Chapter 3

Transformations in the Cartesian plane

CHALLENGE 3

1. Find the rule of the transformation in the Cartesian plane that performs, on a figure,

a) a horizontal scaling of factor 2. ***$(x, y) \rightarrow (2x, y)$***

b) a vertical scaling of factor $\frac{1}{3}$. ***$(x, y) \rightarrow \left(x, \frac{y}{3}\right)$***

c) a horizontal translation of 3 units towards the right. ***$(x, y) \rightarrow (x + 3, y)$***

d) a vertical translation of 2 units downwards. ***$(x, y) \rightarrow (x, y - 2)$***

e) a rotation of 180° about the origin 0. ***$(x, y) \rightarrow (-x, -y)$***

f) a dilatation centred at the origin 0 with ratio $\frac{3}{2}$. ***$(x, y) \rightarrow \left(\frac{3}{2}x, \frac{3}{2}y\right)$***

g) a reflection about the line $y = x$. ***$(x, y) \rightarrow (y, x)$***

2. True or false?

a) A rotation transforms a line into a parallel line. ***False***

b) A reflection preserves orientation of figures. ***False***

c) A scaling preserves parallelism. ***False***

d) A scaling transforms a figure into a similar figure. ***False***

e) A dilatation transforms a figure into a similar figure. ***True***

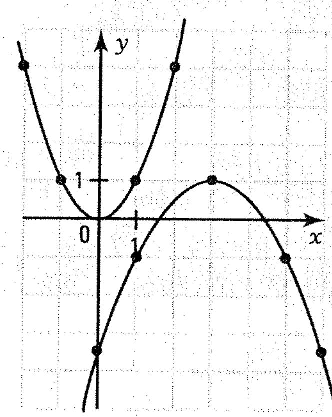
3. We apply the following sequence of transformations to the line $y = x$ and its successive images.

1. A vertical scaling of factor $\frac{1}{2}$.
2. A reflection about the x-axis.
3. A horizontal translation of 3 units towards the right.
4. A vertical translation of 2 units upwards.

What is the equation of the line representing the 4th image? ***$y = -\frac{1}{2}x + \frac{7}{2}$***

4. a) Define a sequence of transformations necessary to draw the graph of the function $y = -\frac{1}{2}(x - 3)^2 + 1$ starting with the graph of the basic quadratic function $y = x^2$.

1. ***The reflection about the x-axis:*** ***$(x, y) \rightarrow (x, -y)$***
2. ***The vertical scaling:*** ***$(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$***
3. ***The horizontal translation:*** ***$(x, y) \rightarrow (x + 3, y)$***
4. ***The vertical translation:*** ***$(x, y) \rightarrow (x, y + 1)$***



b) Draw the graph of $y = -\frac{1}{2}(x - 3)^2 + 1$ and indicate the coordinates of the vertex.

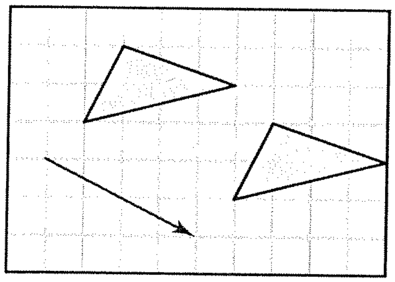
S(3, 1)

3.1 Translation

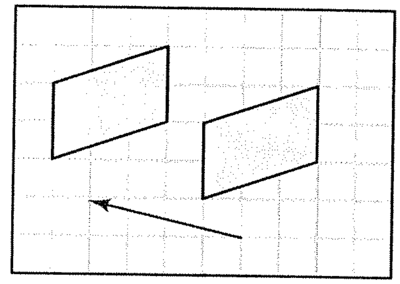
ACTIVITY 1 Translation

a) In each of the following cases, draw the image of each figure under the given translation.

1.



2.



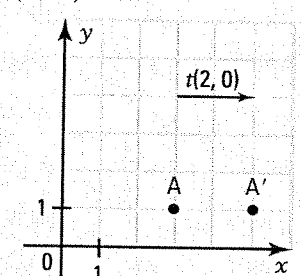
TRANSLATION IN THE CARTESIAN PLANE

- A **horizontal translation** t is defined by the couple $(h, 0)$. It is written: $t(h, 0)$.
  - If $h > 0$, the translation is performed towards the right.
  - If $h < 0$, the translation is performed towards the left.

  The form of the **rule** of a horizontal translation is:

  $$t: (x, y) \rightarrow (x + h, y)$$

  Ex.: Let $t: (x, y) \rightarrow (x + 2, y)$.
  The image of point A(3, 1) is point A′(5, 1).

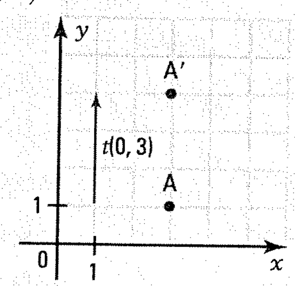


- A **vertical translation** t is defined by the couple $(0, k)$. It is written: $t(0, k)$.
  - If $k > 0$, the translation is performed upwards.
  - If $k < 0$, the translation is performed downwards.

  The form of the **rule** of a vertical translation is:

  $$t: (x, y) \rightarrow (x, y + k)$$

  Ex.: Let $t: (x, y) \rightarrow (x, y + 3)$.
  The image of point A(3, 1) is point A′(3, 4).

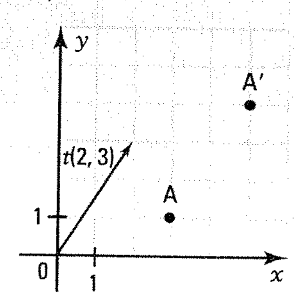


- An **oblique translation** t is defined by the couple (h, k). It is written: $t(h, k)$.

  The form of the **rule** of an oblique translation is:

  $$t: (x, y) \rightarrow (x + h, y + k)$$

  Ex.: Let $t: (x, y) \rightarrow (x + 2, y + 3)$.
  The image of point A(3, 1) is point A′(5, 4).



1. In each of the following cases, indicate the direction of motion (up, down, right, left) then determine the coordinates of point A', image of point A under the given translation.

a) $t: (x, y) \rightarrow (x, y - 2)$
$A(1, -3) \rightarrow$ ***A'(1, 6)***

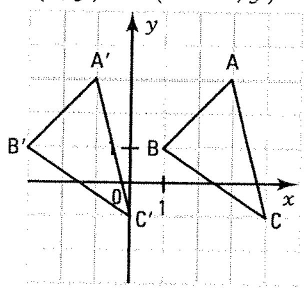
b) $t: (x, y) \rightarrow (x + 3, y)$
$A(2, -1) \rightarrow$ ***A'(-4, 0)***

c) $t: (x, y) \rightarrow (x - 2, y + 3)$
$A(-1, -2) \rightarrow$ ***A'(2, -5)***

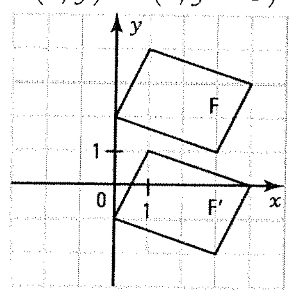
d) $t: (x, y) \rightarrow (x + 1, y - 2)$
$A(2, 3) \rightarrow$ ***A'(4, 0)***

2. Draw the image of each figure under the translation defined by the given rule.

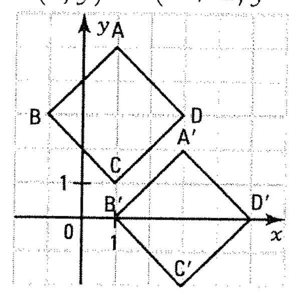
a) $t: (x, y) \rightarrow (x - 4, y)$



b) $t: (x, y) \rightarrow (x, y - 3)$



c) $t: (x, y) \rightarrow (x + 2, y - 3)$



3. Give the rule of the translation which associates point A with point A'.

a) $(2, 5) \rightarrow (2, -1)$ ***(x, y) → (x, y − 6)***

b) $(-2, 1) \rightarrow (1, 1)$ ***(x, y) → (x + 3, y)***

c) $(2, -3) \rightarrow (0, 2)$ ***(x, y) → (x − 2, y + 5)***

d) $(-2, 1) \rightarrow (4, -3)$ ***(x, y) → (x + 6, y − 4)***

4. What is the rule of the translation which associates:

a) figure A with figure B?
(x, y) → (x + 5, y)

b) ligure A with figure C?
(x, y) → (x, y − 6)

c) figure A with figure D?
(x, y) → (x + 5, y − 6)

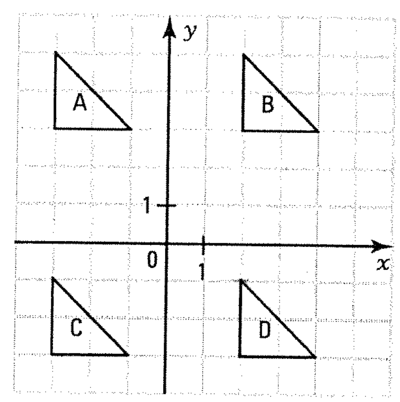
d) figure B with figure A?
(x, y) → (x − 5, y)

e) figure B with figure C?
(x, y) → (x − 5, y − 6)

f) figure B with figure D?
(x, y) → (x, y − 6)

g) figure C with figure A?
(x, y) → (x, y + 6)

h) figure C with figure B?
(x, y) → (x + 5, y + 6)

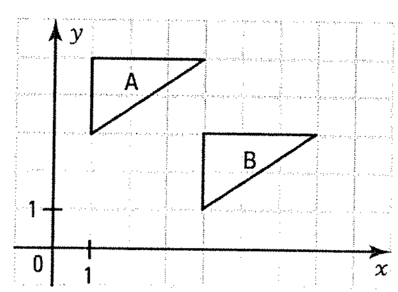


ACTIVITY 2 Inverse translation

a) What is the rule of the translation t which associates figure A with figure B? ***t: (x, y) → (x + 3, y − 2)***

b) 1. What is the rule of the inverse transformation which associates figure B with figure A?
t: (x, y) → (x − 3, y + 2)

2. Is the inverse transformation a translation? ***Yes***

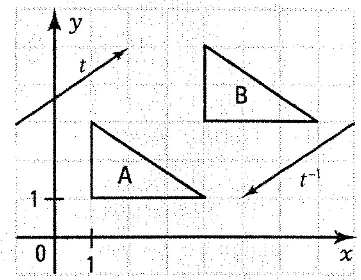


INVERSE TRANSLATION

The inverse of the translation t is a translation written t^{-1}.

If $t: (x, y) \rightarrow (x + a, y + b)$ then $t^{-1}: (x, y) \rightarrow (x - a, y - b)$

Ex.: $t: (x, y) \rightarrow (x + 3, y + 2)$.
$t^{-1}: (x, y) \rightarrow (x - 3, y - 2)$
The image of figure A under translation t is figure B.
The image of figure B under translation t^{-1} is figure A.



5. Define the inverse of the following translations.

a) $t: (x, y) \rightarrow (x + 1, y + 3)$
$t^{-1}: (x, y) \rightarrow (x - 1, y - 3)$

b) $t: (x, y) \rightarrow (x - 1, y - 2)$
$t^{-1}: (x, y) \rightarrow (x + 1, y + 2)$

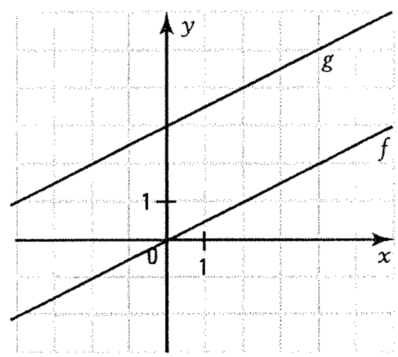
c) $t: (x, y) \rightarrow (x + 3, y - 2)$
$t^{-1}: (x, y) \rightarrow (x - 3, y + 2)$

d) $t: (x, y) \rightarrow (x - 5, y + 4)$
$t^{-1}: (x, y) \rightarrow (x + 5, y - 4)$

ACTIVITY 3 Translation of the graph of a function

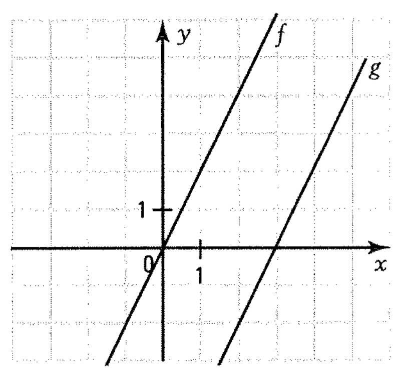
a) Consider on the right the linear function f with rule $y = 0.5x$.

1. Draw the graph of the function g, image of the graph of function f, under the vertical translation: $(x, y) \rightarrow (x, y + 3)$.
2. What is the rule of function g? ***$y = 0,5x + 3$***
3. The graph of a linear function f, with rule $y = ax$, undergoes the vertical translation: $(x, y) \rightarrow (x, y + k)$. The image graph we obtain corresponds to that of a function g. What is the rule of g? ***$y = ax + k$***



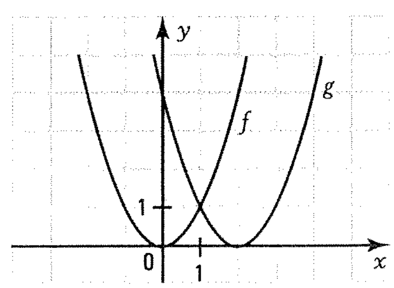
b) Consider on the right the linear function f with rule $y = 2x$.

1. Draw the graph of the function g, image of the graph of function f, under the horizontal translation: $(x, y) \rightarrow (x + 3, y)$.
2. What is the rule of function g? ***$y = 2(x - 3)$***
3. The graph of a linear function f, with rule $y = ax$, undergoes the horizontal translation: $(x, y) \rightarrow (x + h, y)$. The image graph we obtain corresponds to that of a function g. What is the rule of g? ***$y = a(x - h)$***



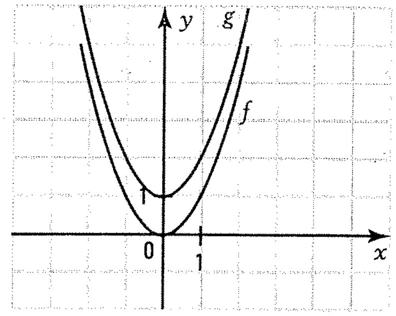
c) Consider on the right the basic quadratic function f with rule $y = x^2$.

1. Draw the graph of the function g, image of the graph of function f, under the horizontal translation: $(x, y) \rightarrow (x + 2, y)$.
2. What is the rule of function g? ***$y = (x - 2)^2$***
3. What is the rule of the function g, image of the graph of function f under the horizontal translation: $(x, y) \rightarrow (x + h, y)$?
$y = (x - h)^2$



d) Consider on the right the basic quadratic function f with rule $y = x^2$.

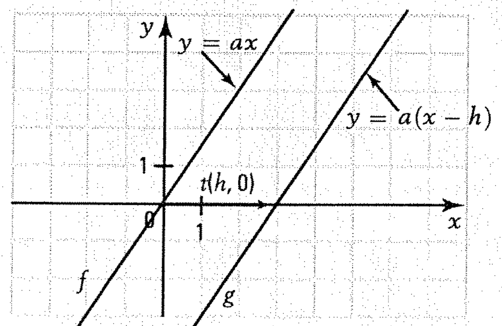
1. Draw the graph of the function g, image of the graph of function f, under the vertical translation: $(x, y) \rightarrow (x, y + 1)$.
2. What is the rule of function g? ***$y = x^2 + 1$***
3. What is the rule of the function g, image of the graph of function f under the vertical translation: $(x, y) \rightarrow (x, y + k)$? ***$y = x^2 + k$***

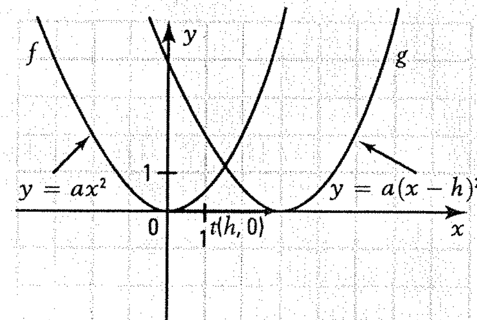


TRANSLATION OF THE GRAPH OF A FUNCTION

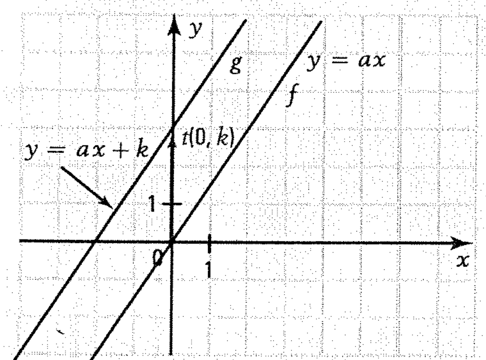
In each of the following cases, the graph of a function f undergoes a translation t giving us the graph of a function g.

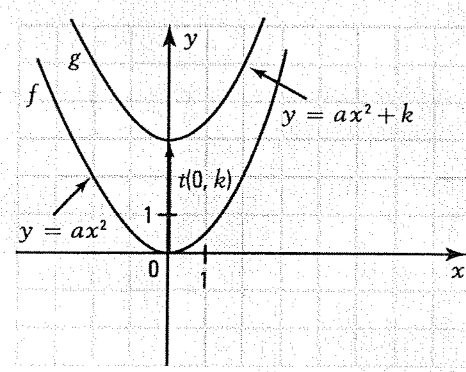
- Horizontal translation t: $(x, y) \rightarrow (x + h, y)$





- Vertical translation t: $(x, y) \rightarrow (x, y + k)$





6. Consider the function f having rule $y = 2x$. In each of the following cases, the graph of function f undergoes a translation t giving us the graph of a function g. Find the rule of g.

a) t: $(x, y) \rightarrow (x, y + 5)$ ***$y = 2x + 5$***

b) t: $(x, y) \rightarrow (x, y - 3)$ ***$y = 2x - 3$***

c) t: $(x, y) \rightarrow (x + 2, y)$ ***$y = 2x - 4$***

d) t: $(x, y) \rightarrow (x - 3, y)$ ***$y = 2x + 6$***

7. Consider the function f having rule $y = 3x^2$. In each of the following cases, the graph of function f undergoes a translation t giving us the graph of a function g. Find the rule of g.

a) t: $(x, y) \rightarrow (x, y + 3)$ ***$y = 3x^2 + 3$***

b) t: $(x, y) \rightarrow (x, y - 1)$ ***$y = 3x^2 - 1$***

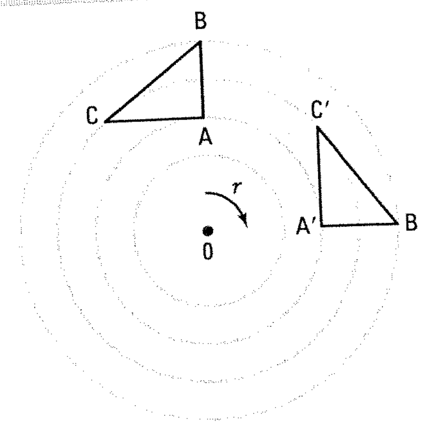
c) t: $(x, y) \rightarrow (x + 2, y)$ ***$y = 3(x - 2)^2$***

d) t: $(x, y) \rightarrow (x - 1, y)$ ***$y = 3(x + 1)^2$***

43.2 Rotation

ACTIVITY 1 Rotation

Circles with common centre 0 are represented on the right. The rotation arrow r centred at 0 indicates the direction and the angle of the rotation.



a) Is the direction of rotation clockwise or counterclockwise?
Clockwise

b) What is the angle of rotation? $\alpha = 90°$

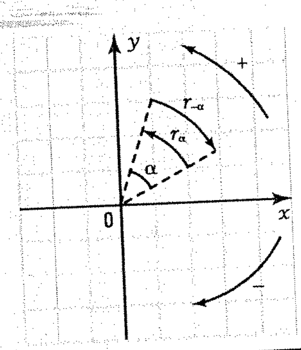
c) Draw the triangle A′B′C′ image of triangle ABC under rotation r.

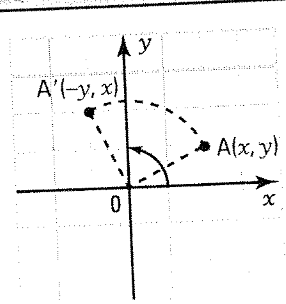
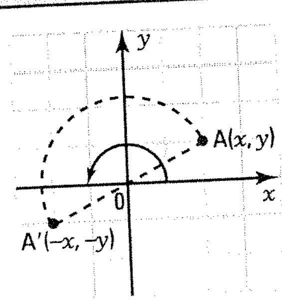
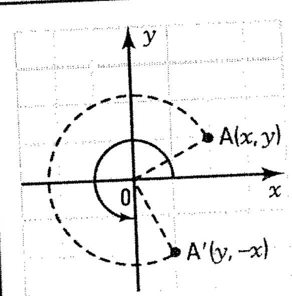
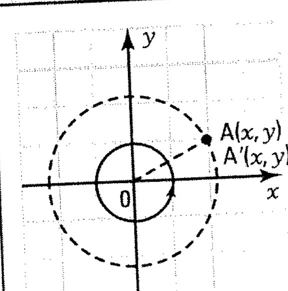
d) Verify that
1. $m\angle A0A' = \alpha$ and $m\overline{0A'} = m\overline{0A}$.
2. $m\angle B0B' = \alpha$ and $m\overline{0B'} = m\overline{0B}$.
3. $m\angle C0C' = \alpha$ and $m\overline{0C'} = m\overline{0C}$.

e) Consider a rotation r of angle α centred at 0. If M′ is the image of point M under rotation r, complete the following properties of the rotation.
1. $m < M0M' =$ α 2. $m\overline{0M'}$ = $m\overline{0M}$

ROTATION CENTRED AT THE ORIGIN

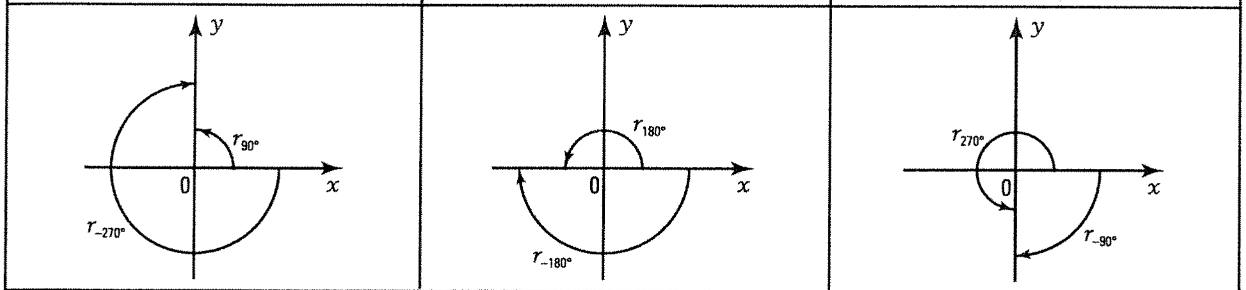
- A rotation centred at 0 (origin of the Cartesian plane) of angle α in the counterclockwise direction (positive direction) is written $r(0, \alpha)$ or r_α.
  A rotation centred at 0 with angle α in the clockwise direction (negative direction) is written $r(0, -\alpha)$ or $r_{-\alpha}$.
- The table below gives, for each rotation with centre 0 and angle α (α is a multiple of 90°), the image A′ of a point A(x, y) and the rule of the rotation.



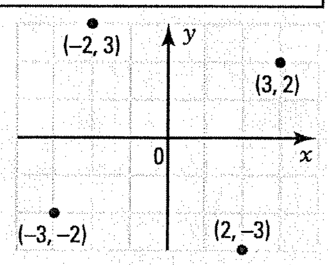
$r_{90°}$	$r_{180°}$	$r_{270°}$	$r_{360°}$
			
$r_{90°}: (x, y) \rightarrow (-y, x)$	$r_{180°}: (x, y) \rightarrow (-x, -y)$	$r_{270°}: (x, y) \rightarrow (y, -x)$	$r_{360°}: (x, y) \rightarrow (x, y)$

- We have the following equivalences:

$r_{90°} = r_{-270°}$	$r_{180°} = r_{-180°}$	$r_{270°} = r_{-90°}$



Ex.: The image of point (3, 2) is:
(–2, 3) under $r_{90°}$; (–3, –2) under $r_{180°}$; (2, –3) under $r_{270°}$;
(3, 2) under $r_{360°}$; (2, –3) under $r_{-90°}$; (–3, –2) under $r_{-180°}$;
(–2, 3) under $r_{-270°}$; (3, 2) under $r_{-360°}$.

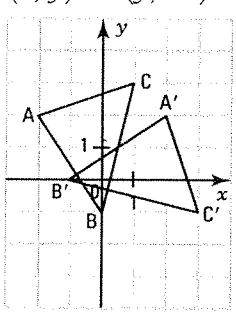


1. Complete the following table. All rotations are centred at the origin of the Cartesian plane.

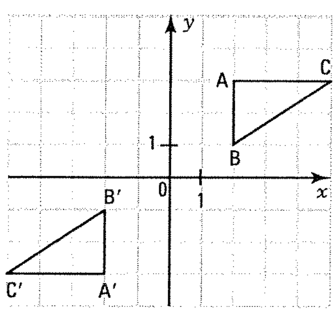
Notation	Description
$r_{270°}$	***270° counterclockwise rotation.***
$r_{-90°}$	90° clockwise rotation.
$r_{90°}$	***90° counterclockwise rotation.***
$r_{180°}$	180° counterclockwise rotation.
$r_{-270°}$	270° clockwise rotation.

2. Draw the image of each figure under the given rotation.

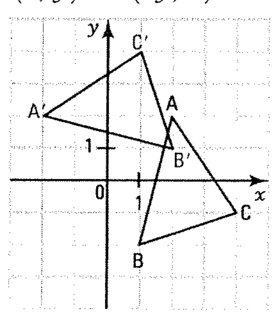
a) $(x, y) \rightarrow (y, -x)$



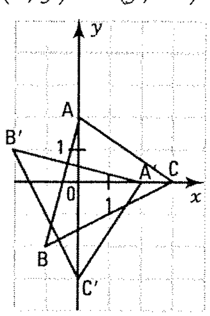
b) $(x, y) \rightarrow (-x, -y)$



c) $(x, y) \rightarrow (-y, x)$



d) $(x, y) \rightarrow (y, -x)$



3. Complete the following table by giving the coordinates of the image point.

	$r_{90°}$	$r_{180°}$	$r_{270°}$	$r_{360°}$	$r_{-90°}$	$r_{-180°}$	$r_{-270°}$	$r_{-360°}$
(5, 2)	***(–2, 5)***	***(–5, –2)***	***(2, –5)***	***(5, 2)***	***(2, –5)***	***(–5, –2)***	***(–2, 5)***	***(5, 2)***
(–2, 3)	***(–3, –2)***	***(2, –3)***	***(3, 2)***	***(–2, 3)***	***(3, 2)***	***(2, –3)***	***(–3, –2)***	***(–2, 3)***
(–3, –1)	***(1, –3)***	***(3, 1)***	***(–1, 3)***	***(–3, –1)***	***(–1, 3)***	***(3, 1)***	***(1, –3)***	***(–3, –1)***
(3, –2)	***(2, –3)***	***(–3, 2)***	***(–2, –3)***	***(3, –2)***	***(–2, –3)***	***(–3, 2)***	***(2, –3)***	***(3, –2)***

4. In each of the following cases, indicate the rotation ($r_{90°}$, $r_{180°}$, $r_{270°}$, $r_{360°}$) which associates point A with its image A′.

a) A(2, 1) and A′(−2, −1) ***$r_{180°}$***

b) (3, 6) and A′(−6, 3) ***$r_{90°}$***

c) A(2, −1) and A′(−1, −2) ***$r_{270°}$***

d) (2, −5) and A′(5, 2) ***$r_{90°}$***

e) A(2, 0) and A′(0, −2) ***$r_{270°}$***

f) (−1, 2) and A′(1, −2) ***$r_{180°}$***

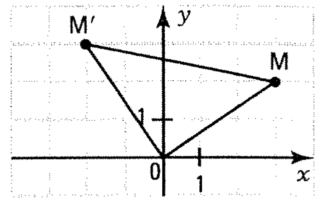
g) A(a, b) and A′($-b$, a) ***$r_{90°}$***

h) (a, b) and A′($-a$, $-b$) ***$r_{180°}$***

i) A(a, b) and A′(a, b) ***$r_{360°}$***

j) (a, b) and A′(b, $-a$) ***$r_{270°}$***

5. Consider on the right the point M(3, 2) and the rotation $r_{90°}$: $(x, y) \rightarrow (-y, x)$.



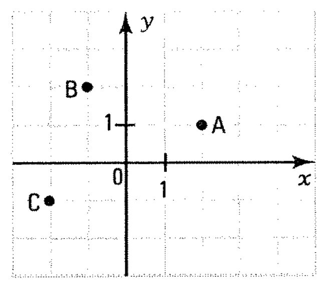
a) Determine the coordinates of the point M′ image of point M under the rotation $r_{90°}$. ***(−2, 3)***

b) Show that $m\overline{0M'} = m\overline{0M}$. ***$m\overline{0M'} = \sqrt{13}$ $m\overline{0M} = \sqrt{13}$***

c) Show that angle M0M′ is equal to the angle of rotation. ***It is sufficient to show, using Pythagoras' relation, that the △MOM′ is right at 0.***

$m\overline{MM'} = \sqrt{26}$; $(m\overline{MM'})^2 = (m\overline{0M})^2 + (m\overline{0M'})^2$

ACTIVITY 2 Inverse rotation



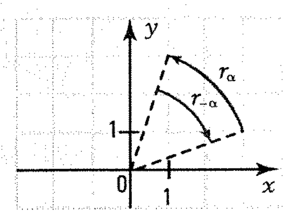
a) The rotation $r_{90°}$ centred at 0 associates point A with point B. What is the inverse rotation which associates point B with point A? ***$r_{-90°}$***

b) The rotation $r_{180°}$ centred at 0 associates point A with point C. What is the inverse rotation which associates point C with point A? ***$r_{-180°}$***

c) Complete: The inverse of the rotation centred at 0 of angle α is the rotation centred at ***0*** of angle ***−α***.

INVERSE ROTATION

- The inverse of the rotation r centred at 0 of angle α is the rotation, written r_{α}^{-1}, centred at 0 of angle −α.



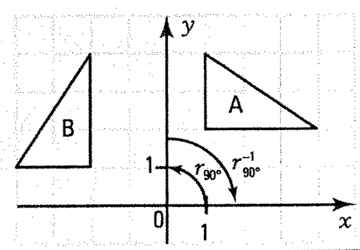
$$r_{\alpha}^{-1} = r_{-\alpha}$$

Ex.: $r_{90°}$: $(x, y) \rightarrow (-y, x)$.

$r_{90°}^{-1}$: $(x, y) \rightarrow (y, -x)$.

The image of figure A under rotation $r_{90°}$ is figure B.

The image of figure B under the inverse rotation $r_{90°}^{-1}$ is figure A.



6. Define the inverse of the following rotations.

a) r: $(x, y) \rightarrow (y, -x)$

r^{-1}: $(x, y) \rightarrow (-y, x)$

b) r: $(x, y) \rightarrow (-x, -y)$

r^{-1}: $(x, y) \rightarrow (-x, -y)$

3.3 Reflection

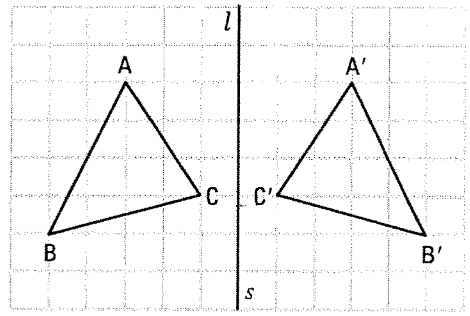
ACTIVITY 1 Reflection

Consider the reflection s whose axis of reflection is the line l.

a) Draw the triangle A′B′C′ image of triangle ABC under reflection s.

b) What does the axis of reflection represent for the line segments AA′, BB′ and CC′?

Line l is the right bisector of each of the line segments.

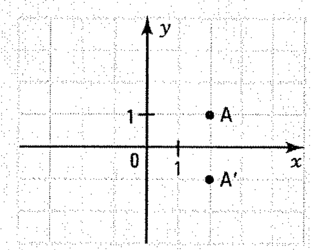


REFLECTION IN THE CARTESIAN PLANE

- The rule of the reflection about the **x-axis**, written s_x, is:

$$s_x : (x, y) \rightarrow (x, -y)$$

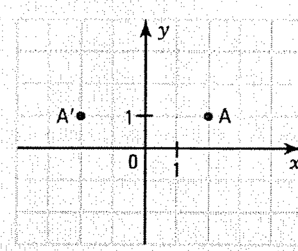
Ex.: The image of point A(2, 1) under s_x is point A′(2, –1).
Points A and A′ are symmetric about the x-axis.



- The rule of the reflection about the **y-axis**, written s_y, is:

$$s_y : (x, y) \rightarrow (-x, y)$$

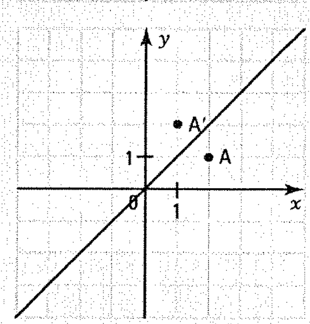
Ex.: The image of point A(2, 1) under s_y is point A′(–2, 1).
Points A and A′ are symmetric about the y-axis.



- The rule of the reflection about the **first bisector** (bisector of quadrants 1 and 3), written $s_{\boxslash}$, is:

$$s_{\boxslash} : (x, y) \rightarrow (y, x)$$

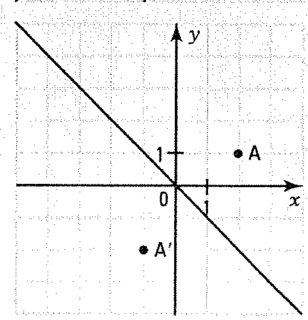
Ex.: The image of point A(2, 1) under $s_{\boxslash}$ is point A′(1, 2).
Points A and A′ are symmetric about the 1st bisector.



- The rule of the reflection about the **second bisector** (bisector of quadrants 2 and 4), written $s_{\boxbslash}$, is:

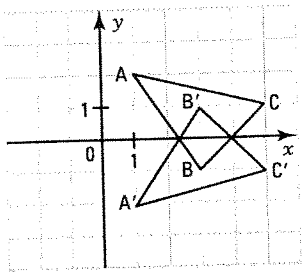
$$s_{\boxbslash} : (x, y) \rightarrow (-y, -x)$$

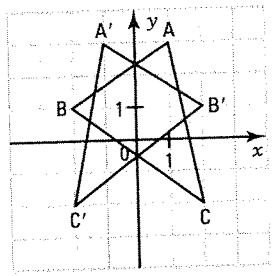
Ex.: The image of point A(2, 1) under $s_{\boxbslash}$ is point A′(–1, –2).
Points A and A′ are symmetric about the 2nd bisector.

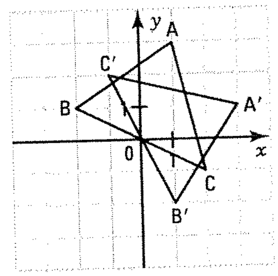


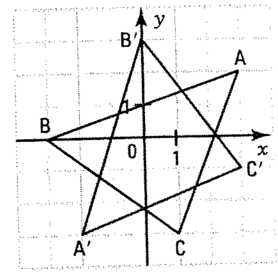
1. Draw the image of each figure under the given reflection.

a) $(x, y) \rightarrow (x, -y)$ **b)** $(x, y) \rightarrow (-x, y)$ **c)** $(x, y) \rightarrow (y, x)$ **d)** $(x, y) \rightarrow (-y, -x)$









2. What is the image of the point (–2, 3) under the reflection about

a) the x-axis? ***(–2, –3)*** **b)** the y-axis? ***(2, 3)***

c) the 1st bisector? ***(3, –2)*** **d)** the 2nd bisector? ***(–3, 2)***

3. Define, with a rule, a reflection which associates

a) point A(–2, –1) with point A′(1, 2) ***(x, y) → (–y, –x)***

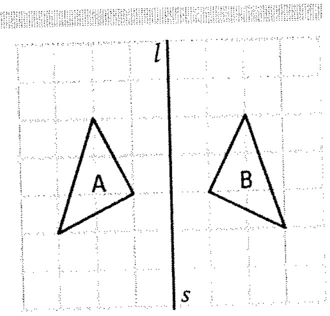
b) point A(1, –3) with point A′(–1, –3) ***(x, y) → (–x, y)***

c) point A(2, –1) with point A′(2, 1) ***(x, y) → (x, –y)***

d) point A(3, 2) with point A′(2, 3) ***(x, y) → (y, x)***

ACTIVITY 2 **Inverse reflection**

Consider the reflection s whose axis of reflection is the line l.



a) Find the image under reflection s of

1. figure A ***B*** 2. figure B ***A***

b) What can be said about the inverse s^{-1} of reflection s? ***$s^{-1} = s$***

INVERSE REFLECTION

- The inverse of reflection s is reflection s.

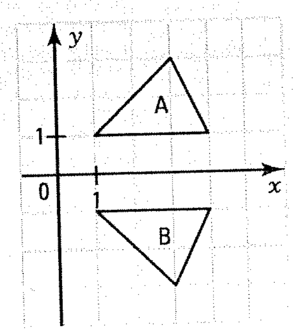
$$s^{-1} = s$$

Ex.: We have $s_x^{-1} = s_x$.

If s_x: $(x, y) \rightarrow (x, -y)$ then s_x^{-1}: $(x, y) \rightarrow (x, -y)$

The image of figure A under reflection s_x is figure B.

The image of figure B under reflection s_x is figure A.



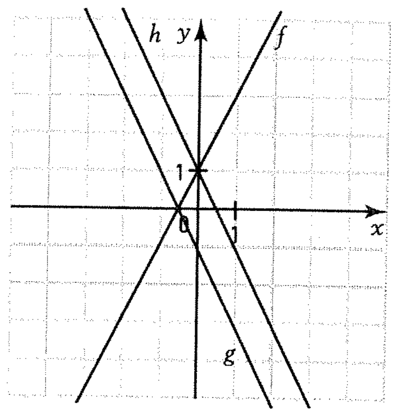
4. Find the inverse of the following reflections.

a) $s_x: (x, y) \to (-x, y)$ $s_y^{-1} = s_y$

b) $s_{\boxslash}: (x, y) \to (y, x)$ $s_{\boxslash}^{-1} = s_{\boxslash}$

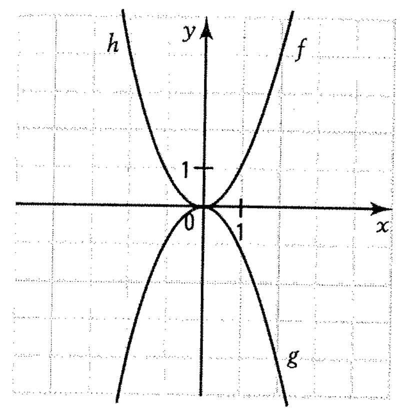
c) $s_{\boxbslash}: (x, y) \to (-y, -x)$ $s_{\boxbslash}^{-1} = s_{\boxbslash}$

ACTIVITY 3 Reflection of the graph of a function



a) Consider on the right the linear function f with rule $y = 2x + 1$.

1. Draw the graph of the function g, image of the graph of function f, under the reflection about the x-axis $s_x: (x, y) \to (x, -y)$.
2. What is the rule of function g? **$y = -2x - 1$**
3. Draw the graph of the function h, image of the graph of function f, under the reflection about the y-axis $s_y: (x, y) \to (-x, y)$.
4. What is the rule of function h? **$y = -2x + 1$**

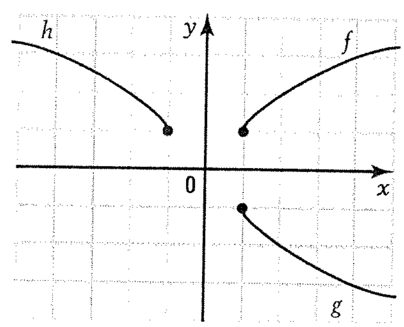


b) Consider on the right the basic quadratic function f with rule $y = x^2$.

1. Draw the graph of the function g, image of the graph of function f, under the reflection about the x-axis $s_x: (x, y) \to (x, -y)$.
2. What is the rule of function g? **$y = -x^2$**
3. Draw the graph of the function h, image of the graph of function f, under the reflection about the y-axis $s_y: (x, y) \to (-x, y)$.
4. Explain why the graphs of functions f and h coincide.
   The y-axis is an axis of symmetry for the parabola.
5. What is the rule of function h? **$y = x^2$**

c) Given the graph of a function $y = f(x)$. Refer to questions a) and b) to verify that

1. the reflection $s_x: (x, y) \to (x, -y)$ applied to the graph of f gives us the graph of a function g such that $g(x) = -f(x)$.
2. the reflection $s_y: (x, y) \to (-x, y)$ applied to the graph of function f gives us the graph of a function h such that $h(x) = f(-x)$.



REFLECTION OF THE GRAPH OF A FUNCTION

Given the graph of a function f,

- the reflection about the x-axis

  $s_x: (x, y) \rightarrow (x, -y)$

  gives us the graph of a function g such that

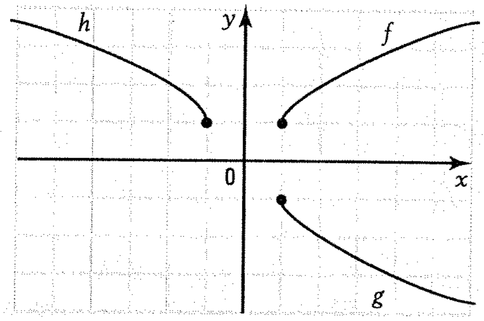
  $g(x) = -f(x)$

- the reflection about the y-axis

  $s_y: (x, y) \rightarrow (-x, y)$

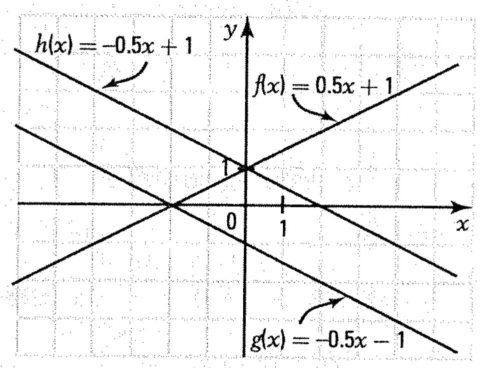
  gives us the graph of a function h such that

  $h(x) = f(-x)$



Ex.: Given the graph of the function f with rule $f(x) = 0.5x + 1$

- Reflection s_x applied to the graph of f gives us the graph of g with rule: $g(x) = -f(x) = -0.5x - 1$.
- Reflection s_y applied to the graph of f gives us the graph of h with rule: $h(x) = f(-x) = 0,5(-x) + 1 = -0,5x + 1$.



5. For each of the following function graphs, we apply a reflection about the x-axis, $s_x: (x, y) \rightarrow (x, -y)$. Find the rule of the function corresponding to the resulting graph.

a) $y = 5x$ ***$y = -5x$***
b) $y = 2x + 3$ ***$y = -2x - 3$***
c) $y = -2x + 5$ ***$y = 2x - 5$***
d) $y = 3x^2$ ***$y = -3x^2$***
e) $y = -5x^2$ ***$y = 5x^2$***
f) $y = \sqrt{x}$ ***$y = -\sqrt{x}$***

6. For each of the following function graphs, we apply a reflection about the y-axis, $s_y: (x, y) \rightarrow (-x, y)$. Find the rule of the function corresponding to the resulting graph.

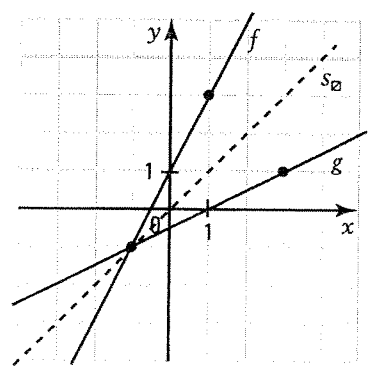
a) $y = 3x$ ***$y = -3x$***
b) $y = 3x + 2$ ***$y = -3x + 2$***
c) $y = -3x + 5$ ***$y = 3x + 5$***
d) $y = 5x^2$ ***$y = 5x^2$***
e) $y = -3x^2$ ***$y = -3x^2$***
f) $y = \sqrt{x}$ ***$y = \sqrt{-x}$***

7. Consider the graph of the function f with rule $y = 2x + 1$ and the reflection about the 1st bisector $s_⊘: (x, y) \rightarrow (y, x)$.

a) Draw the graph of the function g, image of the graph of f, under reflection $s_⊘$.

b) What is the rule of function g? ***$y = \frac{1}{2}x - \frac{1}{2}$***

c) What can be said of functions f and g?
They are inverse of each other.

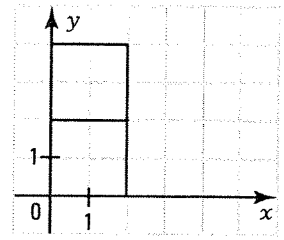


3.4 Scaling

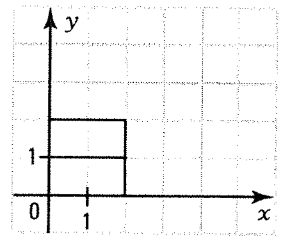
ACTIVITY 1 Impact of a scaling on a figure

a) We call vertical scaling of factor k the transformation, written e_v, with rule:

$$e_v(k): (x, y) \rightarrow (x, ky)$$



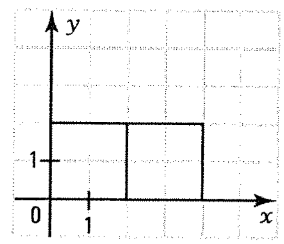
1. Draw the image of the square on the right under the vertical scaling: $(x, y) \rightarrow (x, 2y)$ then indicate if we observe a stretching or a reduction of the initial figure.
   We observe a vertical stretching by a scale factor of 2.



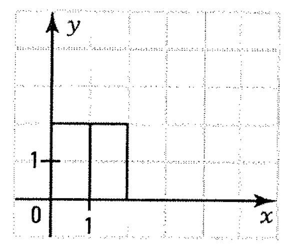
2. Draw the image of the square on the right under the vertical scaling: $(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$ then indicate if we observe a stretching or a reduction of the initial figure.
   We observe a vertical reduction by a scale factor of $\frac{1}{2}$.

b) We call horizontal scaling of factor k the transformation, written e_h, with rule:

$$e_h(k): (x, y) \rightarrow (kx, y)$$



1. Draw the image of the square on the right under the horizontal scaling: $(x, y) \rightarrow (2x, y)$ then indicate if we observe a stretching or a reduction of the initial figure.
   We observe a horizontal stretching by a scale factor of 2.



2. Draw the image of the square on the right under the horizontal scaling: $(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$ then indicate if we observe a stretching or a reduction of the initial figure.
   We observe a horizontal reduction by a scale factor of $\frac{1}{2}$.

c) Consider the vertical or horizontal scaling of factor k. Do we observe a stretching or a reduction of the initial figure if

1. $k > 1$ ***a stretching*** 2. $0 < k < 1$. ***a reduction.***

SCALING IN THE CARTESIAN PLANE

- The vertical scaling e_v of factor k: $(x, y) \rightarrow (x, ky)$ causes
  - a vertical stretching when $k > 1$.
  - a vertical reduction when $0 < k < 1$.

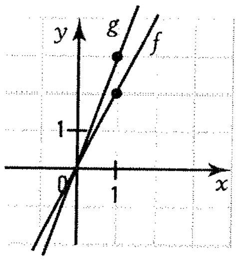
  Ex.: See activity 1 a).
- The horizontal scaling e_h of factor k: $(x, y) \rightarrow (kx, y)$ causes
  - a horizontal stretching when $k > 1$.
  - a horizontal reduction when $0 < k < 1$.

  Ex.: See activity 1 b).

1. In each of the following cases, we apply the indicated scaling to the graph of the function $f(x) = 2x$.

1. Draw the graph of the function g, image of the graph of function f.
2. Find the rule of function g.

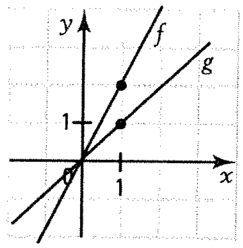
a)



$(x, y) \rightarrow \left(x, \frac{3}{2}y\right)$

g(x) = 3x

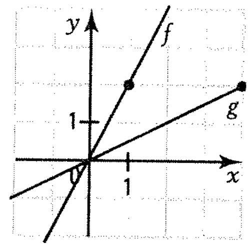
b)



$(x, y) \rightarrow (x, 0{,}5y)$

g(x) = x

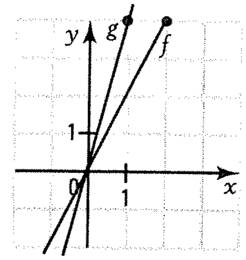
c)



$(x, y) \rightarrow (4x, y)$

g(x) = $\frac{1}{2}x$

d)



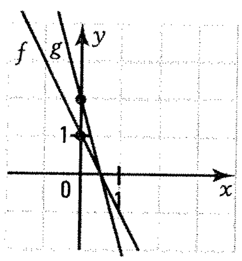
$(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$

g(x) = 4x

2. In each of the following cases, we apply the indicated scaling to the graph of the function $f(x) = -2x + 1$.

1. Draw the graph of the function g, image of the graph of function f.
2. Find the rule of function g.

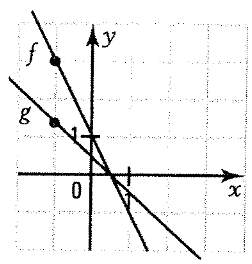
a)



$(x, y) \rightarrow (x, 2y)$

g(x) = −4x + 2

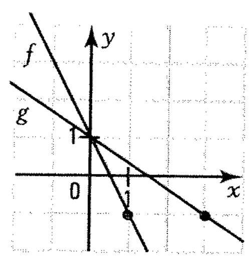
b)



$(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$

g(x) = $-x + \frac{1}{2}$

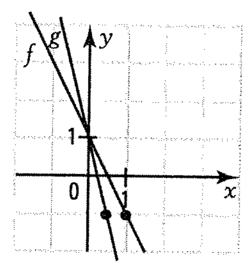
c)



$(x, y) \rightarrow (3x, y)$

g(x) = $-\frac{2}{3}x + 1$

d)



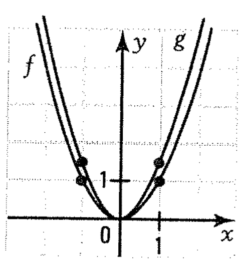
$(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$

g(x) = −4x + 1

3. In each of the following cases, we apply the indicated scaling to the graph of the function $f(x) = x^2$.

1. Draw the graph of the function g, image of the graph of function f.
2. Find the rule of function g.
3. Describe the change caused by the scaling.

a)

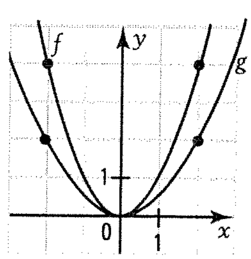


$(x, y) \rightarrow \left(x, \frac{3}{2}y\right)$

g(x) = $\frac{3}{2}x^2$

vertical stretching

b)

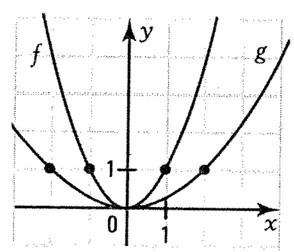


$(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$

g(x) = $\frac{1}{2}x^2$

vertical reduction

c)

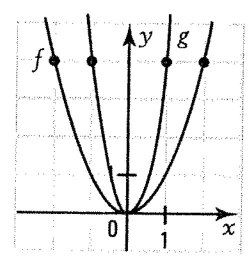


$(x, y) \rightarrow (2x, y)$

g(x) = $\frac{1}{4}x^2$

horizontal stretching

d)



$(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$

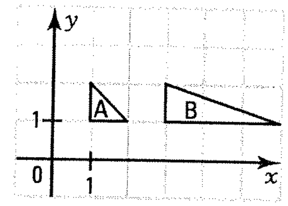
g(x) = $4x^2$

horizontal reduction

4. Given the graph of a function $y = f(x)$, which variable must be multiplied to cause a

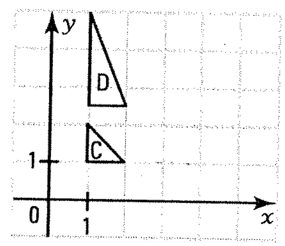
a) horizontal scaling? x b) vertical scaling? y

ACTIVITY 2 Inverse of a scaling



a) Describe the scaling giving

1. figure B starting with figure A.
   A horizontal stretching of factor 3.
   $e_h(3): (x, y) \rightarrow (3x, y)$
2. figure A starting with figure B.
   A horizontal reduction of factor $\frac{1}{3}$.
   $e_h\left(\frac{1}{3}\right): (x, y) \rightarrow \left(\frac{1}{3}x, y\right)$



b) Describe the scaling giving

1. figure D starting with figure C.
   A vertical stretching of factor $\frac{5}{2}$.
   $e_v\left(\frac{5}{2}\right): (x, y) \rightarrow \left(x, \frac{5}{2}y\right)$
2. figure C starting with figure D.
   A horizontal reduction of factor $\frac{2}{5}$.
   $e_v\left(\frac{2}{5}\right): (x, y) \rightarrow \left(x, \frac{2}{5}y\right)$

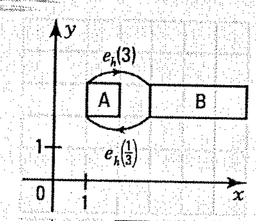
c) Complete:

1. the inverse of a horizontal stretching of factor k is ***a horizontal reduction of factor $\frac{1}{k}$.***
2. the inverse of a vertical stretching of factor k is ***a vertical reduction of factor $\frac{1}{k}$.***

INVERSE OF A SCALING

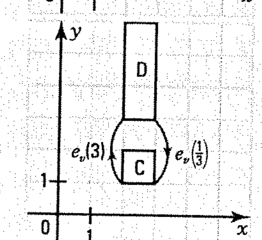
- The inverse of a horizontal stretching of factor k is a horizontal reduction of factor $\frac{1}{k}$.

  If $e_h: (x, y) \rightarrow (kx, y)$ then $e_h^{-1}: (x, y) \rightarrow \left(\frac{1}{k}x, y\right)$



- The inverse of a vertical stretching of factor k is a vertical reduction of factor $\frac{1}{k}$.

  If $e_v: (x, y) \rightarrow (x, ky)$ then $e_v^{-1}: (x, y) \rightarrow \left(x, \frac{1}{k}y\right)$.



5. Define the inverse of the following scalings.

a) $(x, y) \rightarrow (5x, y)$
$(x, y) \rightarrow \left(\frac{x}{5}, y\right)$

b) $(x, y) \rightarrow (x, 2y)$
$(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$

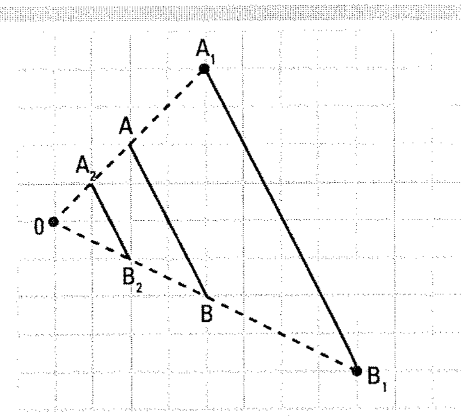
c) $(x, y) \rightarrow \left(\frac{2}{3}x, y\right)$
$(x, y) \rightarrow \left(\frac{3}{2}x, y\right)$

d) $(x, y) \rightarrow \left(x, \frac{3}{4}y\right)$
$(x, y) \rightarrow \left(x, \frac{4}{3}y\right)$

3.5 Dilatation

ACTIVITY 1 Dilatation with positive ratio k

Consider the line segment AB on the right.

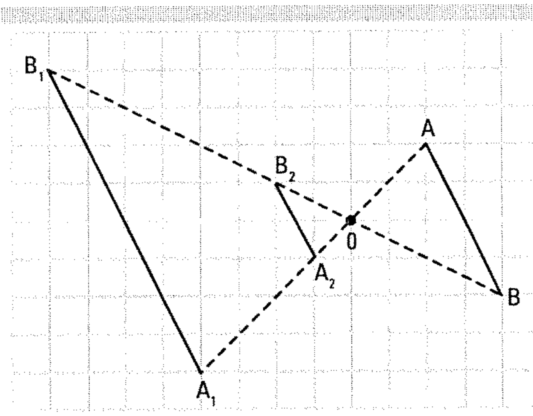


a) Let h_1 be the dilatation centred at 0 with ratio k equal to 2, written $h_1(0, 2)$.
   1. Draw the line segment A_1B_1, image of the line segment AB under dilatation h_1.
   2. Verify the properties of the dilatation:
      1) $m\overline{A_1B_1} = k\,m\overline{AB}$ 2) $\overline{A_1B_1} // \overline{AB}$

b) Let h_2, be the dilatation centred at 0 with ratio k equal to $\frac{1}{2}$, written $h_2\left(0, \frac{1}{2}\right)$.
   1. Draw the line segment A_2B_2, image of the line segment AB under dilatation h_2.
   2. Verify that: 1) $m\overline{A_2B_2} = k\,m\overline{AB}$ 2) $\overline{A_2B_2} // \overline{AB}$

ACTIVITY 2 Dilatation with negative ratio k

Consider the line segment AB on the right.



When the dilatation ratio k is negative, the image A′B′ of the line segment AB verifies the following three properties:

1. The dilatation centred at 0 is located between the initial line segment and the image line segment.
2. $m\overline{A'B'} = |k|\,m\overline{AB}$.
3. $\overline{A'B'} // \overline{AB}$.

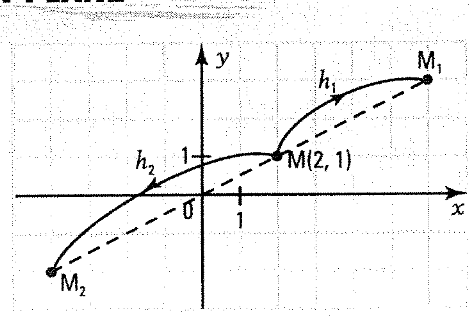
a) Let h_1 be the dilatation centred at 0 with ratio k equal to −2, written $h_1(0, -2)$.
Draw the line segment A_1B_1, image of the line segment AB under dilatation h_1.

b) Let h_2 be the dilatation centred at 0 with ratio k equal to $-\frac{1}{2}$, written $h_2\left(0, -\frac{1}{2}\right)$.
Draw the line segment A_2B_2, image of the line segment AB under dilatation h_2.

DILATATION IN THE CARTESIAN PLANE

- The rule of the **dilatation** h centred at 0 (origin of the Cartesian plane) with ratio k, written $h(0, k)$ is:

$$h(0, k): (x, y) \rightarrow (kx, ky)$$



Ex.: – The rule of the dilatation $h_1(0, 3)$ is: $(x, y) \rightarrow (3x, 3y)$.
The image of M(2, 1) under h_1 is the point $M_1(6, 3)$.
– The rule of the dilatation $h_2(0, -2)$ is: $(x, y) \rightarrow (-2x, -2y)$.
The image of M(2, 1) under h_2 is the point $M_2(-4, -2)$.
Note that the dilatation centred at 0 is located between the initial figure and the image figure when the dilatation ratio is negative.

1. In each of the following cases, determine the coordinates of the point A′, image of point A under the given dilatation h.

a) A(−2, 3) and $h(0, -2)$ ***A′4, −6)***

b) A(2, −4) and $h\left(0, \frac{1}{2}\right)$ ***A′(1, −2)***

c) A(6, −9) and $h\left(0, \frac{-2}{3}\right)$ ***A′(−4, 6)***

d) A(4, −6) and $h\left(0, -\frac{3}{2}\right)$ ***A′(−6, 9)***

2. In each of the following cases, determine the dilatation centred at the origin 0 of the Cartesian plane which transforms point A into point A′.

a) A(−1, 4) and A′ (−2, 8) ***h(0, 2)***

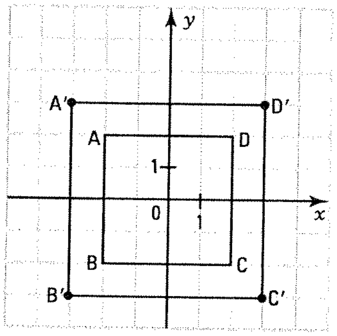
b) A(−6, −9) and A′(−8, −12) $\boldsymbol{h\left(0, \frac{4}{3}\right)}$

c) A(−1, 4) and A′(2, −8) ***h(0, −2)***

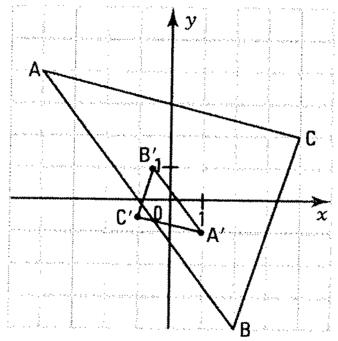
d) A(−6, −9) and A′(4, 6) $\boldsymbol{h\left(0, -\frac{2}{3}\right)}$

3. Draw the image of each of the following figures under the given dilatation.

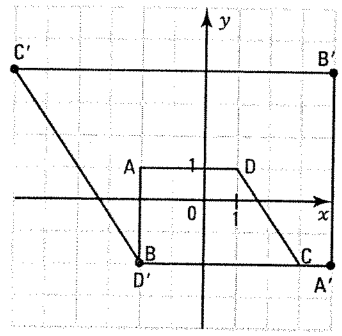
a) $h\left(0, \frac{3}{2}\right)$



b) $h\left(0, -\frac{1}{4}\right)$

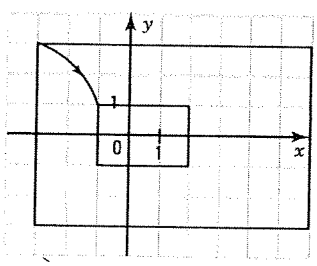


c) $h(0, -2)$



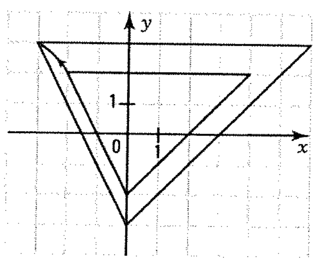
4. Find the rule of the dilatation in each of the following cases.

a)



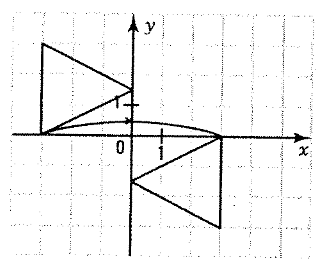
$\boldsymbol{(x, y) \to \left(\frac{1}{3}x, \frac{1}{3}y\right)}$

b)



$\boldsymbol{(x, y) \to \left(\frac{3}{2}x, \frac{3}{2}y\right)}$

c)



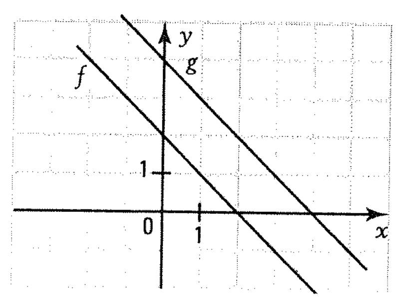
(x, y) → (−x, −y)

5. Consider a dilatation with ratio k. What can be said about $|k|$ if the dilatation causes

a) an enlargement of the initial figure? $\boldsymbol{|k| > 1}$

b) a reduction of the initial figure? $\boldsymbol{|k| < 1}$

6. Consider on the right the linear function f having rule $y = -x + 2$.



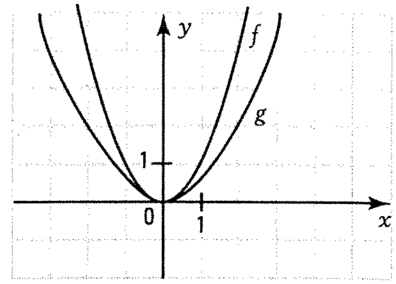
a) Draw the graph of the function g, image of the graph of function f under the dilatation h: $(x, y) \to (2x, 2y)$.

b) Determine the rule of function g. ***y = −x + 4***

c) Explain why functions f and g are represented by parallel lines.

The lines representing f and g have the same slope −1.

7. Consider on the right the quadratic function f having rule $y = x^2$.

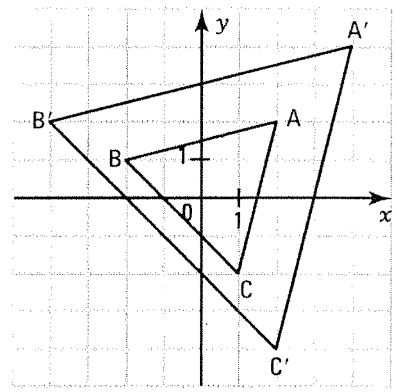


a) Draw the graph of the function g, image of the graph of function f, under the dilatation $h: (x, y) \rightarrow (2x, 2y)$.

b) The representation of function g is a parabola with vertex 0. Find the rule of g.

$y = \frac{1}{2}x^2$

ACTIVITY 3 Inverse of a dilatation



a) What is the rule of the dilatation h which associates triangle ABC with triangle A′B′C′?

$h: (x, y) \rightarrow (2x, 2y)$

b) What is the rule of the inverse dilatation h^{-1} which associates triangle A′B′C′ with triangle ABC?

$h^{-1}: (x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$

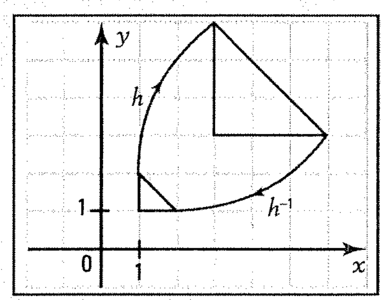
c) 1. Complete: The inverse of a dilatation centred at 0 with ratio k is a dilatation centred at **0** with ratio $\frac{1}{k}$.

2. If the dilatation h is an enlargement, the inverse h^{-1} is a ***reduction*** .

INVERSE DILATATION

- The inverse of the dilatation h centred at 0 with ratio k is the dilatation h^{-1} centred at 0 with ratio $\frac{1}{k}$.

if $h: (x, y) \rightarrow (kx, ky)$ then $h^{-1}: (x, y) \rightarrow \left(\frac{x}{k}, \frac{y}{k}\right)$



- If a dilatation causes an enlargement of factor k, the inverse dilatation causes a reduction of factor $\frac{1}{k}$.

Ex.: The dilatation h represented on the right: $(x, y) \rightarrow (3x, 3y)$ and the reciprocal dilatation $h^{-1}: (x, y) \rightarrow \left(\frac{1}{3}x, \frac{1}{3}y\right)$.

8. Define the inverse of the following dilatations.

a) $(x, y) \rightarrow \left(\frac{3}{2}x, \frac{3}{2}y\right)$

$(x, y) \rightarrow \left(\frac{2}{3}x, \frac{2}{3}y\right)$

b) $(x, y) \rightarrow (-2x, -2y)$

$(x, y) \rightarrow \left(-\frac{1}{2}x, -\frac{1}{2}y\right)$

c) $(x, y) \rightarrow \left(-\frac{5}{2}x, -\frac{5}{2}y\right)$

$(x, y) \rightarrow \left(-\frac{2}{5}x, -\frac{2}{5}y\right)$

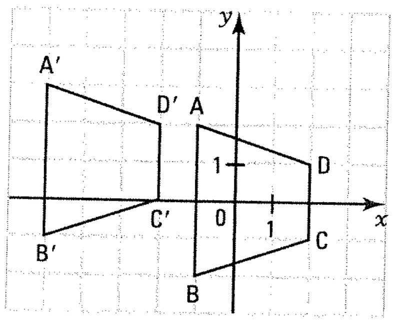
d) $(x, y) \rightarrow \left(\frac{4}{5}x, \frac{4}{5}y\right)$

$(x, y) \rightarrow \left(\frac{5}{4}x, \frac{5}{4}y\right)$

3.6 Sequence of transformations

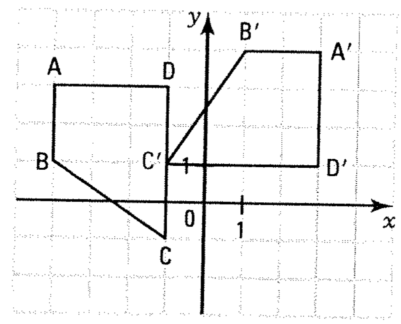
1. Draw the image of each figure under the indicated transformation and describe the transformation..

a) $(x, y) \rightarrow (x - 4, y + 1)$



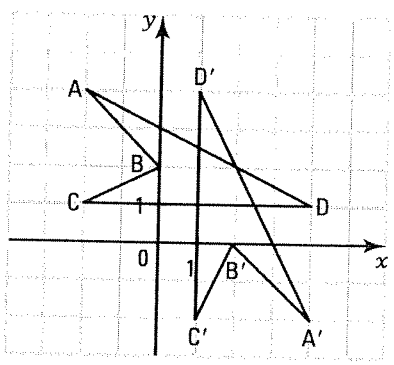
Translation t(−4, 1).

b) $(x, y) \rightarrow (y, -x)$



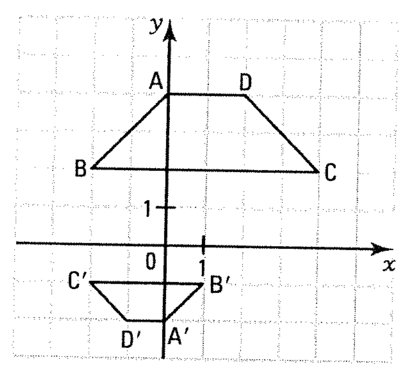
Rotation $r_{-90°}$.

c) $(x, y) \rightarrow (y, x)$



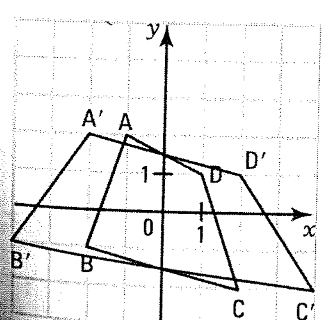
Reflection $s_{\oslash}$.

d) $(x, y) \rightarrow \left(-\frac{1}{2}x, -\frac{1}{2}y\right)$



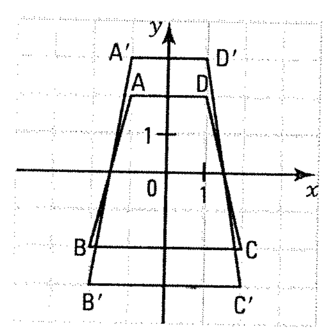
Dilatation $h\left(0, -\frac{1}{2}\right)$.

e) $(x, y) \rightarrow (2x, y)$



Horizontal stretching of factor 2.

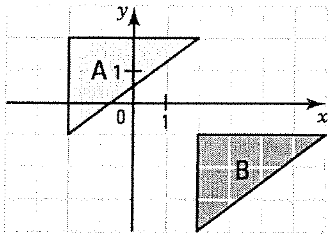
f) $(x, y) \rightarrow \left(x, \frac{3}{2}y\right)$



Vertical stretching of factor $\frac{3}{2}$.

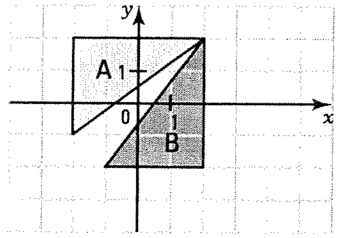
2. In each case, figure B is the image of figure A under a transformation. Determine the rule and specify what type of transformation (translation, rotation, reflection, dilatation) it is.

a)



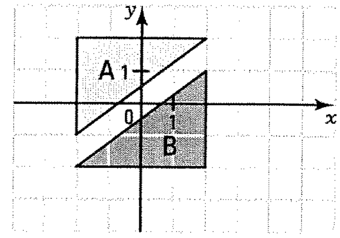
$t_{(4, -3)}: (x, y) \rightarrow (x + 4, y - 3)$

b)



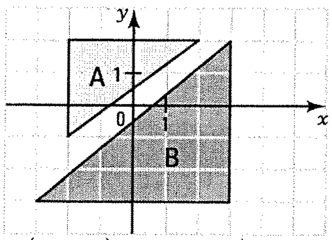
$s_{\boxslash}: (x, y) \rightarrow (y, x)$

c)



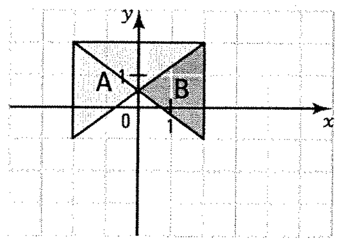
$r_{180°}: (x, y) \rightarrow (-x, -y)$

d)



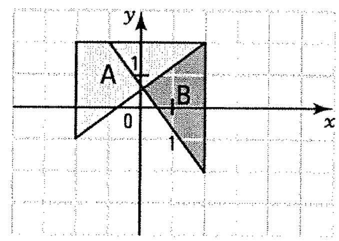
$h\left(0, -\frac{3}{2}\right): (x, y) \rightarrow \left(-\frac{3}{2}x, -\frac{3}{2}y\right)$

e)



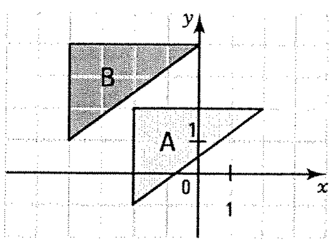
$s_y: (x, y) \rightarrow (-x, y)$

f)



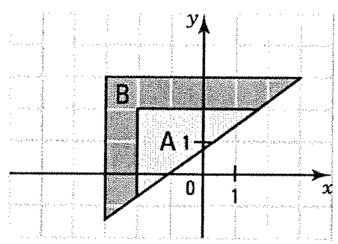
$r_{270°}: (x, y) \rightarrow (y, -x))$

g)



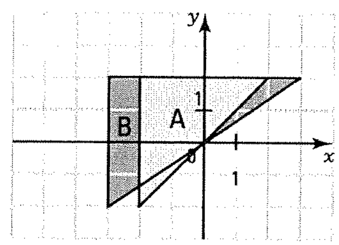
$t_{(-2, 2)}: (x, y) \rightarrow (x - 2, y + 2)$

h)



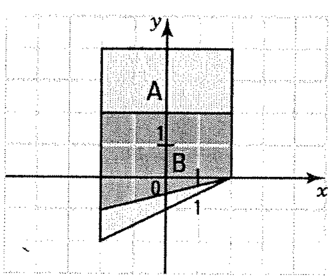
$h\left(0, \frac{3}{2}\right): (x, y) \rightarrow \left(\frac{3}{2}x, \frac{3}{2}y\right)$

i)



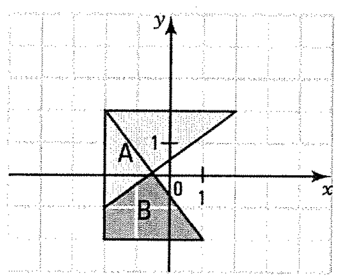
$e_h\left(\frac{3}{2}\right): (x, y) \rightarrow \left(\frac{3}{2}x, y\right)$

j)



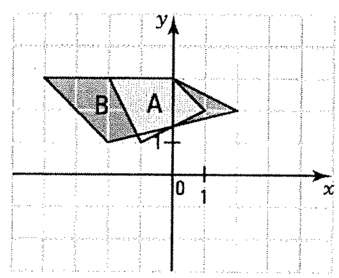
$e_v\left(\frac{1}{2}\right): (x, y) \rightarrow \left(x, \frac{1}{2}y\right)$

k)



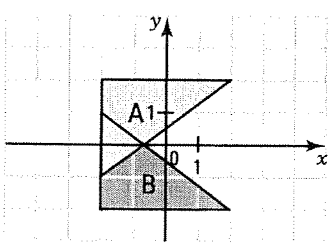
$r_{90°}: (x, y) \rightarrow (-y, x)$

l)



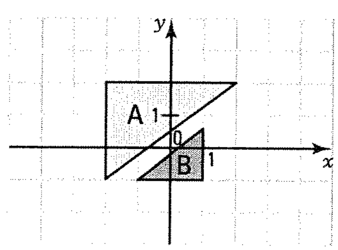
$e_h(2): (x, y) \rightarrow (2x, y)$

m)



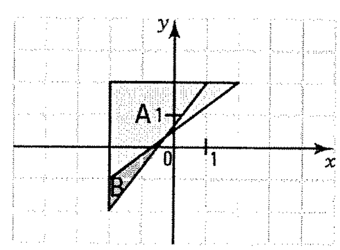
$s_x: (x, y) \rightarrow (x, -y)$

n)



$h\left(0, -\frac{1}{2}\right): (x, y) \rightarrow \left(\frac{-1}{2}x, \frac{-1}{2}y\right)$

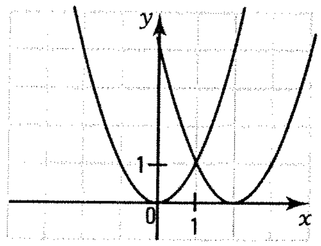
o)



$s_{\boxbslash}: (x, y) \rightarrow (-y, -x)$

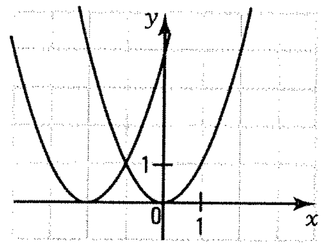
3. Draw the image of each figure under the given transformation then describe the change it caused.

a) $(x, y) \rightarrow (x + 2, y)$



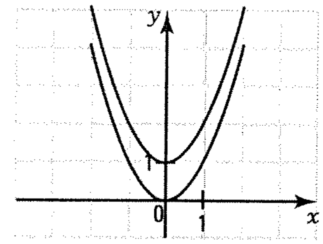
Horizontal translation of 2 units towards the right.

b) $(x, y) \rightarrow (x - 2, y)$



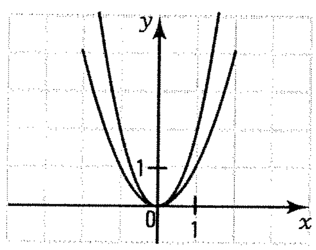
Horizontal translation of 2 units towards the left.

c) $(x, y) \rightarrow (x, y + 1)$



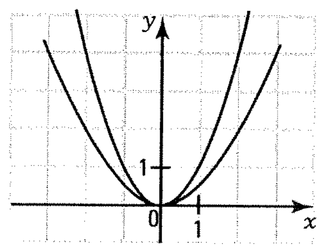
Vertical translation of 1 unit upwards.

d) $(x, y) \rightarrow (x, 2y)$



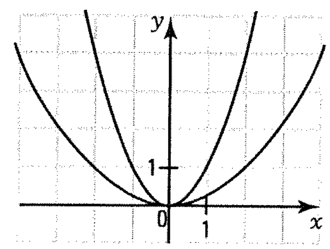
Vertical stretching of factor 2.

e) $(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$



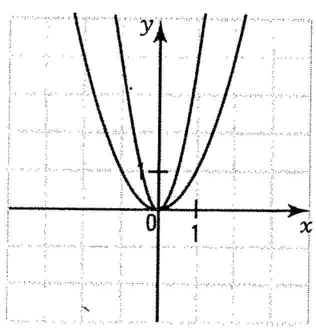
Vertical reduction of factor $\frac{1}{2}$.

f) $(x, y) \rightarrow (2x, y)$



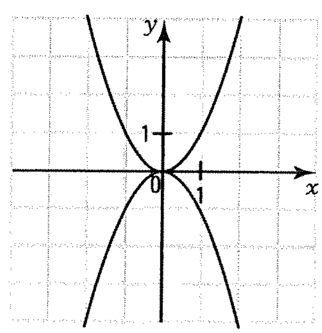
Horizontal stretching of factor 2.

g) $(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$



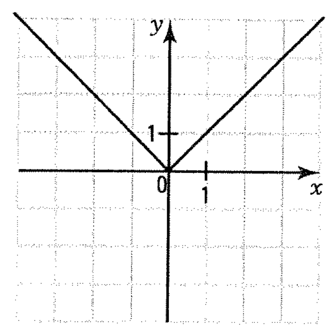
Horizontal reduction of factor $\frac{1}{2}$.

h) $(x, y) \rightarrow (x, -y)$



Reflection about the x-axis.

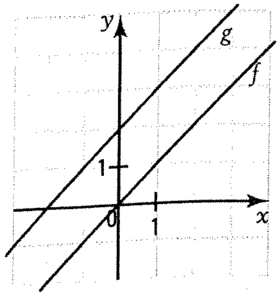
i) $(x, y) \rightarrow (-x, y)$



Reflection about the y-axis.

4. Determine the rule of the transformation which associates the graph of f with the graph of g.

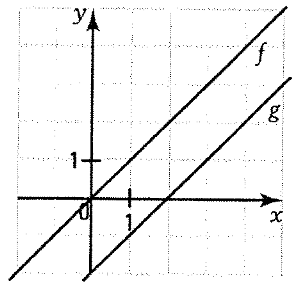
a)



$(x, y) \rightarrow (x, y + 2)$

or $(x, y) \rightarrow (x - 2, y)$

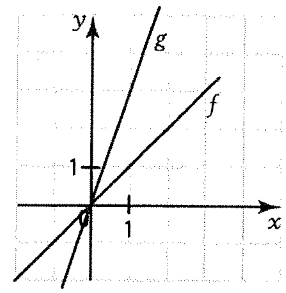
b)



$(x, y) \rightarrow (x, y - 2)$

or $(x, y) \rightarrow (x + 2, y)$

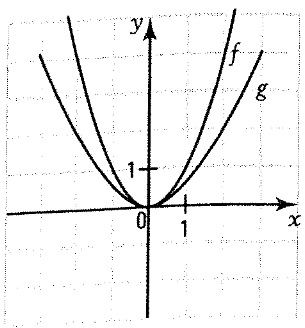
c)



$(x, y) \rightarrow (x, 3y)$

or $(x, y) \rightarrow \left(\frac{x}{3}, y\right)$

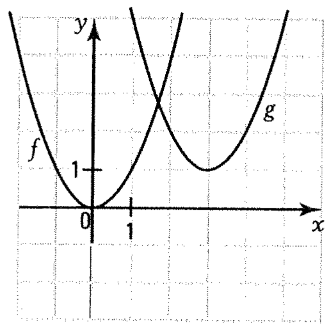
d)



$(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$

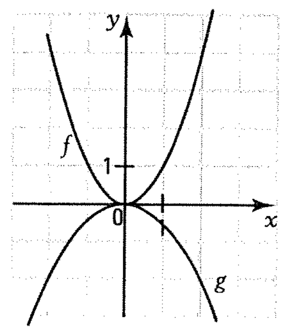
or $(x, y) \rightarrow \left(x\sqrt{2}, y\right)$

e)



$(x, y) \rightarrow (x + 3, y + 1)$

f)

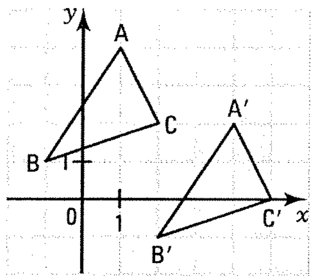


$(x, y) \rightarrow \left(x, -\frac{1}{2}y\right)$

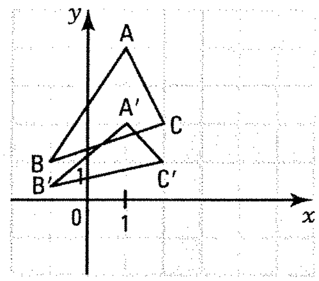
Evaluation 3

1. Draw the triangle A′B′C′, image of triangle ABC under the given transformation.

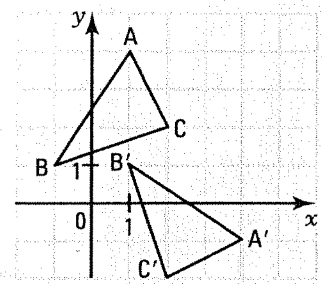
a) $(x, y) \to (x + 3, y - 2)$



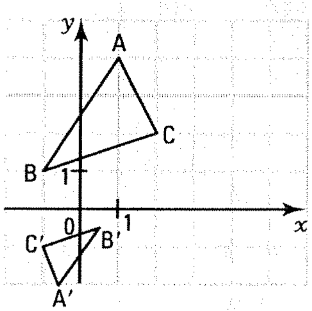
b) $(x, y) \to \left(x, \frac{y}{2}\right)$



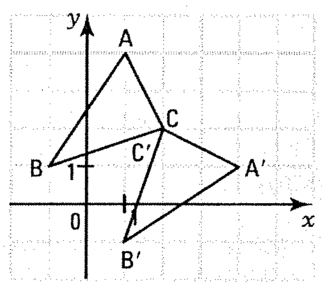
c) $(x, y) \to (y, -x)$



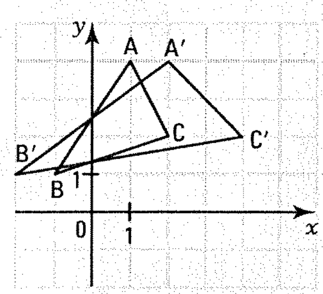
d) $(x, y) \to \left(-\frac{1}{2}x, -\frac{1}{2}y\right)$



e) $(x, y) \to (y, x)$



f) $(x, y) \to (2x, y)$



2. Name and describe each of the transformations of exercise 1.

a) ***Translation*** $t_{(3,\,-2)}$.

b) ***Vertical reduction of factor*** $\frac{1}{2}$; $e_v\left(\frac{1}{2}\right)$.

c) ***Clockwise rotation of angle 90° centred at 0.***

d) ***Dilatation centred at 0 of factor*** $-\frac{1}{2}$; $h\left(0, -\frac{1}{2}\right)$.

e) ***Reflection about the line*** $y = x$; $s_{\boxslash}$

f) ***Horizontal stretching of factor 2;*** $e_h(2)$

3. Find the rule of the inverse of each of the transformations of exercise 1.

a) $(x, y) \to (x - 3, y + 2)$ **b)** $(x, y) \to (x, 2y)$ **c)** $(x, y) \to (-y, x)$

d) $(x, y) \to (-2x, -2y)$ **e)** $(x, y) \to (y, x)$ **f)** $(x, y) \to \left(\frac{x}{2}, y\right)$

4. The image of point M(1, 2) under a transformation is point M′(2, 4).
Find the rule of this transformation if this transformation is

a) a translation. $(x, y) \to (x + 1, y + 2)$ **b)** a dilatation centred at 0. $(x, y) \to (2x, 2y)$

5. The image of point M(0, 5) under a transformation is point M′(5, 0).
Find the rule of this transformation if this transformation is

a) a rotation centred at 0. $(x, y) \to (y, -x)$ **b)** a reflection. $(x, y) \to (y, x)$