CHALLENGE 11

1. What is the perimeter of a square equivalent to a 4 cm by 9 cm rectangle? ***24 cm***

2. What is the total area of a cube equivalent to a rectangular based prism with dimensions 2 cm by 4 cm by 8 cm? ***96 cm²***

3. Of all rectangles with an area of 36 cm^2, what is the perimeter of the rectangle with the smallest perimeter? ***24 cm***

4. Of all rectangles with a perimeter of 36 cm, what is the area of the rectangle with the largest area? ***81*** cm^2

5. Of all rectangular prisms with a total area of 54 cm^2, what is the volume of the prism with the largest volume? ***27*** cm^3

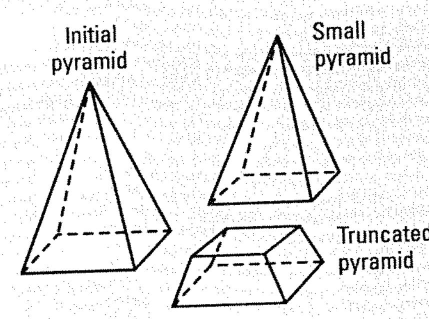
6. Of all rectangular prisms with a volume of 216 cm^3, what is the total area of the prism with the smallest area? ***216*** cm^2

7. You have a surface area of 7200 cm^2 of cardboard to make identical boxes in the shape of rectangular prisms.

Each box must have a capacity of 1000 cm^3. Determine the dimensions of the box that will maximize the number of boxes that you can make with the available cardboard.

12 boxes in the shape of a cube with 10 cm sides.

8. A pyramid has a height of 18 cm and a square base with 12 cm sides. The pyramid is cut along a plane parallel to the base yielding a truncated pyramid and a pyramid similar to the initial one. Determine the volume of the truncated pyramid if the height of the small pyramid is 12 cm.



Scale factor $= \frac{2}{3}$

Volume of initial pyramid = 864 cm^3

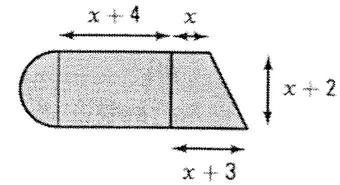
Volume of small pyramid = 256 cm^3

Volume of truncated pyramid = 608 cm^3

11.1 Area and volume of solids

Activity 1 Perimeter and area of a plane figure

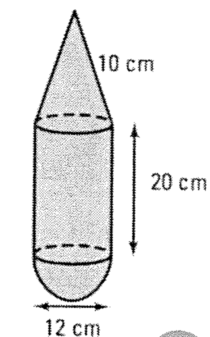
The plane figure on the right is formed by a semi-circle, a rectangle and a trapezoid. What is, to the nearest tenth, its perimeter if the area of the trapezoid is equal to 33 cm²?



$(2x + 3)(x + 2) \div 2 = 33$; $x = 4$; $P = (33,7 + 3\pi)$ cm $\approx 43,1$ cm

Activity 2 Area and volume of a solid

The solid on the right is formed by a hemisphere, a cylinder and a cone.



a) Calculate the total area of this solid.

area of hemisphere: 72π cm²

lateral area of cylinder: 240π cm²

lateral area of cone: 60π cm²

total area of solid: 372π cm²

b) Calculate the volume of this solid.

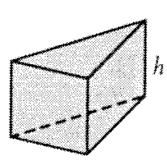
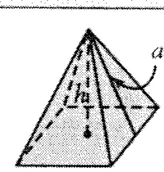
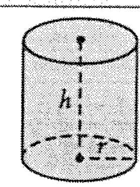
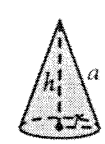
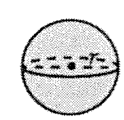
volume of hemisphere: 144π cm³

volume of cylinder: 720π cm³

volume of cone: 96π cm³

volume of solid: 960π cm³

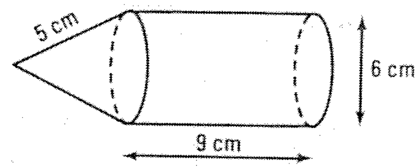
AREA AND VOLUME OF SOLIDS

Symbols	Prism	Pyramid	Cylinder	Cone	Sphere
s: slant height h: height r: radius A_b: area of base P_b: perimeter of base					
Lateral area A_l	$A_l = P_b \times h$	$A_l = \frac{P_b \times s}{2}$	$A_l = 2\pi rh$	$A_l = \pi rs$	
Total area A_t	$A_t = 2A_b + A_l$	$A_t = A_b + A_l$	$A_t = 2A_b + A_l$	$A_t = A_b + A_l$	$A_t = 4\pi r^2$
Volume V	$V = A_b \times h$	$V = \frac{A_b \times h}{3}$	$V = A_b \times h$	$V = \frac{A_b \times h}{3}$	$V = \frac{4}{3}\pi r^3$

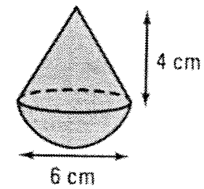
Ex.: Consider the solid on the right:

A_t = area of base of cylinder + lateral area of cylinder + lateral area of cone = $9\pi + 54\pi + 15\pi = 78\pi$ cm^2

V = volume of cylinder + volume of cone
= $81\pi + 12\pi = 93\pi$ cm^3



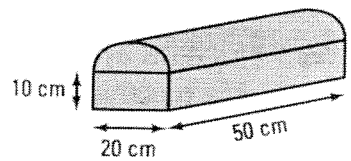
1. A toy is formed by a cone with a height of 4 cm mounted on top of a hemisphere with a radius of 3 cm.



a) Calculate the total area of this toy. ***$18\pi + 15\pi = 33\pi$ cm^2***

b) Calculate its volume. ***$18\pi + 12\pi = 30\pi$ cm^3***

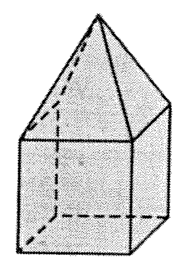
2. A box has the shape of a rectangular prism mounted by a half cylinder.



a) Calculate the total area of this box. ***$1000 + 1400 + 600\pi \approx 4285$ cm^2***

b) Calculate the volume of the box. ***$10\,000 + 2500\pi \approx 17\,854$ cm^2***

3. A sculpture is formed by placing a pyramid with a height of 40 dm on top of a cube with 60 dm sides.



a) Calculate the total area of this sculpture. ***$5 \times 60^2 + \frac{4 \times 60 \times 50}{2} = 24\,000$ dm^2***

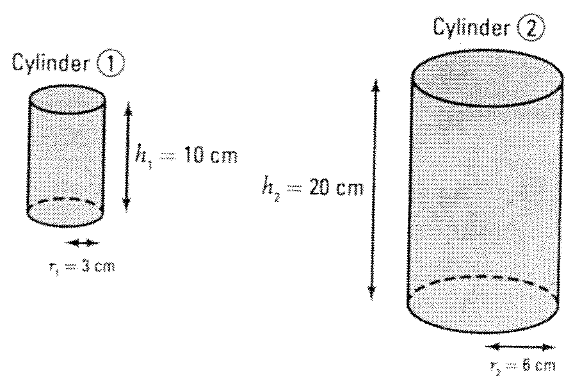
b) Calculate its volume. ***$60^3 + \frac{60^2 \times 40}{3} = 264\,000$ dm^3***

ACTIVITY 3 Similar solids

a) 1. Explain why the cylinders on the right are similar.

The ratio of the corresponding dimensions are the same. $\frac{h_2}{h_1} = \frac{r_2}{r_1} = 2$

2. What is the scale factor? ***2 or $\frac{1}{2}$***



b) Verify that the ratio of the circumferences of the bases is equal to the scale factor.

$C_2 = 12\pi$ cm; $C_1 = 6\pi$ cm; $\frac{C_2}{C_1} = 2$

c) Verify that the ratio of the total areas of the solids is equal to the square of the scale factor.

$A_2 = 312\pi$ cm^2; $A_1 = 78\pi$ cm^2; $\frac{A_2}{A_1} = 2^2$

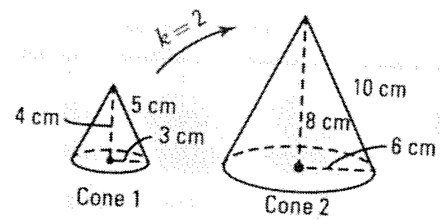
d) Verify that the ratio of the volumes is equal to the cube of the scale factor.

$V_2 = 720\pi$ cm^3; $V_1 = 90\pi$ cm^3; $\frac{V_2}{V_1} = 2^3$

SIMILAR SOLIDS

If k represents the scale factor of two similar solids,
- the ratio of lengths is equal to k;
- the ratio of areas is equal to k^2;
- the ratio of volumes is equal to k^3.

	Circumference of the base	Total area	Volume
Cone 1	6π cm	24π cm^2	12π cm^3
Cone 2	12π cm	96π cm^2	96π cm^3
Ratio	$\frac{12\pi}{6\pi} = 2$	$\frac{96\pi}{24\pi} = 2^2$	$\frac{96\pi}{12\pi} = 2^3$



4. The ratio of the areas of two similar rectangles is equal to $\frac{9}{4}$. What is the perimeter of the large rectangle if the perimeter of the small one is 50 cm? ***75 cm***

5. The ratio of volumes of two similar prisms is equal to 8. What is the area of the big prism if the area of the small one is 150 cm^2? ***600 cm²***

6. A small cylinder with a volume of 8π cm^3 is similar to a larger one with a volume of 27π cm^3. What is the area of the base of the large cylinder if the radius of the small one is 2 cm? ***9π cm²***

7. Two cone shaped containers are similar. What is the volume of the small container if the large container has a radius of 6 cm and a height of 15 cm and if the small container has a radius of 4 cm?

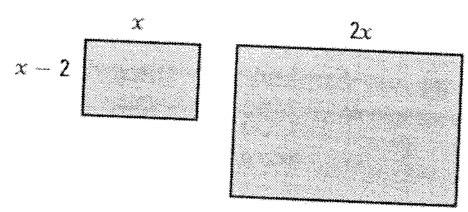
$\frac{160\pi}{3}$ cm³

8. The rectangles on the right are similar. What is the perimeter of the big rectangle if its area is equal to 96 cm^2?

$x(x - 2) = 24$; $x = 6$

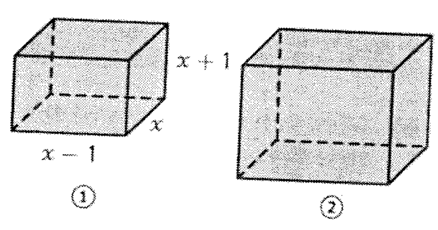
The dimensions of the big rectangle are 12 cm by 8 cm.

Perimeter: 40 cm



9. The rectangular based prisms ① and ② on the right are similar. Let x and $(x - 1)$ represent the dimensions of the base of prism ① and $(x + 1)$ represent its height. Calculate the volume of each prism if the areas of the bases of prisms ① and ② are respectively 12 cm^2 and 48 cm^2.

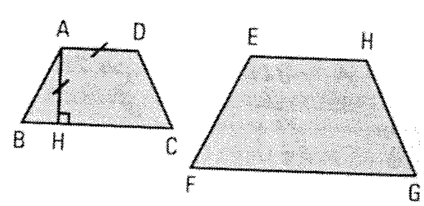
$x(x - 1) = 12$; $x = 4$; volume ① = 60 cm³; volume ② = 480 cm³



10. The trapezoids on the right are similar. The area of trapezoid ABCD is 24 cm^2. If m$\overline{AD}$ = m$\overline{AH}$, m$\overline{BC}$ = 8 cm and m$\overline{EH}$ = 6 cm, calculate the area of trapezoid EFGH.

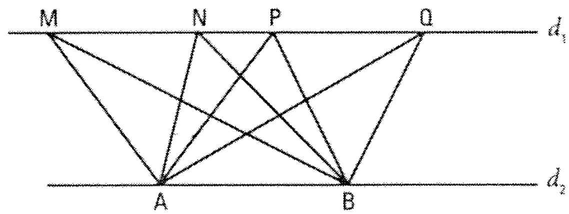
x = mAD; $(8 + x) x \div 2 = 24$; $x = 4$

$k = \frac{3}{2}$; Area of trapezoid EFGH = 54 cm²



11.2 Equivalent plane figures

ACTIVITY 1 Equivalent triangles



In the figure on the right, lines l_1 and l_2 are parallel.

a) Triangles MAB, NAB, PAB and QAB all have the same base AB. Explain why these 4 triangles have the same area.

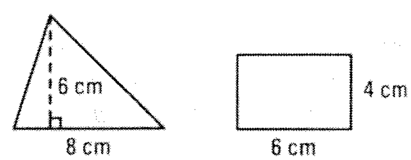
For each triangle, the height relative to the base AB of that triangle is equal to the distance between the two parallel lines l_1 and l_2. The triangles, having the same base and height, therefore have the same area.

b) What is the area of each triangle if the base AB measures 10 cm and the distance between the parallel lines is 8 cm? *40 cm²*

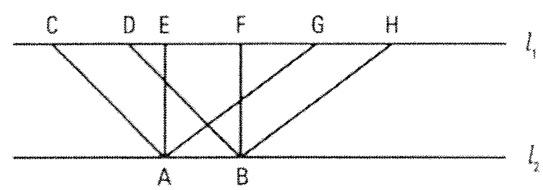
EQUIVALENT PLANE FIGURES

Two plane figures are equivalent if they have the same area.

Ex.: The triangle and the rectangle on the right are equivalent because they both have an area of 24 cm².



1. Lines l_1 and l_2 on the right are parallel. What can be said about the parallelograms ABDC, ABFE and ABHG? Justify your answer.



These parallelograms all have the same base AB and a height equal to the distance between the lines l_1 and l_2. These parallelograms therefore have the same area and are thus equivalent.

2. A triangle has a base of 9 cm and a height of 8 cm. What is the side length of a square equivalent to the triangle? *6 cm*

3. In each of the following cases, find the measure x of the side of the square equivalent to

a) an 8 cm by 12 cm rectangle. *$x = 4$ cm*

b) a right triangle with the sides of the right angle measuring 6 cm and 3 cm. *$x = 3$ cm*

c) a trapezoid with a big base of 12 cm, a small base of 4 cm and a height of 8 cm. ***x = 8 cm***

d) a rhombus with diagonals measuring 5 cm and 10 cm. ***x = 5 cm***

4. True or false?

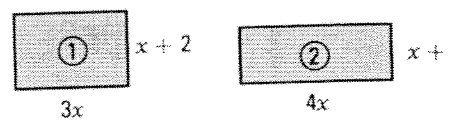
a) If two figures are congruent, then they are equivalent. ***True***

b) If two figures are equivalent, then they are congruent. ***False***

c) If two non-congruent figures are similar, then they are equivalent. ***False***

d) If two figures are equivalent, then they are similar. ***False***

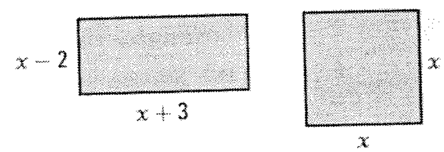
5. The rectangles on the right are equivalent. What is the numerical value of the perimeter of each rectangle?



$3x(x + 2) = 4x(x + 1)$; $x = 2$

$P_1 = 20$ u, $P_2 = 22$ u

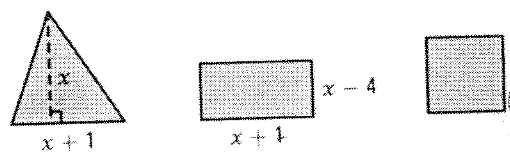
6. The rectangle and the square on the right are equivalent. What is the numerical value of the perimeter of each figure?



$x^2 = (x - 2)(x + 3)$; $x = 6$

Perimeter of rectangle = 26 u; Perimeter of square = 24 u.

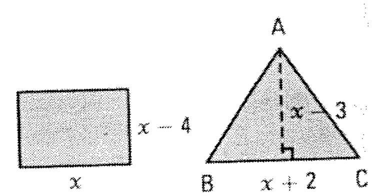
7. The triangle, rectangle and square on the right are equivalent. What is the perimeter of the square?



$\frac{x(x + 1)}{2} = (x + 1)(x - 4)$; $x = 8$

The square has an area of 36 u^2, and therefore a perimeter of 24 u.

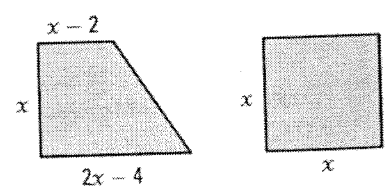
8. The rectangle and isosceles triangle on the right are equivalent. What is the perimeter of the isosceles triangle?



$x(x - 4) = \frac{(x + 2)(x - 3)}{2}$; $x = 6$

$m\overline{BC} = 8$ u; $m\overline{AB} = 5$ u; Perimeter = 18 u

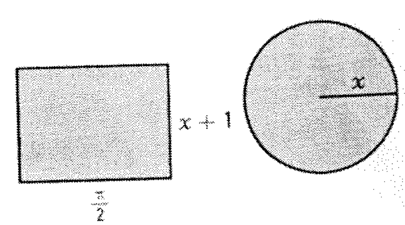
9. The trapezoid and square on the right are equivalent. What is the numerical value of the perimeter of the square?



$[(2x - 4) + (x - 2)]\ x \div 2 = x^2$

$x = 6$ u; Perimeter of square: 24 u

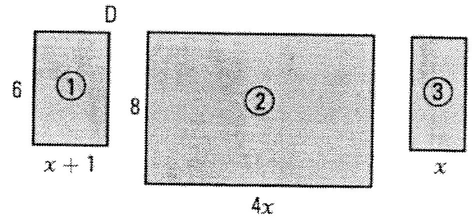
10. The rectangle and the circle on the right are equivalent. What is the numerical value of the circle's circumference?



$\frac{\pi}{2}(x + 1) = \pi x^2$; $x = 1$

Circumference = 2π u

11. Rectangles ① and ② on the right are similar whereas rectangles ① and ③ are equivalent. What is the perimeter of rectangle ③?

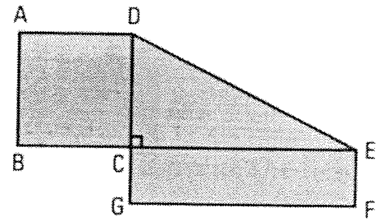


$\frac{4x}{6} = \frac{8}{x+1} \Rightarrow 4x^2 + 4x - 48 = 0;\ x = 3$

Area of rectangle ① = 24 u^2

Length of rectangle ③ = 8 u; Perimeter of rectangle ③: 22 u

12. The square ABCD, the right triangle DCE and the rectangle CGFE are all equivalent. What is the area of each figure if the perimeter of the rectangle is 15 cm?



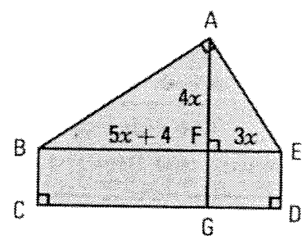
x : side length of square; each figure has an area of x^2; $m\overline{CE} = 2x$;

$mEF = \frac{x}{2}$.

Perimeter of rectangle = 5x; x = 3

Area of each figure is equal to 9 cm^2.

13. The right triangles ABF and EAF on the right are similar. Triangle ABF is equivalent to rectangle BCGF whereas the triangle EAF is equivalent to rectangle DEFG.



a) Calculate the total area of the pentagon ABCDE.

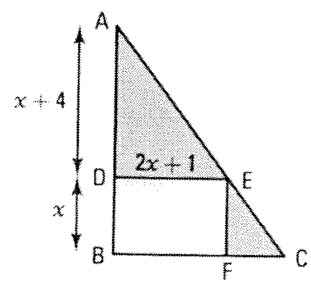
$\frac{4x}{5x+4} = \frac{3x}{4x}$; *x = 12; Area △ABF = 1536 u^2*

Area △AFE = 864 u^2; Area of pentagon = 4800 u^2

b) Calculate the perimeter of the pentagon. *$m\overline{AB} = 80$; $m\overline{AE} = 60$; $m\overline{ED} = 24$; $m\overline{CD} = 100$;*

$m\overline{BC} = 24$; Perimeter of pentagon = 288 u

14. Calculate the numerical value of the area of triangle ABC if triangle ADE and rectangle BDEF are equivalent.



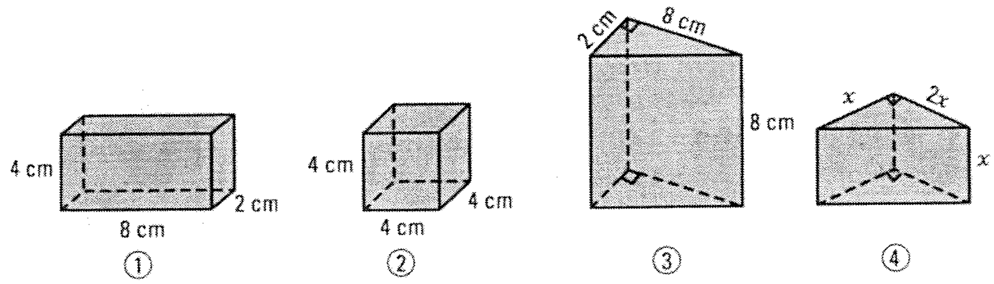
$(x + 4)(2x + 1) \div 2 = x(2x + 1);\ x = 4$

$\triangle ADE \sim \triangle ABC \Rightarrow \frac{8}{12} = \frac{9}{m\overline{BC}}$; *$m\overline{BC}$ = 13,5 u*

Area du △ABC = 81 u^2

11.3 Equivalent solids

ACTIVITY 1 Equivalent prisms



a) Show that prisms ①, ② and ③ above have the same volume.

$V_1 = 8 \times 2 \times 4 = 64$ cm^3; $V_2 = 4 \times 4 \times 4 = 64$ cm^3; $V_3 = \frac{8 \times 2}{2} \times 8 = 64$ cm^3

b) What are the dimensions of prism ④ if the four prisms above have the same volume?

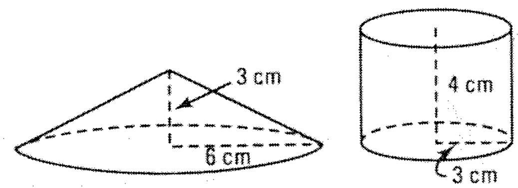
$x = 4$. The base of the prism is a right triangle.

The sides of the base measure 4 cm, 8 cm $\sqrt{80}$ cm; The height of the prism is equal to 4 cm.

EQUIVALENT SOLIDS

Two solids are **equivalent** if they have the same volume.

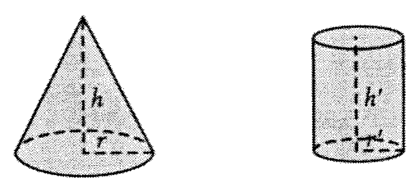
Ex.: The cone and the cylinder on the right are equivalent since they both have a volume of 36π cm^3.



1. A prism with a height of 4 cm has a rectangular base with dimensions 6 cm by 9 cm. What is the measure of a cube's edge that is equivalent to the prism? *6 cm*

2. A cone and a cylinder are equivalent. The radius and the height of the cone measure 6 cm and 10 cm respectively. What is the height of the cylinder if its radius measures 5 cm?

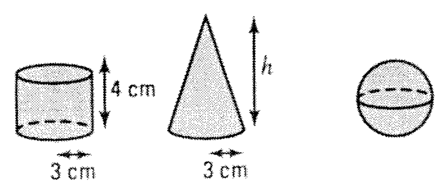
4.8 cm



3. The cylinder, cone and sphere on the right are equivalent. Determine

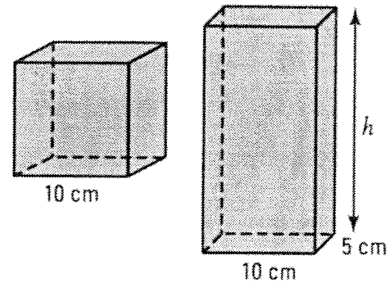
a) the height h of the cone; *$h = 12$ cm*

b) the radius r of the sphere. *$r = 3$ cm*



4. The cube and the rectangular prism on the right are equivalent. If the cube has 10 cm edges and the base of the prism is a 5 cm by 10 cm rectangle, find the height h of the prism.

$10 \times 5 \times h = 10^3 \Rightarrow h = 20$ cm



5. a) A cylinder and a cone with the same height of 6 cm are equivalent. Determine the radius r of the cone if the cylinder's radius measures 3 cm.

$\pi \cdot 3^2 \cdot 6 = \frac{\pi \cdot r^2 \cdot 6}{3} \Rightarrow r = 3\sqrt{3}$ cm

b) A cylinder and a cone with the same radius of 3 cm are equivalent. Determine the height h of the cone if the cylinder's height measures 6 cm.

$\pi \cdot 3^2 \cdot 6 = \frac{\pi \cdot 3^2 h}{3} \Rightarrow h = 18$ cm

6. A sphere, a cylinder and a cone are equivalent and each have a radius of 3 cm. Calculate

a) the height of the cylinder; *$h = 4$ cm*

b) the height of the cone. *$h = 12$ cm*

7. A cone and a cylinder have the same height h and are equivalent. Let r represent the radius of the cylinder. What is the radius of the cone?

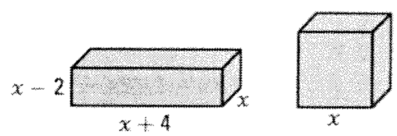
Radius of cone: $r\sqrt{3}$

8. The rectangular prism and the cube on the right are equivalent. By how much does the total area of the prism surpass the total area of the cube?

$x(x + 4)(x - 2) = x^3$; $x = 4$.

Area of prism = 112 u^2; Area of cube = 96 u^2

The area of the prism is 16 u^2 more than the cube.



9. A prism, a cylinder and a cone are equivalent. If the bases of the prism, the cylinder and the cone are equivalent, compare

a) the heights of the prism and the cylinder; *They are equal.*

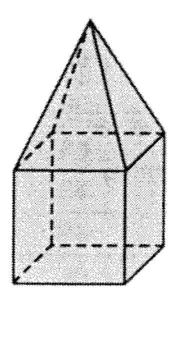
b) the heights of the prism and the cone. *The height of the cone is the triple of the prism's height.*

10. A sculpture is formed by a pyramid mounted on top of a cube. The cube and the pyramid are equivalent. Determine the total height of the sculpture if its volume is equal to 432 cm^3.

Volume of cube = 216 cm^3 $\Rightarrow$ height of cube = 6 cm

Volume of pyramid = $216 = \frac{6^2 \cdot h}{3}$ $\Rightarrow$ height of pyramid = 18 cm

Total height of sculpture = 24 cm



11. A cylinder and a sphere have the same radius and are equivalent.

a) Express the height h of the cylinder as a function of the radius r.

h: height of cylinder; $\pi r^2 h = \frac{4}{3}\pi r^3 \Rightarrow h = \frac{4}{3}r$

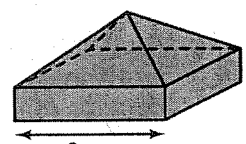
b) Express as a function of r how much the total area of the cylinder surpasses the area of the sphere.

Total area of cylinder= $2\pi r^2 + 2\pi r \cdot \frac{4}{3}r = \frac{14}{3}\pi r^2$***; Area of sphere*** $4\pi r^2$.

The total area of the cylinder is $\frac{2}{3}\pi r^2$ ***u²*** ***greater than the area of the sphere.***

c) Determine the difference between these two areas when $r = 3$ cm. 6π ***cm²***

12. A sculpture is formed by a square based pyramid mounted on top of a prism. The pyramid and the prism are equivalent. Calculate the total area of the sculpture.



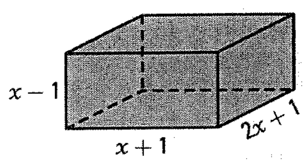
Volume of prism = volume of pyramid = 64 cm³

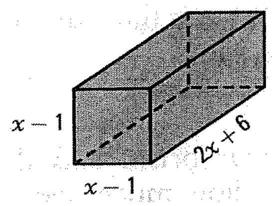
Height of pyramid = 3 cm. Slant height of pyramid = 5 cm.

Lateral area of pyramid = 80 cm²; Lateral area of prism = 32 cm²

Area of prism's base = 64 cm²; Total area of sculpture = 176 cm²

13. The right rectangular prism and square based prism below are equivalent. What is the numerical value of the sum of the volumes of these two prisms?

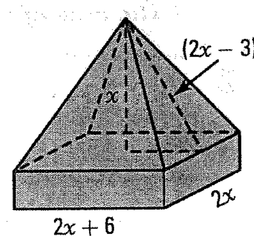




$(x + 1)(x - 1)(2x + 1) = (x - 1)^2(2x + 6); x = 7$

Volume of each prism = 720 cm³; Sum of volumes = 1440 cm³

14. A solid is formed by a pyramid mounted on top of a rectangular prism. The pyramid and prism are equivalent. Let x represent the height of the pyramid and $(2x - 3)$ its slant height. The dimensions of the prism's base are $2x$ and $(2x + 6)$. What is the numerical value of this solid's total volume?



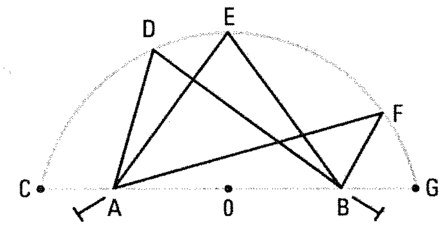
We have: $(2x - 3)^2 = x^2 + (x + 3)^2; x = 9$***u.***

Volume of pyramid = $24 \times 18 \times 9 \div 3 = 1296$ ***cm³***

Total volume= 2592 cm³

11.4 Comparing polygons

ACTIVITY 1 Area of triangles with the same base and same perimeter



Consider the 6 cm segment AB and its mid-point O. Using a piece of taught string attached to two fixed nails at A and B, a semi-ellipse is drawn. With the length of the string constantly 10 cm, the triangles with vertices A, B and any other point on the ellipse all have the same perimeter.

a) What is the perimeter of all these triangles with base AB? ***16 cm***

b) Of all these triangles with the same base, the one with the largest area is the one with the biggest height relative to this base. Determine which triangle has the largest area and indicate which type of triangle it is.

The isosceles triangle AEB.

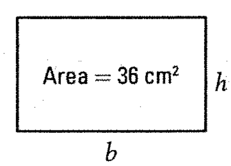
c) What is the area of the triangle with base AB that has the largest area?

$m\overline{EB} = 5$ cm; $m\overline{EO} = 4$ cm; Area $\frac{3 \times 4}{2} = 6$ cm^2

d) Complete: Of all the triangles with the same base and same perimeter, the one with the largest area is ***isosceles*** .

ACTIVITY 2 Perimeter of equivalent rectangles

Consider all rectangles with an area of 36 cm^2. We are seeking the dimensions of the rectangle with the smallest perimeter.



a) Complete the table of values on the right which gives the perimeter of the rectangle as a function of its base b and height h.

b) By observing the table and using a calculator, find the rectangle with the smallest perimeter.

The square with 6 cm sides.

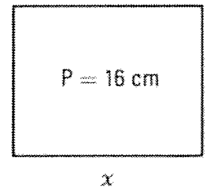
c) What conjecture can we propose as a conclusion to this activity?

For a rectangle with a given area, the square is the one with the smallest perimeter.

Base (cm) b	Height (cm) h	Perimeter (cm) $P = 2b + 2h$
1	36	74
2	18	40
4	9	26
6	6	24
9	4	26
12	3	30
36	1	74

ACTIVITY 3 Area of rectangles with the same perimeter

Consider all rectangles with a perimeter of 16 cm. We are seeking the dimensions of the one with the maximum area.



a) If we let x represent the rectangle's base,

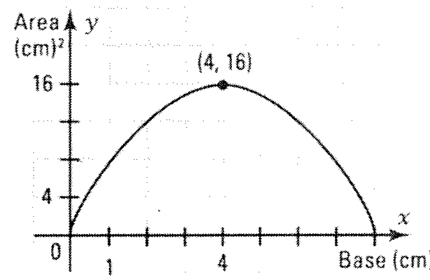
1. express the height as a function of x. $8 - x$
2. express the rectangle's area as a function of x.
   $A(x) = -x^2 + 8x$

b) 1. Complete the table of values on the right which gives the rectangle's area A(x) as a function of its base x.

Base	Height	Area A(x)
1	7	7
2	6	12
3	5	15
4	4	16
5	3	15
6	2	12
7	1	7

2. For what value of x do we get a maximum area?
   4
3. What are the dimensions and the nature of this rectangle when the area is greatest?
   A square with 4 cm sides.

c) 1. Represent, in the Cartesian plane on the right, the function that gives the area A(x) as a function of its base x.

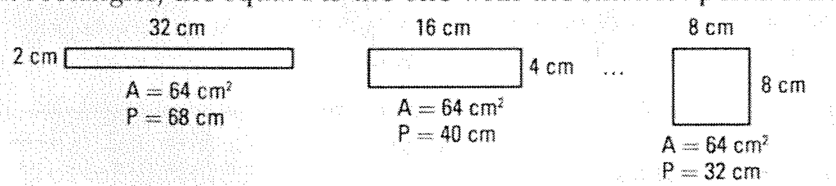


2. Using the graph, indicate for which value of x do we get a maximum area A(x).
   The area is maximized when $x = 4$.
3. What is the maximum area? 16 cm^2

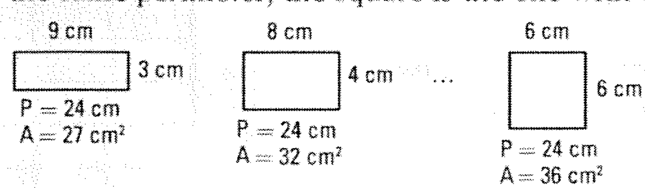
d) Of all rectangles with the same perimeter, what is the nature of the rectangle with the greatest area? *A square*

PERIMETER AND AREA OF RECTANGLES

- Of all equivalent rectangles, the square is the one with the smallest perimeter.

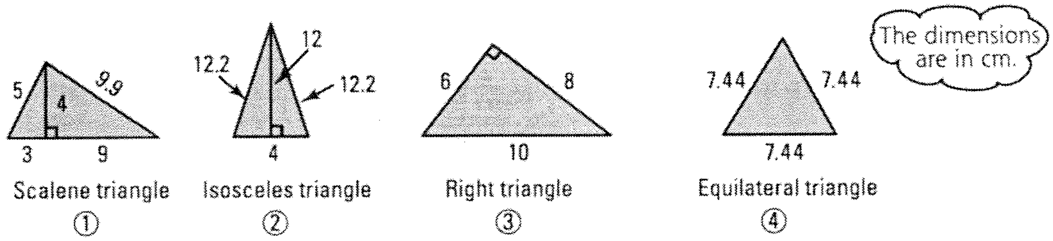


- Of all rectangles with the same perimeter, the square is the one with the largest area.



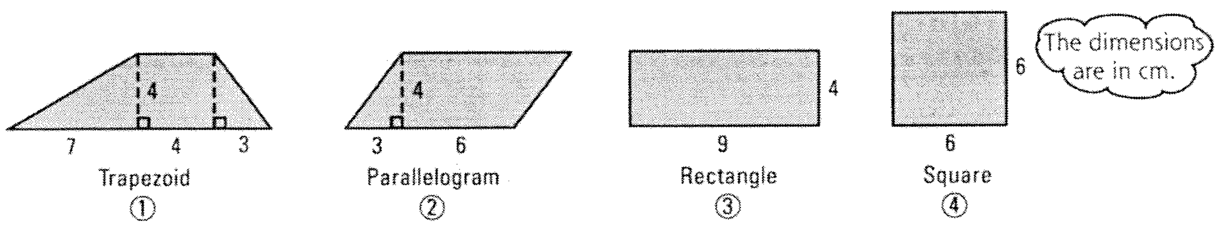
ACTIVITY 4 Perimeter of equivalent n-sided polygons

a) Consider the following equivalent triangles.



1. Verify that the triangles are equivalent. ***They all have an area of 24 cm².***
2. Calculate the perimeter of each triangle. ***①: 26.9 cm ②: 28.4 cm ③: 24 cm ④: 22.32 cm***
3. Of the four triangles, what is the nature of the one with the smallest perimeter?
   The equilateral triangle.

b) Consider the following equivalent quadrilaterals.



1. Verify that the quadrilaterals are equivalent. ***They all have an area of 36 cm².***
2. Calculate the perimeter of each quadrilateral.
   ①: 31 cm ②: 28 cm ③: 26 cm ④: 24 cm
3. Of the four quadrilaterals, what is the nature of the one with the smallest perimeter?
   The square.

PERIMETER OF EQUIVALENT n-SIDED POLYGONS

- Of all equivalent n-sided polygons, the regular polygon is the one with the smallest perimeter.
- Therefore,
  - Of all equivalent triangles, the equilateral triangle has the smallest perimeter.
  - Of all equivalent quadrilaterals, the square has the smallest perimeter.
  - Of all equivalent pentagons, the regular pentagon has the smallest perimeter.

ACTIVITY 5 Perimeter of equivalent regular polygons

Consider all regular polygons with an area of 100 cm^2. The table of values below gives the perimeter of each regular polygon as a function of the number of its sides.

a) As the number of sides increases, does the perimeter of the polygon increase or decrease?
It decreases.

b) To what number does the perimeter seem to be approaching as the number of sides increases? *35.449 cm*

c) What geometric shape does the regular polygon become as the number of its sides becomes infinite? *A circle*

d) Verify that the circle with a perimeter of 35.449 cm has an approximate area of 100 cm^2 ($\pi = 3.1416$).
Radius ≈ 5.6419 cm; Area ≈ 99.999 3 cm^2

Number of sides	Area (cm^2)	Side (cm)	Perimeter (cm)
3	100	15.197	45.590
4	100	10	40
5	100	7.624	38.119
6	100	6.204	37.224
10	100	3,.605	36.051
100	100	0.355	35.455
1000	100	0.035	35.44914
10 000	100	0.004	35.44908
⋮			

ACTIVITY 6 Area of regular polygons with the same perimeter

Consider all regular polygons with a perimeter of 100 cm. The table of values below gives the area of each regular polygon as a function of the number of its sides.

a) As the number of sides increases, does the area of the polygon increase or decrease?
It increases.

b) To what number does the area seem to be approaching as the number of sides increases?
795.7747 cm^2

c) The polygon approaches a circle as its number of sides increases. What is the approximate value of the radius of this circle?
15.9155 cm

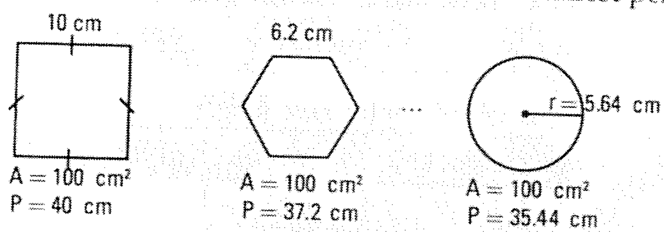
d) Verify that the circle with the radius established in c) has approximately the area established in b).
$\pi(15.9155)^2 \approx 795.7747$

Number of sides	Perimeter (cm)	Apothem (cm)	Area (cm^2)
3	100	28.87	481.125
4	100	12.5	625
5	100	13.764	688.191
6	100	14.434	721.688
10	100	15.388	769.421
100	100	15.910	795.513
1000	100	15.915	795.772
10 000	100	15.9155	795.7747
⋮			

PERIMETER AND AREA OF REGULAR POLYGONS

- Of all equivalent regular polygons, the one with the smallest perimeter is the one with the largest number of sides.

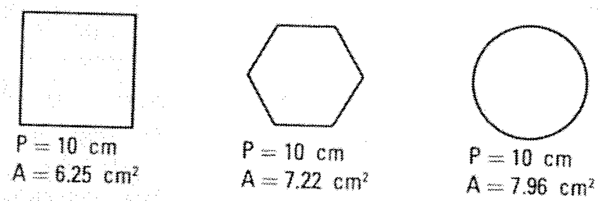
  Taken to the limit, the equivalent circle is the one with the smallest perimeter.



When the number n of sides increases, the perimeter P decreases. (n ↗ P ↘)

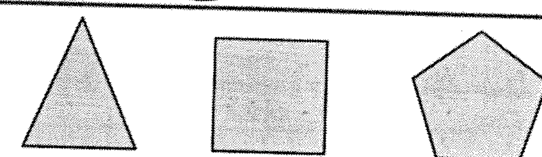
- Of all regular polygons with the same perimeter, the one with the largest area is the one with the largest number of sides.

  Taken to the limit, the circle with the same perimeter is the one with the largest area.

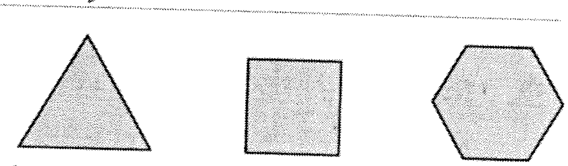


When the number n of sides increases, the area A increases. (n ↗ A ↗)

1. The following regular polygons have the same perimeter.
Which one has the largest area? Justify your answer.
The pentagon since it is the one with the greatest number of sides.



2. The following regular polygons are equivalent.
Which is the one with the smallest perimeter? Justify your answer.
The hexagon since it is the one with the greatest number of sides.



3. a) Of all triangles with the same base and perimeter, what is the nature of the triangle with the largest area? ***The isosceles triangle.***

b) What is the maximum area of a triangle with a 12 cm base and a perimeter of 32 cm?
48 cm²

4. a) 1. Of all equivalent rectangles, what is the nature of the one with the smallest perimeter?
The square.

2. What is the smallest perimeter of a rectangle with an area of 100 cm²?
40 cm

b) 1. Of all rectangles with the same perimeter, what is the nature of the one with the greatest area? ***The square.***

2. What is the greatest area of a rectangle with a perimeter of 100 cm? ***625 cm²***

5. Using 100 m of fence, a farmer wants to make a rectangular enclosure for a herd of sheep.

a) What must he do if he wants his enclosure to have the greatest area?
He must make a square enclosure.

b) What are the dimensions of the enclosure with the greatest area?
A square with 25 m sides.

c) What is the maximum area of the enclosure? ***625 m^2***

6. **a)** Consider an equilateral triangle with side length x.

1. What is its height? $\frac{x\sqrt{3}}{2}$ 2. What is its area? $\frac{x^2\sqrt{3}}{4}$

b) Consider all triangles with an area of $\sqrt{3}$ cm^2.

1. What is the nature of the triangle with the smallest perimeter?
An equilateral triangle.
2. What is the minimum perimeter of a triangle with an area of $\sqrt{3}$ cm^2? ***6 cm***

7. **a)** A regular pentagon and hexagon are equivalent. Which one has the greatest perimeter?
The pentagon.

b) A regular pentagon and hexagon have the same perimeter. Which one has the greatest area? ***The hexagon.***

8. **a)** What is the maximum area, to the nearest hundredth, of a triangle with a 4 cm base and a perimeter of 16 cm? ***11.31 cm^2***

b) What is the maximum area of a rectangle with a perimeter of 16 cm? ***16 cm^2***

c) What is the maximum area, to the nearest hundredth, of a triangle with a perimeter of 12 cm? ***6.93 cm^2***

d) What is the maximum area, to the nearest hundredth, of an hexagon with a perimeter of 12 cm? ***10.39 cm^2***

9. Using 100 m of fence, a yard in the shape of a regular polygon is to be surrounded.

a) Complete: As we increase the number of sides, the area of the yard ***increases.***

b) To the limit, what is the shape of the yard with the biggest area? What is this area?
A circular yard with an approximate area of 796 m^2.

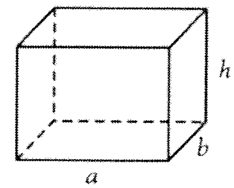
10. We want to use 100 m of fence to enclose a yard in the shape of a regular polygon. Which of the two shapes, square or hexagon, will have the largest area? Justify your answer

1. without calculating the two areas. ***Of all regular polygons with the same perimeter, the one with the largest area is the one with the largest number of sides.***
2. by calculating the two areas.
Area of square: 625 m^2; area of hexagon: 722 m^2

11.5 Comparing solids

ACTIVITY 1 Volume of rectangular prisms with the same total area

a) The prisms ① to ⑤ with the dimensions given below all have the same total area. Calculate the volume of each prism.



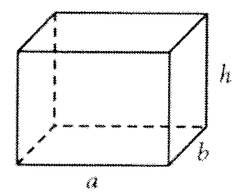
	a (cm)	b (cm)	h (cm)	Area (cm²)	Volume (cm³)
Prism 1	6	6	1	96	***36***
Prism 2	6	4.5	2	96	***54***
Prism 3	6	4	2.4	96	***57.6***
Prism 4	3	3	6.5	96	***58.5***
Prism 5	4	4	4	96	***64***

b) Of these prisms with the same total area, which is the one with the largest volume?

Prism 5; a cube.

ACTIVITY 2 Total area of equivalent rectangular prisms

a) The prisms ① to ⑤ with the dimensions given below are all equivalent. Calculate the total area of each prism.



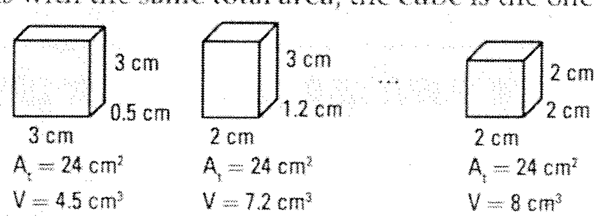
	a (cm)	b (cm)	h (cm)	Volume	Area (cm²)
Prism 1	64	1	1	64	***258***
Prism 2	32	2	1	64	***196***
Prism 3	16	2	2	64	***136***
Prism 4	8	4	2	64	***112***
Prism 5	4	4	4	64	***96***

b) Of these equivalent prisms, which is the one with the smallest total area?

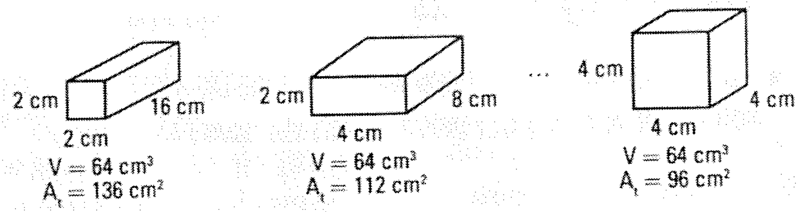
Prism 5; a cube.

TOTAL AREA AND VOLUME OF RECTANGULAR PRISMS

- Of all rectangular prisms with the same total area, the cube is the one with the largest volume.

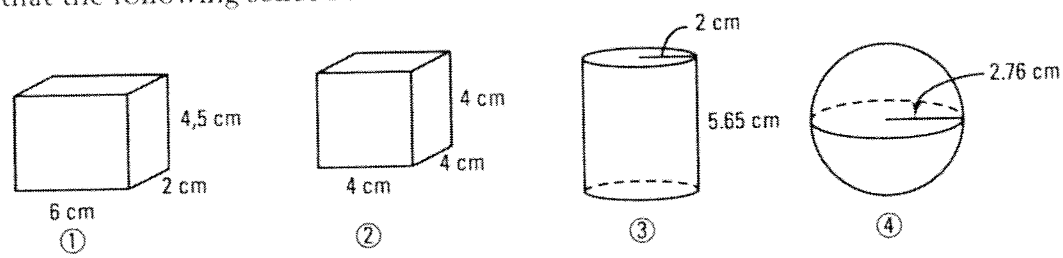


- Of all rectangular prisms with the same volume, the cube is the one with the smallest total area.



ACTIVITY 3 Volume of solids with the same total area

a) Verify that the following solids have the same total area, to the nearest unit, of 96 cm^2.



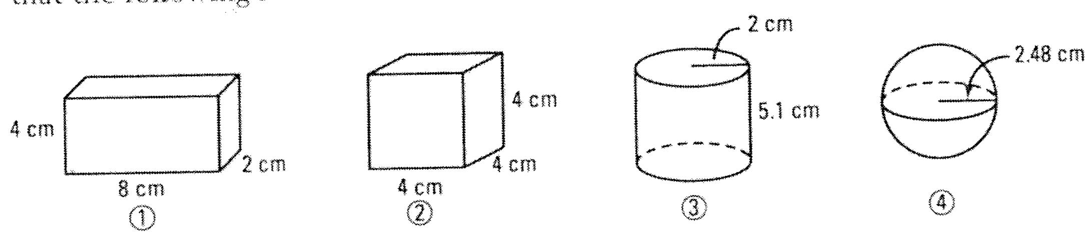
b) Of these solids, verify that the sphere has the largest volume.

$V_1 = 54$ cm³; $V_2 = 64$ cm³; $V_3 = 71$ cm³; $V_4 = 88{,}1$ cm³

The sphere has the largest volume.

ACTIVITY 4 Total area of solids with the same volume

a) Verify that the following solids have the same volume, to the nearest unit, of 64 cm^3.



b) Of these solids, which one has the smallest total area?

① $A_t = 112$ cm²; ② $A_t = 96$ cm²; ③ $A_t = 89.221$ cm²; ④ $A_t = 77.29$ cm²

The sphere has the smallest total area.

TOTAL AREA AND VOLUME OF SOLIDS

- Of all solids with the same total area, the sphere is the one with the largest volume.
- Of all solids with the same volume, the sphere is the one with the smallest total area.

1. A company uses boxes shaped like rectangular prisms to package its products.

a) Indicate the nature and the dimensions of the least expensive box that has a volume of 8 dm^3. ***The box must be a cube with 2 dm edges.***

b) If the least expensive box is used, what will be the production costs of 100 boxes if the cardboard used to make the boxes costs 0.01 ¢/cm^2?
Area of one box = 24 dm^2 = 2400 cm^2; Cost of one box = 24 ¢; Cost of 100 boxes = $240.

2. Using cardboard with an area of 600 cm^2, we want to make a box in the shape of a rectangular prism.

a) What is the volume of a box with a base whose dimensions are 10 cm by 6 cm? ***900 cm^3***

b) What are the dimensions and the volume of the box with the greatest volume?
A cube with 10 cm edges and a volume of 1000 cm^3.

3. a) Of all solids with a total area of 6 cm^2, what is the shape of the solid with the greatest volume? What is this volume? ***A sphere with a 1.38 cm radius and an approximate volume of 11 cm^3***

b) Of all solids with a total volume of 8 cm^3, what is the shape of the solid with the smallest total area? ***A sphere with a 1.24 cm radius and an approximate area of 19.32 cm^2***

4. a) What is the maximum volume of a rectangular prism with a total area of 150 cm^2?
125 cm^3

b) What is the maximum volume, to the nearest hundredth, of a solid with a total area of 1256 cm^2? ***4185.6 cm^3***

c) What is the minimum area of a rectangular prism with a volume of 512 cm^3? ***384 cm^2***

d) What is the minimum total area, to the nearest hundredth, of a solid with a volume of 113 cm^3? ***113 cm^2***

5. A pear, a banana and an orange have peels with the same total area. Of these three fruits, which one has the greatest volume? ***The orange.***

6. A toy is to be painted in gold and can take the shape of a cone or a sphere. If the two shapes have the same volume, which one will cost less to cover in gold? ***The sphere.***

7. A sphere and a cube each have an area of 5024 cm^2. Which of the two solids has the greatest volume? Justify your answer

a) without calculating the two volumes.
Of all solids with the same total area, the sphere has the greatest volume.

b) by calculating the two volumes.
Cube: edge ≈ 28.94 cm; volume ≈ 24 230 cm^3; Sphere: radius ≈ 20 cm; volume ≈ 33 485 cm^3

8. We want to manufacture boxes in the shape of rectangular prisms with the following dimensions: 20 cm by 10 cm by 5 cm. The material used to manufacture these boxes has a total area of 42 dm^2.

a) What is the maximum number of boxes that can be made? ***6 boxes.***

b) Determine the dimensions of the box that will enable you to make the maximum number of boxes with this material. ***7 cubic boxes with 10 cm edges.***

Evaluation 11

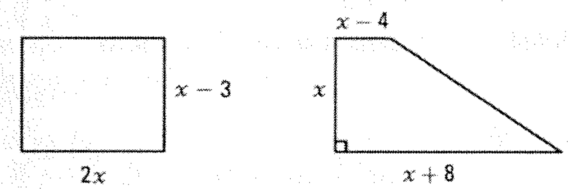
1. A rectangle with a 6 cm length and 4 cm width is equivalent to a triangle with an 8 cm base. What is the triangle's height relative to this base? ***6 cm***

2. A 6 cm by 4 cm by 2 cm rectangular prism is equivalent to a square based pyramid. If the side length of the pyramid's base is 4 cm, calculate its height. ***9 cm***

3. A cone and a cylinder have congruent circular bases and are equivalent. What can be said about the heights of these two solids?

The height of the cone is three times the cylinder's height.

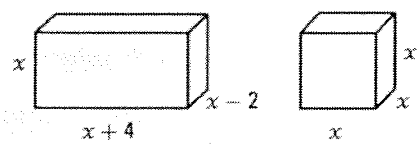
4. The rectangle and right trapezoid on the right are equivalent. What is the numerical value of the rectangle's perimeter?



$2x(x - 3) = (2x + 4) \cdot x \div 2;\ x = 8;$

Perimeter of rectangle = 42 u.

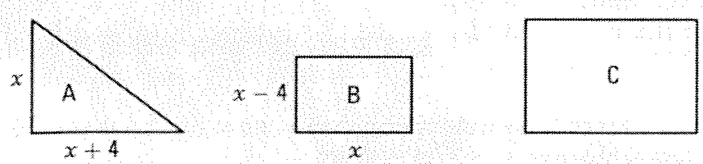
5. The rectangular prism and cube on the right are equivalent. What is the numerical value of the area of each solid?



$x(x + 4)(x - 2) = x^3;\ x = 4$

Area of prism: 112 u²; Area of cube: 96 u²

6. Figures A and B below are equivalent whereas figures B and C are similar. The area of figure C is 24 cm² greater than twice the area of figure A. Determine the perimeter of figure C.

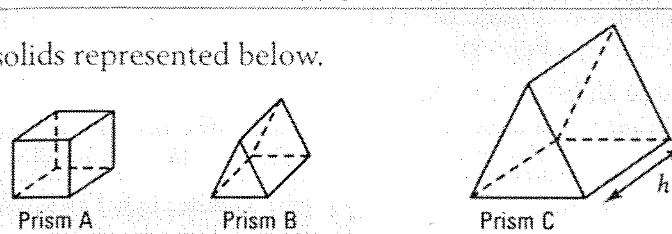


$\frac{x(x + 4)}{2} = x(x - 4);\ x = 12;$ ***Area of figure B = 96 cm²; Area of figure C = 216 cm²***

Scale factor $= \frac{3}{2}$

Perimeter of figure B = 40 cm; Perimeter of figure C = 60 cm.

7. Consider the three solids represented below.



Prisms A and B are equivalent whereas prisms B and C are similar. The volume of prism C is 8 times greater than the volume of prism A. The total volume of all three prisms is 10 dm³. What is the height h of prism C if the area of prism B's base is 100 cm²?

v: volume of prism A; We have: $v + v + 8v = 10\,000;\ v = 1000\ \text{cm}^3$.

h': height of prism B; We have: $1000 = 100 \times h;\ h' = 10$ cm. $k = 2$. *Therefore,* $h = 20$ cm.

8. A rectangular plot of land with dimensions 45 m by 20 m and a square plot of land have the same area. The cost, per metre of fence, is \$25.

a) Show that the square plot of land is cheaper to fence in.
Rectangular plot: \$3250. Square plot: \$3000.

b) Explain why, of all plots with the same area as the rectangular plot, the plot with the minimal cost to fence will be a square plot.
It has been proven that of all equivalent rectangles, the square is the one with the smallest perimeter.

9. We want to construct a box in the shape of a rectangular prism with 216 dm^2 of material.

a) We choose to make a box with a 60 cm by 40 cm base. What will its volume be?
x: height; $4800 + 120x + 80x = 21\ 600$; $x = 84$; volume = 201.6 dm^3
The volume of such a box is 201.6 dm^3.

b) 1. What must the shape of the box be to maximize its volume? Justify your answer.
A cube, since of all rectangular prisms with the same total area, the cube is the one that has the greatest volume.

2. What are the dimensions of the box that satisfies these conditions?
x: edge of cube; $6x^2 = 216$; $x = 6$
The box is a cube with 6 dm edges.

3. What is the maximum volume of the box that can be constructed from the given material? ***216 dm^3***

10. A sphere and a cube each have a volume of 8000 cm^3. Which of the two solids has the smallest total area? Justify your answer

a) without calculating the two volumes.
Of all equivalent solids, the sphere has the smallest total area.

b) by calculating the two volumes.
Cube: 2400 cm^2; sphere 1934 cm^2.

Chapter 12

Matrices

CHALLENGE 12

1. A company owns a warehouse in Montreal and another one in Quebec City. The Montreal warehouse has an inventory of 850 units of product A, 720 units of product B and 250 units of product C. The Quebec warehouse has an inventory of 650 units of product A, 420 units of product B and 210 units of product C.

In the Montreal warehouse, posted prices per unit are \$25 for product A, \$20 for product B and \$30 for product C.

In the Quebec warehouse, posted prices per unit are \$26 for product A, \$22 for product B and \$28 for product C..

a) Using a table, represent the inventory of both warehouses.

	Montreal	*Quebec*
A	*850*	*650*
B	*720*	*420*
C	*250*	*210*

b) Using a table, represent the revenue obtained by the sale of the units in each warehouse.

	Montreal	*Quebec*
A	*\$21 250*	*\$16 900*
B	*\$14 400*	*\$9 240*
C	*\$7 500*	*\$5 880*

2. The matrix product $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ defines a geometric transformation.

a) Consider triangle ABC with vertices A(1, 4), B(−2, 1) and C(3, 2). What are the coordinates of the vertices of triangle A′B′C′, image of triangle ABC under this transformation?

A′(1, 8); B′(−2, 2); C′(3, 4)

b) Describe this transformation.

Vertical scaling of factor 2.

c) Determine the matrix product corresponding to the inverse transformation.

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

12.1 Matrices

Activity 1 Representing a situation using a matrix

An electronics store has three branches. On a Saturday of a given weekend, we observe that 6 televisions, 4 dvd players, 3 video cameras and 5 cd players were sold at the first branch, 5 televisions, 3 dvd players, 4 video cameras and 4 cd players were sold at the second branch and 4 televisions, 5 dvd players, 4 video cameras and 6 cd players were sold at the third branch.

a) Complete the table below indicating, for each branch, the number of appliances of each model that were sold.

	Branch 1	Branch 2	Branch 3
Televisions	6	5	4
Dvd players	4	3	5
Video cameras	3	4	4
Cd players	5	4	6

b) Such a table arranged into rows and columns is called matrix.
This matrix has 4 rows and 3 columns. We say that its dimension or size is 4 by 3, written 4×3.
This matrix, denoted by A, is represented on the right.
Complete the columns of matrix A.

$$A = \begin{bmatrix} 6 & 5 & 4 \\ 4 & 3 & 5 \\ 3 & 4 & 4 \\ 5 & 4 & 6 \end{bmatrix}$$

c) The values 6, 4, 3 and 5 are the elements of the 1st column of the matrix.
We denote by i the number of a row and by j the number of a column.
a_{ij} represents the element located on the ith row and jth column.
The element 4 located on the 2nd row and 1st column is denoted by a_{21}.
Thus, $a_{21} = 4$. Determine the following elements.

1. $a_{12} =$ 5 2. $a_{22} =$ 3 3. $a_{32} =$ 4 4. $a_{23} =$ 5

MATRICES

- A matrix is a table of numbers arranged into rows and columns. We denote by m the number of rows and by n the number of columns. The numbers in a matrix are called elements. We usually denote a matrix using an uppercase letter such as A, and each element using a lowercase letter a_{ij} where i denotes the row and j the column where element a_{ij} is located.

Matrix A on the right has 3 rows ($m = 3$) and 4 columns ($n = 4$). We say that it is of dimension or size 3 by 4.
This dimension is written 3×4.
Note that: $a_{23} = 2$ and $a_{32} = 0$.

$$A_{3 \times 4} = \begin{bmatrix} 2 & 5 & 4 & 0 \\ 3 & -1 & 2 & 1 \\ 1 & 0 & -2 & 3 \end{bmatrix}$$

- A zero matrix is a matrix whose elements are all zero. This matrix is written: $0_{m \times n}$.

  Ex.: $0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- A row matrix is a matrix with only one row.

  Ex.: $A_{1\times 3} = \begin{bmatrix} 2 & 5 & 0 \end{bmatrix}$ is a row matrix.

- A column matrix is a matrix with only one column.

  Ex.: $A_{3\times 1} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ is a column matrix.

- A square matrix of order n, written $A_{n \times n}$, is a matrix with n rows and n columns. The elements $a_{11}, a_{22}, \ldots, a_{nn}$ form the main diagonal of the matrix.

  Ex.: $A_{3\times 3} = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix}$ is a square matrix. The main diagonal is formed by elements 2, 1 and 0.

- A diagonal matrix is a square matrix where all the elements not on the main diagonal are zero.

  Ex.: $A_{3\times 3} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonal matrix.

- An identity matrix, written $I_{n \times n}$, is a diagonal matrix where all the elements on the main diagonal are equal to 1.

  Ex.: $I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1. David buys 2 kg of bananas, 3 kg of oranges and 4 kg of apples. Valerie buys 1 kg of bananas, 4 kg of oranges and 2 kg of apples.

a) Complete the matrices describing these purchases.

1. A =

	bananas	*oranges*	*apples*
David	**2**	**3**	**4**
Valerie	**1**	**4**	**2**

2. B =

	David	*Valerie*
bananas	2	1
oranges	3	4
apples	4	2

b) Identify the size of

1. matrix A. ***2 × 3***
2. matrix B.. ***3 × 2***

2. Consider the following matrices.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 5 & 6 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

a) Determine the size of each matrix.

1. A ***3 × 2*** 2. B ***2 × 2*** 3. C ***2 × 3*** 4. D ***1 × 3*** 5. E ***2 × 1***

b) Identify, among the matrices above,

1. a square matrix. ***B*** 2. a row matrix. ***D*** 3. ua column matrix. ***E***

3. Consider the following matrices.

$A = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

a) Identify the matrices which are square. ***A, B, C and D.***

b) Determine the diagonal matrices. ***B and D.***

c) Explain why matrix C is not an identity matrix. ***C is not diagonal.***

d) Identify the identity matrix among the given matrices. ***D***

4. Determine the following matrices.

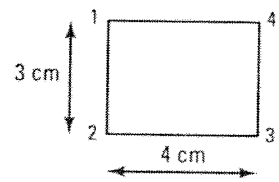
a) $0_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ b) $0_{3 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ c) $I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ d) $I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5. Determine the matrix.

a) $A_{2 \times 3}$ such that $a_{ij} = i + j$. b) $B_{2 \times 2}$ such that $b_{ij} = ij$. c) $C_{3 \times 3}$ such that $c_{ij} = (-1)^{i+j}$.

$A_{2 \times 3} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ $B_{2 \times 2} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $C_{3 \times 3} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

6. The rectangle on the right has a length of 4 cm and a width of 3 cm. The vertices of this rectangle are named 1, 2, 3 and 4.



a) Determine the matrix $A_{4 \times 4}$ where each element a_{ij} represents the distance between vertices i and j.

$A = \begin{bmatrix} 0 & 3 & 5 & 4 \\ 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 4 & 5 & 3 & 0 \end{bmatrix}$

b) Explain why

1. $a_{ii} = 0$ for all i. ***The distance between two points that are the same is zero. d(A, A) = 0.***
2. $a_{ij} = a_{ji}$ for all i and all j.
The distance between two points is symmetrical, i.e. d(A, B) = d(B, A).

7. Consider matrix A of size 2 by 3, $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 4 \end{bmatrix}$.

a) Determine the matrix B having the same size as A such that $a_{ij} = b_{ij}$, i and j.

Matrices A and B are called equal. $B = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 4 \end{bmatrix}$

b) Determine the matrix C having same size as A such that $a_{ij} = -c_{ij}$, i and j.

Matrices A and C are called opposite. $C = \begin{bmatrix} -2 & -3 & 1 \\ 0 & 5 & -4 \end{bmatrix}$

8. Consider matrix A of size 2 by 3, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

The transpose of matrix A is the matrix B of size 3 by 2 such that $b_{ij} = a_{ji}$.

a) Determine matrix B. $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

b) What can be said of matrices A and B of exercise 1? ***One is the transpose of the other.***

12.2 Operations between matrices

ACTIVITY 1 Addition of two matrices

Consider again activity 1 of page 434.

Matrix A describing Saturday's sales at each branch of an electronics store is described on the right.

A =

	Br. 1	Br. 2	Br. 3
Tel.	6	5	4
Dvd.	4	3	5
Vid.	3	4	4
Cd.	5	4	6

Matrix B describing Sunday's sales at each branch of this store is described on the right.

B =

	Br. 1	Br. 2	Br. 3
Tel.	3	4	2
Dvd.	4	4	3
Vid.	2	5	1
Cd.	6	4	4

a) Determine the matrix C describing the total sales for the weekend.

b) Explain how we determine each element c_{ij} of matrix C.

$c_{ij} = a_{ij} + b_{ij}$, i and j.

$$C = \begin{bmatrix} 9 & 9 & 6 \\ 8 & 7 & 8 \\ 5 & 9 & 5 \\ 11 & 8 & 10 \end{bmatrix}$$

ACTIVITY 2 Multiplication of a matrix by a real number

A company has two branches, one in Quebec, the other in Ontario. Matrix A on the right gives the hourly wage of first year employees according to whether or not they hold a high school diploma.

A =

	Quebec	Ontario
Diploma	14	15
No diploma	10	12

a) The company decides to raise the hourly wage of each employee by 5%. Determine the matrix B giving the new hourly wage for employees.

$$B = \begin{bmatrix} 14.70 & 15.75 \\ 10.50 & 12.60 \end{bmatrix}$$

b) Explain how we determine each element b_{ij} of matrix B.

$b_{ij} = 1,05 \times a_{ij}$

ADDITION OF MATRICES AND MULTIPLICATION OF A MATRIX BY A REAL NUMBER

- Let $A_{m \times n}$ and $B_{m \times n}$, be two matrices of same size.
  - The sum A + B of these two matrices is the matrix C of size $m \times n$ such that

    $$c_{ij} = a_{ij} + b_{ij}, \quad i \text{ and } j.$$

  - The difference A − B of these two matrices is the matrix D of size $m \times n$ such that

    $$d_{ij} = a_{ij} - b_{ij}, \quad i \text{ and } j.$$

- Let $A_{m \times n}$ be a matrix and k a real number. The product of matrix A by the real number k is a matrix P of size $m \times n$ such that

  $$p_{ij} = ka_{ij}, \quad i \text{ and } j.$$

Ex.: Given matrices $A_{2\times3} = \begin{bmatrix} 2 & 3 & 0 \\ -3 & 4 & -1 \end{bmatrix}$ and $B_{2\times3} = \begin{bmatrix} 5 & -2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$

We have: $A + B = \begin{bmatrix} 7 & 1 & 4 \\ -3 & 5 & -3 \end{bmatrix}$; $A - B = \begin{bmatrix} -3 & 5 & -4 \\ -3 & 3 & 1 \end{bmatrix}$; $2A = \begin{bmatrix} 4 & 6 & 0 \\ -6 & 8 & -2 \end{bmatrix}$.

1. Consider matrices $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}$. Determine

a) $A + B$ $\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$ **b)** $A - B$ $\begin{bmatrix} 1 & 4 \\ 2 & -5 \end{bmatrix}$ **c)** $-5A$ $\begin{bmatrix} -10 & -15 \\ 0 & 5 \end{bmatrix}$

d) $2A + 3B$ $\begin{bmatrix} 7 & 3 \\ -6 & 10 \end{bmatrix}$ **e)** $3A - 2B$ $\begin{bmatrix} 4 & 11 \\ 4 & -11 \end{bmatrix}$ **f)** $0A$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

g) $-A$ $\begin{bmatrix} -2 & -3 \\ 0 & 1 \end{bmatrix}$ **h)** $A + (-A)$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **i)** $B + 0_{2\times2}$ $\begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}$

2. Matrix A_1 on the right gives the grades (out of 100) Anita, Donita and Claire got on their 1st term mathematics and science exams. Matrices A_2, A_3 and A_4 give the grades Anita, Donita and Claire got on their mathematics and science exams for the following three terms.

	Math	*Sciences*
Anita	70	85
Donita	84	76
Claire	74	78

$A_2 = \begin{bmatrix} 76 & 82 \\ 80 & 84 \\ 84 & 80 \end{bmatrix}$; $A_3 = \begin{bmatrix} 68 & 76 \\ 82 & 80 \\ 76 & 84 \end{bmatrix}$; $A_4 = \begin{bmatrix} 70 & 80 \\ 72 & 90 \\ 88 & 82 \end{bmatrix}$

The weight of the exams is the same for both subjects. The first two terms count for 20%, the third term counts for 25% and the fourth term counts for 35% of the final grade.

a) Determine the matrix F giving the final grade of each student by subject.

$F = \begin{bmatrix} 70,7 & 80,4 \\ 78,5 & 83,5 \\ 81,4 & 81,3 \end{bmatrix}$

b) Which one of the three students has the best grade in

1. mathematics? ***Claire*** 2. science? ***Donita***

ACTIVITY 3 Multiplication of matrices

Matrix $A_{3 \times 4}$ below indicates for each of the grocery stores G_1, G_2 and G_3 the cost (in $) of a kilogram of different fruits.

Karen wishes to buy 1 kg of bananas, 2 kg of oranges, 3 kg of apples and 1 kg of grapes. Valerie wants to buy 2 kg of bananas, 3 kg of oranges, 2 kg of apples and 1 kg of grapes. Karen and Valerie's purchases are indicated by the matrix $B_{4 \times 2}$ below.

$$A_{3\times 4} = \begin{array}{c} \\ G_1 \\ G_2 \\ G_3 \end{array} \begin{array}{c} \begin{array}{cccc} \textit{bananas} & \textit{oranges} & \textit{apples} & \textit{grapes} \end{array} \\ \begin{bmatrix} 0.30 & 0.80 & 0.75 & 1.80 \\ 0.25 & 0.70 & 0.80 & 2.00 \\ 0.40 & 0.90 & 0.70 & 2.50 \end{bmatrix} \end{array} \qquad B_{4\times 2} = \begin{array}{c} \\ \textit{bananas} \\ \textit{oranges} \\ \textit{apples} \\ \textit{grapes} \end{array} \begin{array}{c} \begin{array}{cc} \textit{Karen} & \textit{Valerie} \end{array} \\ \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 2 \\ 1 & 1 \end{bmatrix} \end{array}$$

a) What is the total cost Karen must pay if she shops at grocery store E_1? ***$5.95***

b) To determine the total cost Karen must pay, we multiplied, in order, element by element, each element of the 1st row of A with each element of the 1st column of B then we computed the sum of these products, in other words:

$$\begin{bmatrix} 0.30 & 0.80 & 0.75 & 1.80 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = 0.30 \times 1 + 0{,}.80 \times 2 + 0.75 \times 3 + 1.80 \times 1.$$

Indicate which row of A and which column of B we should multiply in order to determine the total cost paid by

1. Karen, if she shops at grocery store 2. ***2nd row of A and 1st column of B.***
2. Valerie, if she shops at grocery store 3. ***3rd row of A and 2nd column of B.***

c) Matrix $C_{3 \times 2}$ on the right indicates, according to the chosen grocery store, the total cost of Karen and Valerie's purchases.
We get an element c_{ij} of matrix C by multiplying the ith row of A by the jth column of B.
Determine all the missing elements of matrix C.

$$C_{3\times 2} = \begin{array}{c} \\ G_1 \\ G_2 \\ G_3 \end{array} \begin{array}{c} \begin{array}{cc} \textit{Karen} & \textit{Valerie} \end{array} \\ \begin{bmatrix} 5.95 & \mathbf{6.30} \\ 6.05 & \mathbf{6.20} \\ \mathbf{6.80} & 7.40 \end{bmatrix} \end{array}$$

PRODUCT OF MATRICES

- Let $A_{m \times p}$ and $B_{p \times n}$ be two matrices such that the number of columns in A equals the number of rows in B.

  The product of A and B, written AB, is a matrix C of dimension $m \times n$ where m is the number of rows in A and n is the number of columns in B.

  To determine an element c_{ij} of matrix C, we proceed as follows:

  1. we multiply, in order, element by element, each element of the ith row of A with each element of the jth column of B.
  2. we compute the sum of these products.

  Ex.: $A_{4\times 3} \times B_{3\times 2} = C_{4\times 2}$

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \\ \boxed{1 \quad 0 \quad 2} \\ 4 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & \boxed{4} \\ 1 & \boxed{3} \\ 3 & \boxed{1} \end{bmatrix} = \begin{bmatrix} 10 & 18 \\ 11 & 19 \\ 8 & \boxed{6} \\ 15 & 21 \end{bmatrix}$$

  Element c_{32} of matrix C is obtained by multiplying the 3rd row of A with the 2nd column of B.

$$c_{32} = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = 1 \times 4 + 0 \times 3 + 2 \times 1 = 6$$

- Note that the matrix product AB can be determined when the number of columns in A is equal to the number of rows in B.

3. Consider the matrices $A_{2\times 3}$, $B_{4\times 2}$, $C_{3\times 4}$ and $D_{2\times 2}$. Determine, when possible, the dimension of the following matrices.

a) AC ***2 × 4*** b) AB ***impossible*** c) DA ***2 × 3*** d) CB ***3 × 2***

e) AD ***impossible*** f) BD ***4 × 2*** g) (CB)A ***3 × 3*** h) (DA)(CB) ***2 × 2***

4. Determine, if possible, the product AB.

a) $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$ $AB = \begin{bmatrix} 11 & 16 \\ 4 & 5 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ $AB = \begin{bmatrix} 1 & 12 \\ 1 & -2 \end{bmatrix}$

c) $A = \begin{bmatrix} 3 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}$ $AB = \begin{bmatrix} 8 & -7 \\ 0 & 9 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -2 & 1 \\ 3 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 4 & 1 & 2 \end{bmatrix}$ $AB = \begin{bmatrix} 2 & 10 & 1 & 4 \\ 2 & 6 & -5 & 0 \\ 7 & -1 & -3 & -3 \end{bmatrix}$

e) $A = \begin{bmatrix} -2 & 1 & 3 \\ 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$ $AB = \begin{bmatrix} -5 & -2 \\ 5 & 12 \\ -2 & 3 \end{bmatrix}$

5. Consider $A = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

a) Find

1. $AB = \begin{bmatrix} -1 & -3 \\ 4 & 10 \end{bmatrix}$ 2. $BA = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$

b) Is matrix multiplication commutative? ***No***

6. Soit $A_{2\times2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a) Calcule

1. $AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ 2. $IA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

b) Is it true that the identity matrix $I_{2\times2}$ is the neutral element of the set $\mathcal{M}$ of 2×2 matrices with the matrix multiplication operation, in other words is it true that for any matrix A in $\mathcal{M}$, we have: $AI = A$ and $IA = A$?

Yes

c) What is the neutral element of the set $\mathcal{M}$ of 3×3 matrices with the matrix multiplication operation?

The identity matrix $I_{3\times3}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. Consider matrices $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$.

Verify that multiplication is distributive over addition, i.e.: $A(B + C) = AB + AC$.

1. $A(B + C) = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 17 & 22 \end{bmatrix}$

2. $AB + AC = \begin{bmatrix} 2 & 4 \\ 8 & 18 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 17 & 22 \end{bmatrix}$

We have: $A(B + C) = AB + AC$

Activity 4 Geometric transformations in the Cartesian plane and matrix product

a) Consider the matrix product: $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

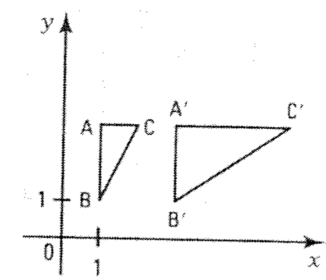
We interpret this matrix product as follows.

– The column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ represents the coordinates of a point M(x, y).

– The column matrix $\begin{bmatrix} x' \\ y' \end{bmatrix}$ represents the coordinates of the image point M$'(x', y')$.

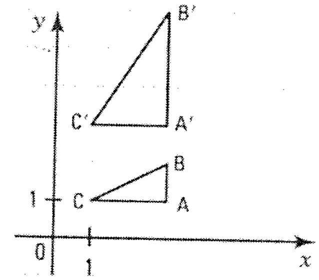
– The square matrix $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ transforms point M into point M$'$.

1. Using this matrix product, calculate the coordinates of points A$'$, B$'$ et C$'$, respective images of the vertices of triangle ABC.
   A'(3, 3); B'(3, 1); C'(6, 3)
2. Draw the image triangle A$'$B$'$C$'$.
3. This matrix product corresponds to a geometric transformation. Define this transformation.
   Horizontal scaling: (x, y) (3x, y).



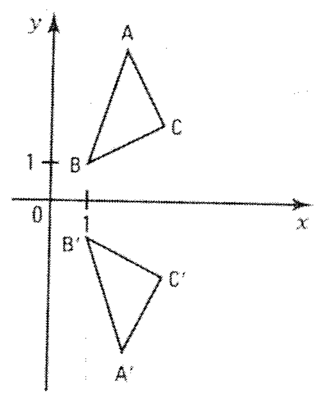
b) Consider the matrix product: $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

1. Using this matrix product, determine and then draw the triangle A$'$B$'$C$'$ image of triangle ABC. ***A'(3, 3); B'(3, 6); C'(1, 3)***
2. Define the geometric transformation which corresponds to this matrix product. ***Vertical scaling: (x, y) (x, 3y)***



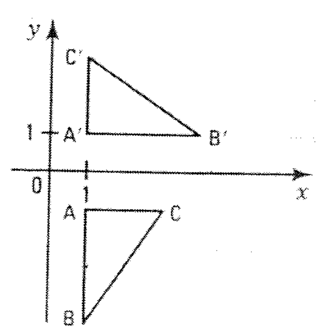
c) Consider the matrix product: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

1. Using this matrix product, determine and then draw the triangle A$'$B$'$C$'$ image of triangle ABC.
   A'(2, −4); B'(1, −1); C'(3, −2)
2. Define the geometric transformation which corresponds to this matrix product.
   The reflection about the x-axis: (x, y) (x, −y).



d) Consider the matrix product: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

1. Using this matrix product, determine and then draw the triangle A$'$B$'$C$'$ image of triangle ABC.
   A'(1, 1); B'(4, 1); C'(1, 3)
2. Define the geometric transformation which corresponds to this matrix product.
   The counterclockwise rotation of angle 90° centred at 0: (x, y) (−y, x).



GEOMETRIC TRANSFORMATIONS AND MATRIX PRODUCT

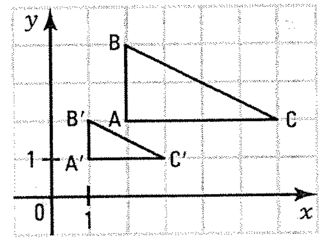
The matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ defines the geometric transformation:

$$(x, y) \qquad (x', y') = (ax + by, cx + dy)$$

- The column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ represents the coordinates of a point M(x, y).
- The column matrix $\begin{bmatrix} x' \\ y' \end{bmatrix}$ represents the coordinates of the point M$'(x', y')$, image of point M under this transformation.
- The square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called transformation matrix.
- The table below indicates, for some transformations, the matrix product that defines it.

Transformations	Matrix product
Horizontal scaling of factor k. $(x, y) \quad (kx, y)$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$
Vertical scaling of factor k. $(x, y) \quad (x, ky)$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$
Reflection about the x-axis. $(x, y) \quad (x, -y)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$
Reflection about the y-axis. $(x, y) \quad (-x, y)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$
Reflection about the 1st bisector. $(x, y) \quad (y, x)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$
Reflection about the 2nd bisector. $(x, y) \quad (-y, -x)$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$
Dilatation centred at 0 with ratio k. $(x, y) \quad (kx, ky)$	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$
Counterclockwise rotation of angle 90° centred at 0. $(x, y) \quad (-y, x)$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

8. Consider triangle ABC with vertices A(2, 2), B(2, 4) and C(6, 2) and the following matrix product.



$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

a) Using this matrix product, determine and then draw the triangle A′B′C′, image of triangle ABC.

A′(1, 1); B′(1, 2); C′(3, 1)

b) Define the geometric transformation which corresponds to this matrix product.

The dilatation centred at 0 with ratio $\frac{1}{2}$: $(x, y) \mapsto \left(\frac{x}{2}, \frac{y}{2}\right)$

c) Consider the transformation matrix $A_{2 \times 2} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $B_{2 \times 3}$ the matrix where each column corresponds respectively to the coordinates of the vertices A, B and C of triangle ABC.

$B = \begin{bmatrix} 2 & 2 & 6 \\ 2 & 4 & 2 \end{bmatrix}$. Calculate and interpret the product AB.

$$A \cdot B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 & 6 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

The columns of matrix AB correspond respectively to the coordinates of the vertices of the image triangle A′B′C′.

9. The matrix product $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ defines a counterclockwise rotation of angle θ centred at 0.

a) Determine the matrix product when

1. $\theta = \frac{\pi}{2}$ $\quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

2. $\theta = \pi$ $\quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

3. $\theta = \frac{3\pi}{2}$ $\quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

4. $\theta = \frac{\pi}{6}$ $\quad \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

b) Consider triangle ABC with vertices A(2, 4), B(0, 2) and C(6, 0). Find the coordinates of the triangle A′B′C′, image of triangle ABC under the counterclockwise rotation centred at 0 of angle

1. $\theta = \frac{\pi}{2}$ ***A′(−4, 2); B′(−2, 0); C′(0, 6)***

2. $\theta = \pi$ ***A′(−2, −4); B′(0, −2); C′(−6, 0)***

3. $\theta = \frac{3\pi}{2}$ ***A′(4, −2); B′(2, 0); C′(0, −6)***

4. $\theta = \frac{\pi}{6}$ ***A′***$(\sqrt{3} - 2, 1 + 2\sqrt{3})$, ***B′***$(-1, \sqrt{3})$, ***C′***$(3\sqrt{3}, 3)$

Evaluation 12

1. Consider matrices $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$.

Determine

a) $A + B$ $\begin{bmatrix} 1 & 3 \\ -1 & 7 \end{bmatrix}$ **b)** $A - C$ $\begin{bmatrix} -2 & 4 \\ -4 & 4 \end{bmatrix}$ **c)** $-2A$ $\begin{bmatrix} -2 & -4 \\ 6 & -8 \end{bmatrix}$

d) $3A - 2B$ $\begin{bmatrix} 3 & 4 \\ -13 & 6 \end{bmatrix}$ **e)** $A \cdot B$ $\begin{bmatrix} 4 & 7 \\ 8 & 9 \end{bmatrix}$ **f)** $(A + B)\,C$ $\begin{bmatrix} 6 & -2 \\ 4 & 2 \end{bmatrix}$

2. Consider matrices $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$.

a) Verify that $AB = AC$.

$AB = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ $AC = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$

b) Is the following implication: $AB = AC \Rightarrow B = C$ true or false?

It is false.

3. Let $A = [\,1\ \ 2\ \ 3\,]$ and $B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. Calculate

a) $AB =$ $[\,8\,]$ **b)** $BA =$ $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

4. Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 6 & 2 \end{bmatrix}$.

a) Calculate

1. AB $\begin{bmatrix} 6 & 2 \\ 4 & 6 \end{bmatrix}$ 2. BA $\begin{bmatrix} 6 & 2 \\ 4 & 6 \end{bmatrix}$

b) Compare AB and BA. ***AB = BA***

c) Is it true that matrix multiplication is commutative? If not, give a counterexample.

No. Indeed, $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 11 & 16 \end{bmatrix}$ ***and*** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 4 & 18 \end{bmatrix}$

5. The matrix product $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ describes a geometric transformation.

a) Describe this transformation. ***Horizontal scaling of factor 2.***

b) Determine the matrix product corresponding to the inverse geometric transformation.

$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

6. A company owns three warehouses, one in Montreal, one in Toronto and one in Calgary. The warehouse in Montreal has an inventory of 820 units of product A and 650 units of product B, the warehouse in Toronto has 750 units of product A and 825 of product B and the warehouse in Calgary has 950 units of product A and 720 of product B. The prices per unit for the Montreal warehouse are \$20 for product A and \$25 for product B. For the Toronto warehouse, the prices per unit are \$19 for product A and \$26 for product B while for the Calgary warehouse, the prices are \$18 for product A and \$24 for product B.

a) Let Q denote the matrix of dimension 2×3 describing the inventory of each warehouse. Determine matrix Q.

	M	***T***	***C***
A	***820***	***750***	***950***
B	***650***	***825***	***720***

b) Let P denote the matrix of dimension 3×2 describing for each warehouse the price per unit. Determine matrix P.

	A	***B***
M	***20***	***25***
T	***19***	***26***
C	***18***	***24***

c) Let R denote the row matrix indicating for each warehouse the revenue generated by the sale of all units. Determine matrix R.

M	***T***	***C***
32 650	***35 700***	***34 380***

7. A study was performed on a sample of 100 individuals. Matrix A below indicates the distribution of individuals according to gender and the category adult or child.

The quantities (in grams) of proteins, lipids and carbohydrates consumed daily per individual are given by matrix B below.

$$A = \begin{array}{c} \\ \textit{Male} \\ \textit{Fem} \end{array}\begin{array}{c} \textit{Adult} \quad \textit{Child} \\ \begin{bmatrix} 30 & 25 \\ 25 & 20 \end{bmatrix}\end{array}; \quad B = \begin{array}{c} \\ \textit{Adult} \\ \textit{Child} \end{array}\begin{array}{c} \textit{Protein} \quad \textit{Lipids} \quad \textit{Carbohydr.} \\ \begin{bmatrix} 15 & 20 & 25 \\ 20 & 15 & 30 \end{bmatrix}\end{array}$$

a) Calculate matrix C if C = AB.

	Protein	***Lipids***	***Carbohydr.***
Male	***950***	***975***	***1500***
Fem	***775***	***800***	***1225***

b) Interpret

1. C_{12} ***Male individuals in the sample consume a total of 975 g of lipids.***
2. C_{21} ***Female individuals in the sample consume a total of 775 g of proteins.***