...$m\overline{PB} \times m\overline{PB'}$	***In a ratio, the cross-products are equal.***

...ecant to the circle intersecting the circle successively at point M then at ...duct $m\overline{PM} \times m\overline{PM'}$ remain equal to the product $m\overline{PA} \times m\overline{PA'}$? ***Yes***

...the circle and a secant always passing through point ...ircle at point M then at point M', does the product ...nstant whatever the position of the secant?



...n on the right, point P is located 7 cm from the ...ant passing through P intersecting the circle at ...Complete the table below giving the measure of ...ure of PM.

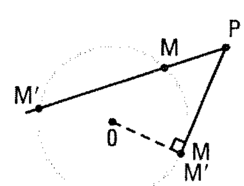


...	3	3.2	3.6	4	$\sqrt{24}$
...	***8***	***7.5***	***$6.\bar{6}$***	***6***	***$\sqrt{24}$***

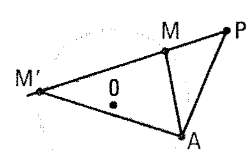
...M when $m\overline{PM} = \sqrt{24}$?

...e at point M.

...cle and a secant always passing through point ...oint M then at point M', does the product ...when, in the limit, the secant assumes the ...words when M and M' are merged?



...e drawn from a point P a secant to the ...point A. Justify the steps showing that

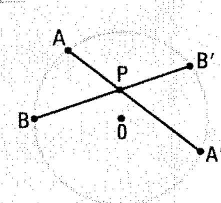


Consider triangles PAM and PM'A.

	Steps	Justifications
1.	$\angle MPA \cong \angle APM'$	***Common angle of the two triangles.***
2.	$\angle PM'A \cong \angle AMP$	***Inscribed angle intercepting the same arc $\overset{\frown}{AM}$, therefore they are congruent.***
3.	$\Delta PAM \sim \Delta PM'A$	***Similarity case AA.***
4.	$\dfrac{m\overline{PA}}{m\overline{PM'}} = \dfrac{m\overline{PM}}{m\overline{PA}}$	***The corresponding sides of 2 similar triangles are proportional.***
5.	$(m\overline{PA})^2 = m\overline{PM} \times m\overline{PM'}$	***In a ratio, the cross-products are equal.***

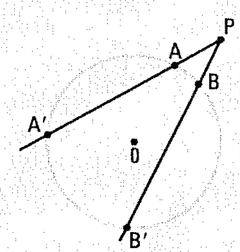
CONSTANT PRODUCT THEOREM

- When two chords intersect inside a circle, the product of the measures of the segments of one of them equals the product of the segments of the other.



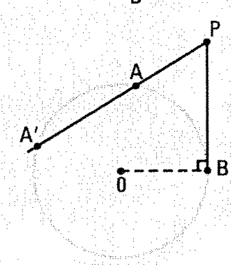
$$m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$$

- If, from a point P outside a circle, we draw two secants intersecting the circle at points A and A' and at points B and B' respectively then



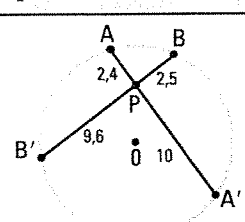
$$m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$$

- If, from a point P outside a circle, we draw a secant intersecting the circle at points A and A' and a tangent to the circle at point B then



$$m\overline{PA} \times m\overline{PA'} = (m\overline{PB})^2$$

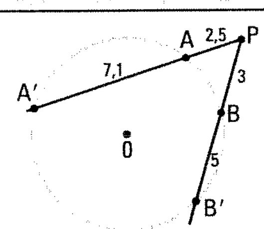
- In conclusion, the product of the distances of a point P to each of the intersection points of a secant passing through P with the circle is constant. This product is called **power of point P** with respect to the circle.



$m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$

2.4 × 10 = 2.5 × 9.6

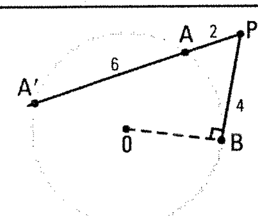
Power of point P = 24



$m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$

2.5 × 9.6 = 3 × 8

Power of point P = 24



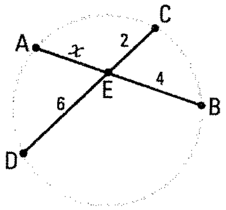
$m\overline{PA} \times m\overline{PA'} = (m\overline{PB})^2$

$2 \times 8 = 4^2$

Power of point P = 16

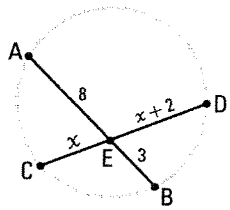
1. In each of the following cases, determine x.

a)



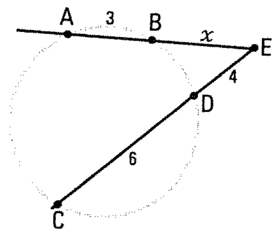
x = 3

b)



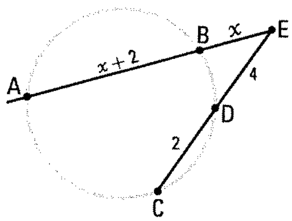
x = 4

c)



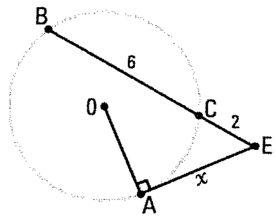
x = 5

d)



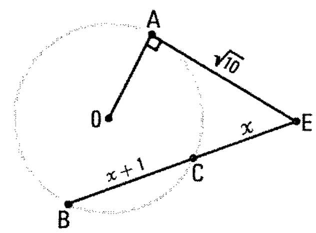
x = 3

e)



x = 4

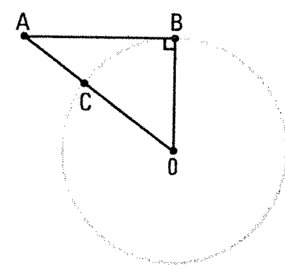
f)



x = 2

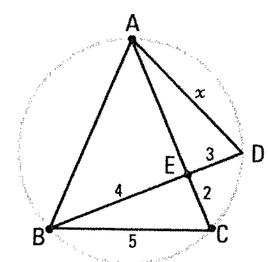
2. From point A, a line segment AB tangent to the circle centred at 0 was drawn.

Knowing that $m\overline{AB} = 12$ cm and that $m\overline{AC} = 6$ cm, determine the radius of the circle.



$r \cdot (r + 6) = 12^2$; $r = 9$ cm.

3. Determine the perimeter of triangle AED on the figure on the right.



$m\overline{AE} = 6$ u

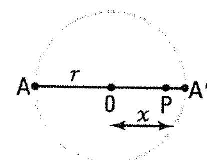
$\triangle AED \sim \triangle BEC$ (case AA)

$\frac{m\overline{AD}}{m\overline{BC}} = \frac{m\overline{ED}}{m\overline{EC}} \Rightarrow m\overline{AD} = 7.5$ u

Perimeter = 16.5 u.

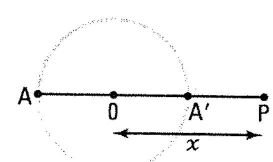
4. a) Consider a point P inside a circle of radius r. We denote by x the distance between point P and the centre of the circle.

Show that the power of point P with respect to the circle is equal to $r^2 - x^2$.



Power of point P = $m\overline{PA} \times m\overline{PA'} = (r + x) \cdot (r - x) = r^2 - x^2$

b) Consider a point P outside a circle of radius r. We denote by x the distance between point P and the centre of the circle. Show that the power of point P with respect tot the circle is equal to $x^2 - r^2$.



Power of point P = $m\overline{PA} \times m\overline{PA'} = (x - r)(x + r) = x^2 - r^2$

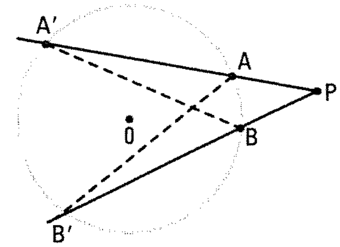
c) What must be the position of a point P with respect to a circle in order for the power of point P with respect to the circle to be zero?

$x = r$; point P is located on the circle.

ACTIVITY 2 Another invariant product

a) Consider a circle centred at 0 and a point P exterior to the circle. From point P, we draw two secants to the circle intersecting the circle at points A and A′ and at points B and B′ respectively. Justify the steps showing that $m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$.

Consider triangles PA′B and PB′A.

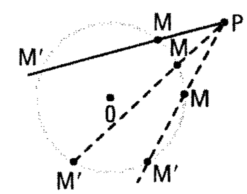


	Steps	Justifications
1.	$\angle A'PB \cong \angle B'PA$	***Common angle of the two triangles.***
2.	$\angle BA'P \cong \angle AB'P$	***Inscribed angle intercepting the same arc AB, therefore they are congruent.***
3.	$\triangle PA'B \sim \triangle PB'A$	***Similarity case AA.***
4.	$\dfrac{m\overline{PA'}}{m\overline{PB'}} = \dfrac{m\overline{PB}}{m\overline{PA}}$	***The corresponding sides of 2 similar triangles are proportional.***
5.	$m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$	***In a ratio, the cross-products are equal.***

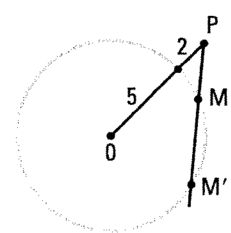
b) Draw, from point P, a secant to the circle intersecting the circle successively at point M then at point M′. Does the product $m\overline{PM} \times m\overline{PM'}$ remain equal to the product $m\overline{PA} \times m\overline{PA'}$? ***Yes***

c) Given a point P outside the circle and a secant always passing through point P and intersecting the circle at point M then at point M′, does the product $m\overline{PM} \times m\overline{PM'}$ remain constant whatever the position of the secant?

Yes



d) In the circle of radius 5 cm on the right, point P is located 7 cm from the centre 0. Consider any secant passing through P intersecting the circle at point M then at point M′. Complete the table below giving the measure of $\overline{PM'}$ according to the measure of $\overline{PM}$.



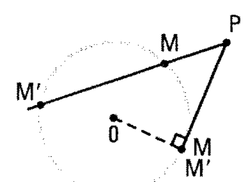
$m\overline{PM}$	2	2.4	2.5	3	3.2	3.6	4	$\sqrt{24}$
$m\overline{PM'}$	12	***10***	***9.6***	***8***	***7.5***	***$6.\overline{6}$***	***6***	***$\sqrt{24}$***

e) What is the position of line PM when $m\overline{PM} = \sqrt{24}$?

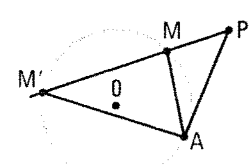
Line PM is tangent to the circle at point M.

f) Given a point P outside the circle and a secant always passing through point P and intersecting the circle at point M then at point M′, does the product $m\overline{PM} \times m\overline{PM'}$ remain constant when, in the limit, the secant assumes the position of the tangent, in other words when M and M′ are merged?

Yes



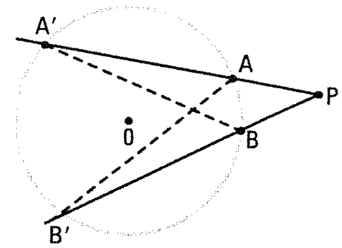
g) On the circle on the right, we have drawn from a point P a secant to the circle at M and M′ and a tangent at point A. Justify the steps showing that $(m\overline{PA})^2 = m\overline{PM} \times m\overline{PM'}$.



ACTIVITY 2 Another invariant product

a) Consider a circle centred at 0 and a point P exterior to the circle. From point P, we draw two secants to the circle intersecting the circle at points A and A′ and at points B and B′ respectively. Justify the steps showing that $m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$.

Consider triangles PA′B and PB′A.

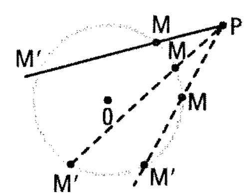


	Steps	Justifications
1.	$\angle A'PB \cong \angle B'PA$	***Common angle of the two triangles.***
2.	$\angle BA'P \cong \angle AB'P$	***Inscribed angle intercepting the same arc AB, therefore they are congruent.***
3.	$\Delta PA'B \sim \Delta PB'A$	***Similarity case AA.***
4.	$\frac{m\overline{PA'}}{m\overline{PB'}} = \frac{m\overline{PB}}{m\overline{PA}}$	***The corresponding sides of 2 similar triangles are proportional.***
5.	$m\overline{PA} \times m\overline{PA'} = m\overline{PB} \times m\overline{PB'}$	***In a ratio, the cross-products are equal.***

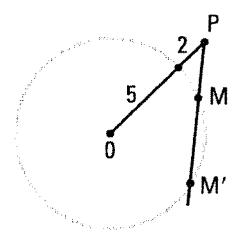
b) Draw, from point P, a secant to the circle intersecting the circle successively at point M then at point M′. Does the product $m\overline{PM} \times m\overline{PM'}$ remain equal to the product $m\overline{PA} \times m\overline{PA'}$? ***Yes***

c) Given a point P outside the circle and a secant always passing through point P and intersecting the circle at point M then at point M′, does the product $m\overline{PM} \times m\overline{PM'}$ remain constant whatever the position of the secant?

Yes



d) In the circle of radius 5 cm on the right, point P is located 7 cm from the centre 0. Consider any secant passing through P intersecting the circle at point M then at point M′. Complete the table below giving the measure of $\overline{PM'}$ according to the measure of $\overline{PM}$.



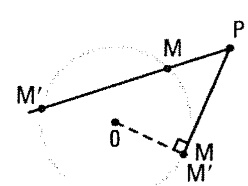
$m\overline{PM}$	2	2.4	2.5	3	3.2	3.6	4	$\sqrt{24}$
$m\overline{PM'}$	12	***10***	***9.6***	***8***	***7.5***	***$6.\bar{6}$***	***6***	***$\sqrt{24}$***

e) What is the position of line PM when $m\overline{PM} = \sqrt{24}$?

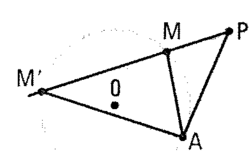
Line PM is tangent to the circle at point M.

f) Given a point P outside the circle and a secant always passing through point P and intersecting the circle at point M then at point M′, does the product $m\overline{PM} \times m\overline{PM'}$ remain constant when, in the limit, the secant assumes the position of the tangent, in other words when M and M′ are merged?

Yes

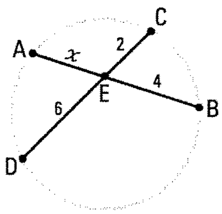


g) On the circle on the right, we have drawn from a point P a secant to the circle at M and M′ and a tangent at point A. Justify the steps showing that $(m\overline{PA})^2 = m\overline{PM} \times m\overline{PM'}$.



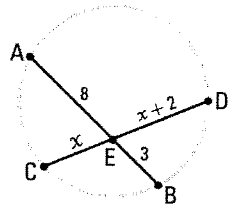
1. In each of the following cases, determine x.

a)



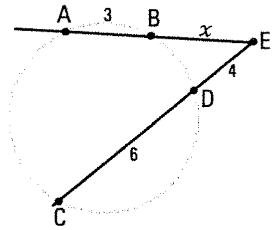
$x = 3$

b)



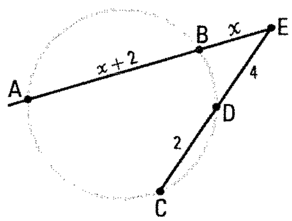
$x = 4$

c)



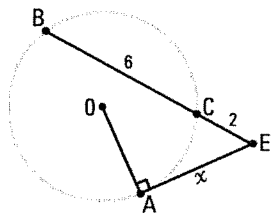
$x = 5$

d)



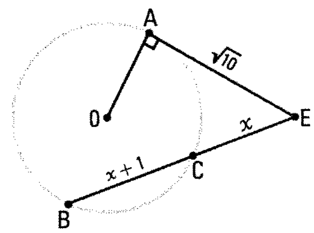
$x = 3$

e)



$x = 4$

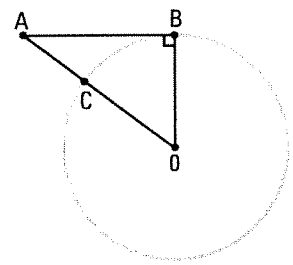
f)



$x = 2$

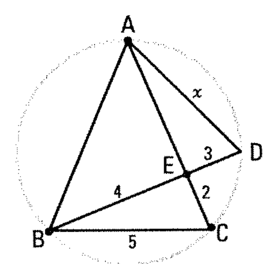
2. From point A, a line segment AB tangent to the circle centred at 0 was drawn.

Knowing that $m\overline{AB} = 12$ cm and that $m\overline{AC} = 6$ cm, determine the radius of the circle.



r · (r + 6) = 12²; r = 9 cm.

3. Determine the perimeter of triangle AED on the figure on the right.



$m\overline{AE}$ = 6 u

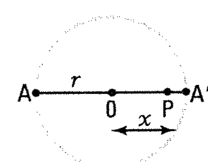
$\triangle AED \sim \triangle BEC$ (case AA)

$\dfrac{m\overline{AD}}{m\overline{BC}} = \dfrac{m\overline{ED}}{m\overline{EC}} \Rightarrow m\overline{AD} = 7.5$ u

Perimeter = 16.5 u.

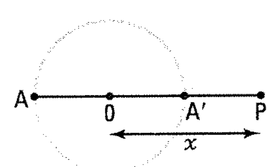
4. a) Consider a point P inside a circle of radius r. We denote by x the distance between point P and the centre of the circle.

Show that the power of point P with respect to the circle is equal to $r^2 - x^2$.



Power of point P = $m\overline{PA} \times m\overline{PA'} = (r + x) \cdot (r - x) = r^2 - x^2$

b) Consider a point P outside a circle of radius r. We denote by x the distance between point P and the centre of the circle. Show that the power of point P with respect tot the circle is equal to $x^2 - r^2$.



Power of point P = $m\overline{PA} \times m\overline{PA'} = (x - r)(x + r) = x^2 - r^2$

c) What must be the position of a point P with respect to a circle in order for the power of point P with respect to the circle to be zero?

x = r; point P is located on the circle.

Evaluation 10

1. Complete each of the following statements.

a) The perpendicular bisector of a chord of a circle passes through ***the centre of the circle***.

b) Any line perpendicular to the endpoint of the radius of a circle is ***tangent to the circle***.

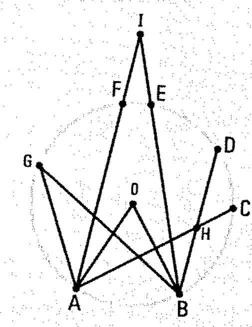
c) Two congruent central angles intercept ***two congruent arc***.

d) Two congruent arcs are subtended by ***two congruent chords***.

e) Two chords located at the same distance from the centre are ***congruent***.

f) The arcs between two parallel chords are ***congruent***.

2. On the figure on the right, we have:
$m\overset{\frown}{AB} = x$, $m\overset{\frown}{CD} = y$ and $m\overset{\frown}{EF} = z$..
Determine



a) m ∠AOB x

b) m ∠AGB $\frac{x}{2}$

c) m ∠AHB $\frac{x+y}{2}$

d) m ∠AIB $\frac{x-z}{2}$

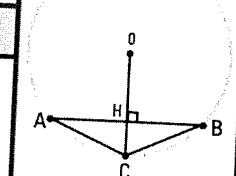
3. a) What is the measure of an inscribed angle which intercepts a half-circle? ***90°***

b) From a point P outside a circle centred at 0, the two line segments PA and PB tangent the circle are drawn. What can be said of
1. line segments PA and PB? ***They are congruent.***
2. angles 0PA and 0PB? ***They are congruent.***

c) In a circle centred at 0, a radius 0C is perpendicular to a chord $\overline{AB}$ at H. What can be said of
1. line segments HA and HB? ***They are congruent.***
2. arcs CA and CB? ***They are congruent.***

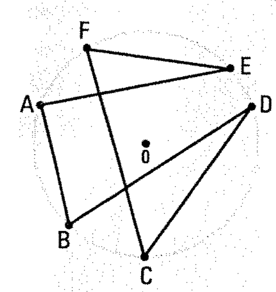
d) What can be said of the intersection point of the perpendicular bisectors of any two chords of a circle? ***The intersection point is the centre of the circle.***

4. We have drawn on the right the radius 0C perpendicular to the chord AB at H. Show that triangles HAC and HBC are congruent.



	Steps	Justifications
1.	$\overline{HA} \cong \overline{HB}$	*Any radius perpendicular to a chord divides the chord into two congruent line segments (theorem).*
2.	∠AHC ≅ ∠BHC	*Right angles.*
3.	$\overline{HC} \cong \overline{HC}$	*Common side.*
4.	△HAC ≅ △HBC	*Congruence case (SAS).*

5. In the circle on the right, chords AB and CF are parallel. Show that angles AEF and BDC are congruent.

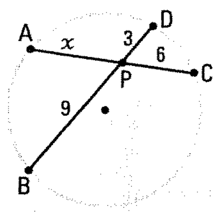


1. $\overset{\frown}{AF} \cong \overset{\frown}{BC}$***, arcs between two parallel chords AB and CF are congruent (theorem).***

2. $\angle AEF \cong \angle BDC$***; two inscribed angles intercepting two congruent arcs are necessarily congruent.***

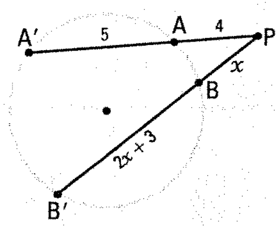
6. In each of the following cases, determine x.

a)



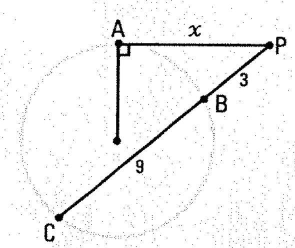
x = 4,5

b)



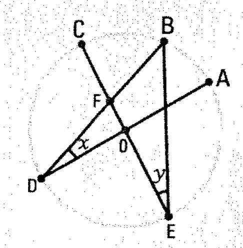
x = 3

c)



x = 6

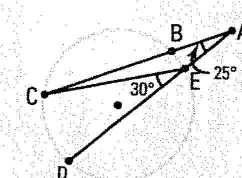
7. In the circle centred at 0 on the right $\overline{AD}$ and $\overline{CE}$ are diameters.
m $\angle ADB = x$ and m $\angle BEC = y$.
Express, as a function of x and y, the measure of the interior angle BFC.



$m\overset{\frown}{AB} = 2x$; $m\overset{\frown}{BC} = 2y$; $m\overset{\frown}{DE} = 2x + 2y$

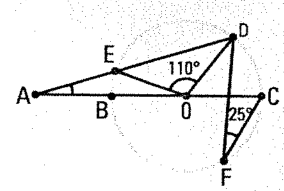
$m\ \angle BFC = \frac{1}{2}\left(m\overset{\frown}{BC} + m\overset{\frown}{DE}\right) = x + 2y.$

8. On the figure on the right, angle CED measures 30° and exterior angle CAD measures 25°. What is the measure of arc BE?



$m\overset{\frown}{CD} = 60°$; $m\overset{\frown}{BE} = x$; $25° = \frac{60° - x}{2}$; $x = 10°$.

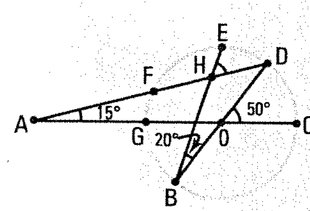
9. On the figure on the right, $\overline{BC}$ is a diameter.
We have: m $\angle CFD = 25°$ and m $\angle D0E = 110°$.
What is the measure of angle CAD?



$m\overset{\frown}{CD} = 50°$; $m\overset{\frown}{DE} = 110°$; $m\overset{\frown}{EB} = 20°$.

$m\ \angle CAD = \frac{50° - 20°}{2} = 15°.$

10. On the figure on the right, $\overline{GC}$ and $\overline{BD}$ are diameters.
We have: m $\angle C0D = 50°$, m $\angle EBD = 20°$ and m $\angle CAD = 15°$.
What is the measure of angle EHD?



$m\overset{\frown}{CD} = 50°$; $m\overset{\frown}{DE} = 40°$; $m\overset{\frown}{GB} = 50°$; $m\overset{\frown}{FG} = x$

$15° = \frac{50 - x}{2}$; $x = 20°$; $m\overset{\frown}{FB} = 70°$;

$m\ \angle EHD = \frac{40° - 70°}{2} = 55°.$

Chapter 11

Equivalent figures

CHALLENGE 11

11.1 Area and volume of solids

11.2 Equivalent plane figures

11.3 Equivalent solids

11.4 Comparing polygons

11.5 Comparing solids

EVALUATION 11