Chapter 4

Polynomial functions

CHALLENGE 4

4.1 Function

4.2 Polynomial functions

4.3 Quadratic function – Standard form

4.4 Quadratic function – General form

4.5 Quadratic function – Factored form

4.6 Study of a quadratic function

EVALUATION 4

CHALLENGE 4

1. Given $f(x) = 2x - 3$ and $g(x) = x^2 - 2x + 3$. Find the rule of the function

a) $g \circ f$ ***$g \circ f(x) = 4x^2 - 16x + 18$***

b) $f \circ g$ ***$f \circ g(x) = 2x^2 - 4x + 3$***

2. For each of the following functions, indicate if its inverse is a function. If not, explain why. If it is, find the rule of the inverse.

a) $f(x) = 2x - 3$ ***f^{-1} is a function; $f^{-1}(x) = \frac{x + 3}{2}$***

b) $g(x) = x^2 - 2x + 3$ ***g^{-1} is not a function; $g(0) = 3$ and $g(2) = 3$.***
The image of 3 in g^{-1} is not unique, therefore g^{-1} is not a function.
Also, g^{-1} is not a function because the horizontal line $y = 3$ intersects the graph of g at more than one point ((0, 3) and (2, 3)).

3. What are the coordinates of the vertex of the parabola defined by the equation $y = ax^2 + bx + c$?

$V\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

4. What are the coordinates of the vertex of the parabola passing through the points A(1, 0), B(3, 0) and C(4, 6)?

V(2, −2)

5. Study the variation of the function $f(x) = x^2 - 4x + 3$.

$f \searrow$ over $]-\infty, 2]$; $f \nearrow$ over $[2, +\infty[$

6. Consider the quadratic function $f(x) = 2x^2 - 4x - 6$.
Write the rule of this function in the following forms:

a) standard form.

$f(x) = 2(x - 1)^2 - 8$

b) factored form.

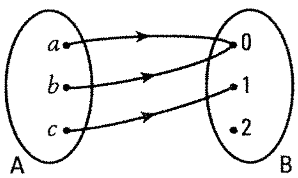
$f(x) = 2(x + 1)(x - 3)$

4.1 Function

ACTIVITY 1 Definition of a function

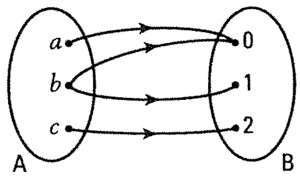
In each of the following cases, indicate if the relation is a function. If not, explain why.

a)



Yes

b)

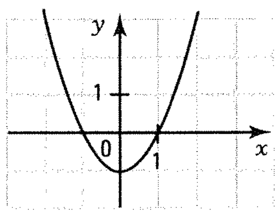


No, since b has two images 0 and 1.

c) $\{(a, 1), (b, 2), (a, 3)\}$

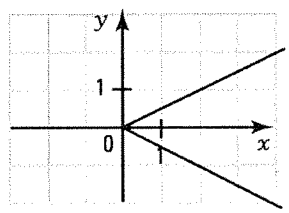
No, since a has two images 1 and 3.

d)



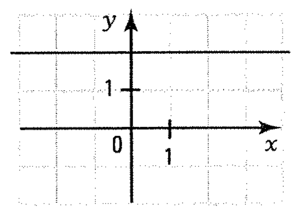
Yes

e)



No, since 2 has two images 1 and −1.

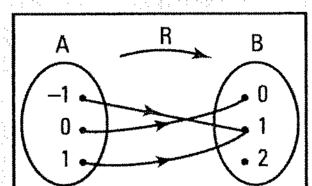
f)



Yes

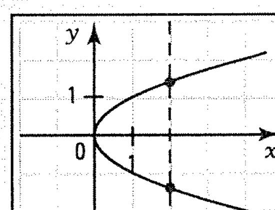
DEFINITION OF A FUNCTION

- A relation given by a source set A to a target set B is a **function** if each element from A is associated with **at most** one element from B.
- Mapping diagram
  Given the mapping diagram of a relation, this relation is a function if, from each element of the source set, at most one arrow is drawn.



R is a function.

- Cartesian graph
  Given the Cartesian graph of a relation, this relation is a function if any vertical line intersects the graph of this relation in at most one point.



R is not a function.

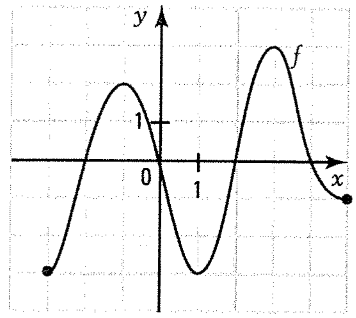
- Set of ordered pairs
  Given a relation's set of ordered pairs, this relation is a function if the first coordinate of each pair verifying the relation appears only once.

$G_R = \{(\underline{a}, 0), (b, 1), (\underline{a}, 2)\}$

R is not a function.

ACTIVITY 2 Properties of functions

Consider the function f represented on the right.



a) What is the domain of f? *[-3, 5]*

b) What is the range of f? *[-3, 3]*

c) What are the zeros of f? *-2, 0, 2 and 4*

d) What is the initial value of f? *0*

e) Over what interval is the function f

1. positive? *[-2, 0] ∪ [2, 4]* 2. negative? *[-3, -2] ∪ [0, 2] ∪ [4, 5]*

f) Over what interval is the function f

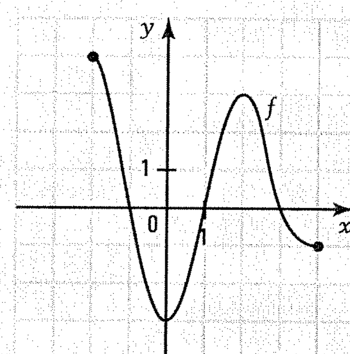
1. increasing? *[-3, -1] ∪ [1, 3]* 2. decreasing? *[-1, 1] ∪ [3, 5]*

g) What is, for function f, its

1. absolute maximum? *3* 2. absolute minimum? *-3*

PROPERTIES OF FUNCTIONS

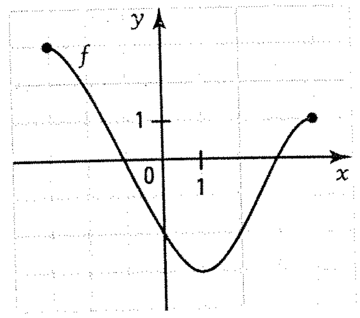
Consider the function f represented on the right.



- The **domain** of a function f is the subset of the elements of the source set which have an image in f.
  dom $f = [-2, 4]$
- The **range** of a function f is the subset of the elements of the target set which are images by f.
  ran $f = [-3, 4]$
- The **zeros** of the function f are the values of x for which the function is equal to zero. The zeros of f are: –1, 1 and 3.
- The **initial value** of the function f is the value of y when $x = 0$. The initial value of f is –3.
- Studying the **sign** of a function consists of finding the values of x for which the function is positive or those for which the function is negative.

  $f(x) \geqslant 0$ if $x \in [-2, -1] \cup [1, 3]$.
  $f(x) \leqslant 0$ if $x \in [-1, 1] \cup [3, 4]$.
- Studying the **variation** of a function consists of finding the values of x for which the function is increasing or those for which the function is decreasing.
  f is increasing if $x \in [0, 2]$.
  f is decreasing if $x \in [-2, 0] \cup [2, 4]$.
- The **absolute maximum** (or **minimum**) of a function is the highest image (or the lowest image) when it exists.
  max $f = 4$, min $f = -3$

1. Consider the function f represented on the right. Determine



a) 1. dom $f =$ ***[-3, 4]*** 2. ran $f =$ ***[-3, 3]***

b) 1. the zeros of f: ***-1 and 3***

2. the initial value: ***-2***

c) the values of x for which the function f is

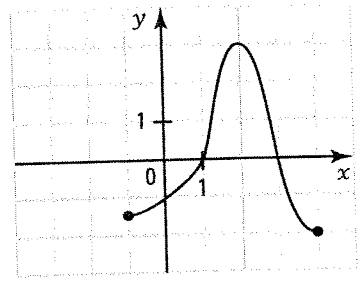
1. positive: ***[-3, -1] ∪ [3, 4]*** 2. negative: ***[-1, 3]***

d) the values of x for which the function f is

1. increasing: ***[1, 4]*** 2. decreasing: ***[-3, 1]***

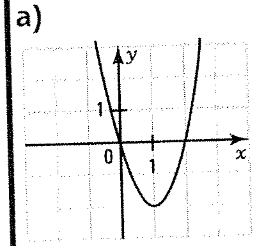
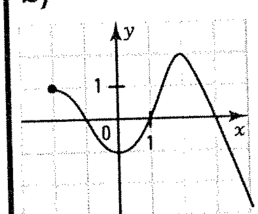
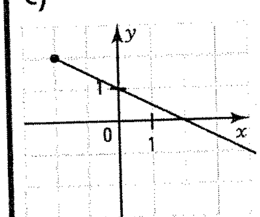
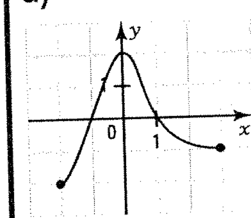
e) 1. the maximum of f: ***3*** 2. the minimum of f: ***-3***

2. Draw the graph of a function satisfying the following conditions.



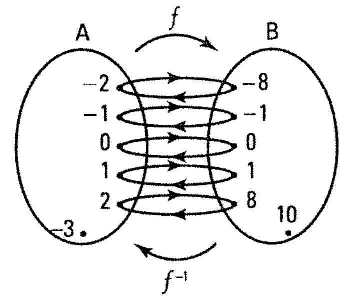
1. dom $f = [-1, 4]$.
2. ran $f = [-2, 3]$.
3. The zeros of f are: 1 and 3.
4. The initial value is -1.
5. The function is negative when $x \in [-1, 1] \cup [3, 4]$.
6. The function is increasing when $x \in [-1, 2]$ and decreasing when $x \in [2, 4]$.
7. $\max f = 3$ and $\min f = -2$.

3. Study the following functions by completing the table below.

	a) 	b) 	c) 	d) 
Domain	ℝ	***[-2, +∞[***	***[-2, +∞[***	***[-2, 3]***
Range	***[-2, +∞[***	***]-∞, 2]***	***]-∞, 2]***	***[-2, 2]***
Zeros	***0 and 2***	***-1, 1 and 3***	***2***	***-1 and 1***
Initial value	***0***	***-1***	***1***	***2***
$f(x) \geq 0$ if $x \in$	***]-∞, 0] ∪ [2, +∞[***	***[-2, -1] ∪ [1, 3]***	***[-2, 2]***	***[-1, 1]***
$f(x) \leq 0$ if $x \in$	***[0, 2]***	***[-1, 1] ∪ [3, +∞[***	***[2, +∞[***	***[-2, -1] ∪ [1, 3]***
$f \nearrow$ if $x \in$	***[1, +∞[***	***[0, 2]***	***never***	***[-2, 0]***
$f \searrow$ if $x \in$	***]-∞, 1]***	***[-2, 0] ∪ [2, +∞[***	***[-2, +∞[***	***[0, 3]***
Extrema	***min f = -2***	***max f = 2***	***max f = 2***	***max f = 2, min f = -2***

ACTIVITY 3 Inverse of a function

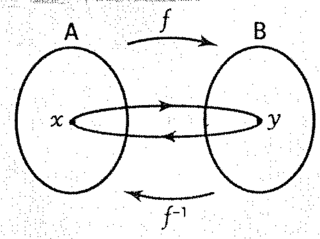
a) Consider the mapping diagram of function f.



1. Deduce the mapping diagram of the inverse f^{-1}.
2. Explain why f^{-1} is a function.
   There is at most one arrow from each element of the source set B.
3. Determine
   1) dom f. *{-2, -1, 0, 1, 2}* 2) ran f. *{-8, -1, 0, 1, 8}*
   3) dom f^{-1}. *{-8, -1, 0, 1, 8}* 4) ran f^{-1}. *{-2, -1, 0, 1, 2}*
4. Verify that
   1) dom f = ran f^{-1}. 2) ran f = dom f^{-1}.

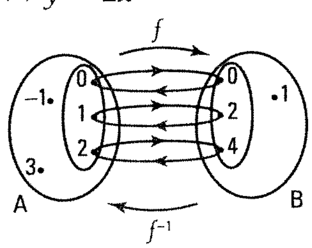
b) Is the inverse of a function always a function? *No*

INVERSE OF A FUNCTION



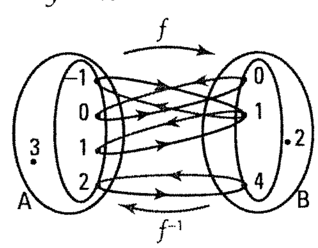
- If f is the function of a source set A toward a target set B, the **inverse** of f, written f^{-1}, has the source set B and the target set A.
- The inverse of a function is not necessarily a function.

Ex.: f: A → B
$x \mapsto y = 2x$



f^{-1} is a function.
dom f = ran f^{-1} = {0, 1, 2}
ima f = dom f^{-1} = {0, 2, 4}

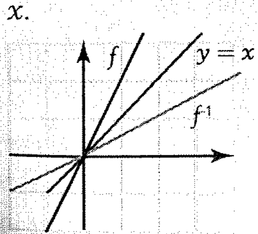
Ex.: f: A → B
$x \mapsto y = x^2$



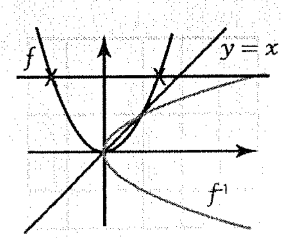
f^{-1} is not a function.
dom f = ran f^{-1} = {−1, 0, 1, 2}
ran f = dom f^{-1} = {0, 1, 4}

For any function f, we have: **dom f = ran f^{-1}** and **ran f = dom f^{-1}**

The Cartesian graphs of a function and its inverse are **symmetrical** about the line with the equation $y = x$.



f^{-1} is a function.



f^{-1} is not a function.

...se of a function f is not a function when a **horizontal line** can be drawn to intersect ...h of f at more than one point.

ACTIVITY 4 Rule of the inverse

A salesman in a store receives a weekly base salary of \$250 and a sales commission of \$10 per item sold for the week.

a) Let a represent the number of items sold for the week, and s represent the total weekly salary. Determine the rule of

1. the function f which gives the total salary s as a function of the number of items sold a.
   $s = 250 + 10a$
2. the function f^{-1} which associates, to a given salary s, the number of items sold a. $a = \frac{s - 250}{10}$

b) Complete the table of values on the right for the functions f and f^{-1}.

a	0	5	10	15	20
s	250	300	350	400	450

(f: $a \to s$; f^{-1}: $s \to a$)

RULE OF THE INVERSE

Given the function f with the rule: $y = 2x + 6$. To determine the rule of the inverse f^{-1},

1. we isolate x in the rule of f.
   $y = 2x + 6$
   $2x = y - 6$
   $x = \frac{1}{2}y - 3$
2. we switch the letters x and y.
   $y = \frac{1}{2}x - 3$
   f^{-1} therefore has the rule: $y = \frac{1}{2}x - 3$.

We interchange the letters x and y to respect the convention of function notation which assigns x as elements of the source set and y as elements of the target set.

4. For each of the following rules of functions, find the rule of its inverse.

a) $y = 5x$
$y = \frac{x}{5}$

b) $y = 3x - 6$
$y = \frac{x}{3} + 2$

c) $y = -2x + 10$
$y = \frac{-x}{2} + 5$

d) $y = 0.1x + 100$
$y = 10x - 1000$

e) $y = \frac{2}{3}x - 6$
$y = \frac{3}{2}x + 9$

f) $y = -\frac{3}{4}x + 12$
$y = \frac{-4}{3}x + 16$

5. A capital of \$1000 is invested on January 1st, 2009 at an annual interest rate of 10%. Find the rule which associates

a) a given number of elapsed years t since the beginning, to the accumulated capital C.
$C = 1000 + 100t$

b) a given accumulated capital C, to the number of elapsed years t. $t = 0.01C - 10$

6. A car's gas tank initially contains 60 litres of gas. This car consumes on average 12 litres/100 km. Find the rule of the function which associates,

a) a given distance traveled d (in km), to the quantity q of gas remaining in the tank.
$q = -0.12d + 60$

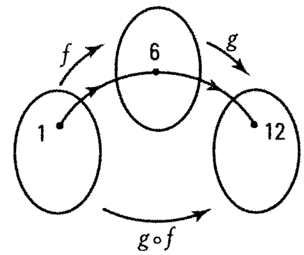
b) a given quantity q of gas remaining in the tank, to the distance traveled d (in km).
$d = -\frac{25}{3}q + 500$

ACTIVITY 5 Composition of functions

Consider the function f defined by the rule $f(x) = x + 5$ and the function g defined by the rule $g(x) = 2x$.

a) Determine

1. $f(1)$ ___6___ 2. $g(f(1))$ ___12___

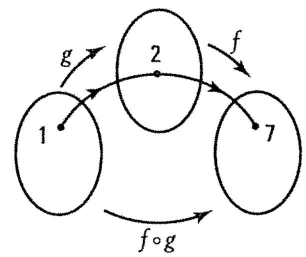


b) The composition of f by g, written $g \circ f$ is defined by $g \circ f(x) = g(f(x))$.

1. Calculate $g \circ f(1)$ ___12___
2. Determine the rule of $g \circ f$. ___$g \circ f(x) = g(f(x)) = g(x + 5) = 2x + 10$___

c) Determine

1. $g(1)$ ___2___ 2. $f(g(1))$ ___7___



d) The composition of g by f, written $f \circ g$, is defined by $f \circ g(x) = f(g(x))$.

1. Calculate $f \circ g(1)$ ___7___
2. Determine the rule of $f \circ g$. ___$f \circ g(x) = f(g(x)) = f(2x) = 2x + 5$___

e) Compare the rules of $g \circ f$ and of $f \circ g$.

___$g \circ f(x) \neq f \circ g(x)$___

COMPOSITION OF FUNCTIONS

- Given two functions f and g,
  - the composition of f by g, written $g \circ f$, is defined by the rule:

$$g \circ f(x) = g(f(x))$$

  - the composition of g by f, is defined by the rule:

$$f \circ g(x) = f(g(x))$$

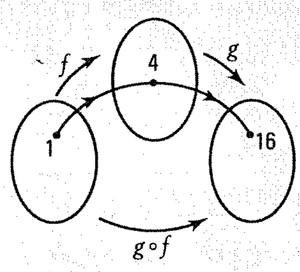
Ex.: Given $f(x) = x + 3$ and $g(x) = x^2$, we have:

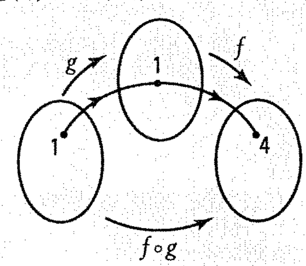
$g \circ f(1) = g(f(1)) = g(4) = 16$;

$g \circ f(x) = g(x + 3) = (x + 3)^2$;

$f \circ g(1) = f(g(1)) = f(1) = 4$

$f \circ g(x) = f(x^2) = x^2 + 3$





Note that, in general, $g \circ f(x) \neq f \circ g(x)$.

7. Consider the functions $f(x) = 3x - 5$ and $g(x) = -2x + 8$. Determine

a) $g \circ f(2) =$ ___6___ **b)** $f \circ g(-1) =$ ___25___ **c)** $f \circ g(4) =$ ___-5___

d) $g \circ f(0) =$ ___18___ **e)** $g \circ g(7) =$ ___20___ **f)** $f \circ g(-5) =$ ___49___

8. Consider the functions $f(x) = -2x + 5$ and $g(x) = 4x - 3$.
Determine the rules of the following functions.

a) $f \circ g(x) =$ ***$f(g(x)) = f(4x - 3) = -2(4x - 3) + 5 = -8x + 11$***

b) $g \circ f(x) =$ ***$g(f(x)) = g(-2x + 5) = 4(-2x + 5) - 3 = -8x + 17$***

c) $f \circ f(x) =$ ***$f(f(x)) = f(-2x + 5) = -2(-2x + 5) + 5 = 4x - 5$***

d) $g \circ g(x) =$ ***$g(g(x)) = g(4x - 3) = 4(4x - 3) - 3 = 16x - 15$***

9. Consider the functions $f(x) = 2x + 3$ and $g(x) = 3x - 2$.

a) Determine the rule of
1. $g \circ f$. ***$g \circ f(x) = 6x + 7$***
2. $f \circ g$. ***$f \circ g(x) = 6x - 1$***

b) Verify that $g \circ f(x) \neq f \circ g(x)$.

10. Consider the functions $f(x) = x + 5$ and $g(x) = x - 2$. In each case, verify that, $g \circ f(x) = f \circ g(x)$.
$g \circ f(x) = x + 3$, $f \circ g(x) = x + 3$

11. Consider the function $f(x) = 2x + 8$.

a) Determine the rule of the inverse f^{-1}. ***$f^{-1}(x) = \frac{1}{2}x - 4$***

b) 1. Determine the rule of the composite $f^{-1} \circ f$.
$f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(2x + 8) = \frac{1}{2}(2x + 8) - 4 = x$

2. Determine the rule of the composite $f \circ f^{-1}$.
$f \circ f^{-1}(x) = f(f^{-1}(x)) = f\left(\frac{1}{2}x - 4\right) = 2\left(\frac{1}{2}x - 4\right) + 8 = x$

3. Verify that $f^{-1} \circ f(x) = f \circ f^{-1}(x) = x$.

12. In Quebec, every purchase is taxable. The goods and services tax (GST) is 5%. The Quebec sales tax (QST) is 7.5%.
Let f be the function which associates a given purchase amount x to the amount y including GST.
Let g be the function which associates a given purchase amount x to the amount y including QST.

a) Determine the rule of the function
1. f: ***$y = 1.05x$***
2. g: ***$y = 1.075x$***

b) 1. Determine the rule of the function $g \circ f$. ***$g \circ f(x) = 1.12875x$***
2. Determine the rule of the function $f \circ g$. ***$f \circ g(x) = 1.12875x$***

c) Compare the rules of the functions $g \circ f$ and $f \circ g$. What can you conclude?
The rules are equal. To calculate the final price of a product, it doesn't matter if you apply the GST first and then the QST, or the QST first and then the GST.

d) 1. What is the final price of a product with a $39.80 price tag? ***$44.92***
2. What is the initial price tag of a product if the final cost paid is $56.44? ***$50***

4.2 Polynomial functions

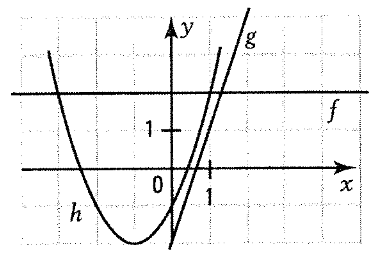
ACTIVITY 1 Polynomial functions

a) Among the following functions, indicate which ones are polynomial functions.

1. $P(x) = -5x + 8$ ***Yes, 1st degree***
2. $P(x) = -4x^2 - 5x$ ***Yes, 2nd degree***
3. $P(x) = \frac{5}{x} + 3$ ***No***
4. $P(x) = -3$ ***Yes, degree 0***
5. $P(x) = \sqrt{x} - 7$ ***No***
6. $P(x) = x^3 + 4x^2 - 5x + 3$ ***Yes, 3rd degree***

b) Represent the following polynomial functions in the Cartesian plane on the right.

1. $f(x) = 2$ 2. $g(x) = 3x - 2$ 3. $h(x) = (x + 1)^2 - 2$



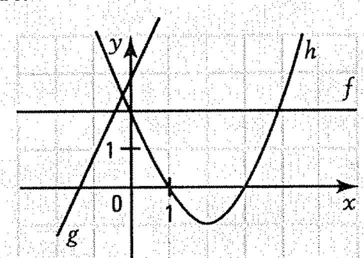
POLYNOMIAL FUNCTIONS

- A **polynomial function** is any function with a polynomial for a rule.

  Ex.: $f(x) = 2$ is a zero degree polynomial function.

  $g(x) = 2x + 3$ is a 1st degree polynomial function.

  $h(x) = x^2 - 4x + 3$ is a 2nd degree polynomial function.

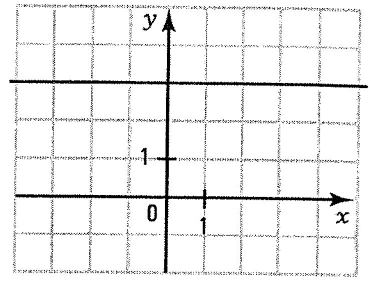


- The following table classifies polynomial functions according to their degree.

Degree	Basic polynomial function	Transformed polynomial function	Name
0	$f(x) = 1$	$f(x) = b$ where $b \in \mathbb{R}$	constant function
1	$f(x) = x$	$f(x) = ax$ where $a \in \mathbb{R}^*$	direct variation linear function
		$f(x) = ax + b$ where $a, b \in \mathbb{R}^*$	partial variation linear function
2	$f(x) = x^2$	$f(x) = ax^2 + bx + c$ where $a \in \mathbb{R}^*$	quadratic function
3	$f(x) = x^3$	$f(x) = ax^3 + bx^2 + cx + d$ where $a \in \mathbb{R}^*$	cubic function

ACTIVITY 2 Constant functions

Consider the function f given by the rule $y = 3$.



a) Represent this function in the Cartesian plane.

b) Determine

1. dom $f =$ ℝ 2. ran $f =$ {3}
3. the zeros of f if they exist. **No zeros**
4. the y-intercept. **3**
5. the sign of f. **$f(x) \geqslant 0$ over ℝ**
6. the variation of f. **f is a constant function** 7. the extrema of f. **max f = min f = 3**

c) What is the rate of change between two random points on the graph of f? **It is zero.**

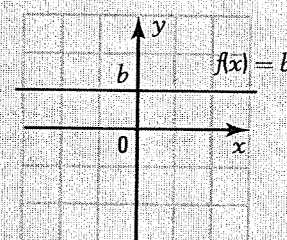
CONSTANT FUNCTIONS

- A **constant function** is a zero degree polynomial function. It is described by a rule of the form:

$$\boxed{f(x) = b},\ b \in \mathbb{R}$$

- The Cartesian graph of a constant function is a horizontal line with the equation $y = b$.

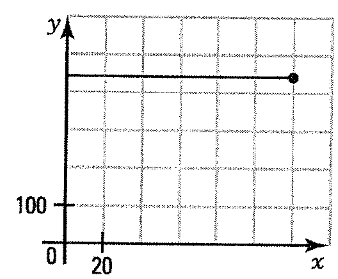
Study of a constant function



– dom $f = \mathbb{R}$
– ran $f = \{b\}$
– The constant function has no zero unless $b = 0$.
– $f(x) > 0$ over ℝ if $b > 0$
  $f(x) < 0$ over ℝ if $b < 0$
– $\max f = \min f = b$

- The rate of change of any constant function is zero.
- A **zero function** is a constant function described by the rule $f(x) = 0$. Its Cartesian graph is represented by the x-axis.

1. A ski resort is open 120 days during the ski season. The cost of a season pass is \$450. Consider the function f which gives the total cost y as a function of the number x of days of skiing.



a) How much does it cost to ski for 12 days? **\$450**

b) What is the rule of function f? **y = 450**

c) Represent function f in the Cartesian plane.

d) Determine

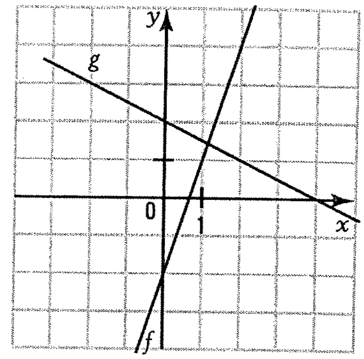
1. dom $f =$ **[0, 120]** 2. ran $f =$ **{450}**

ACTIVITY 3 Linear function – Functional form

Consider the functions $f(x) = 3x - 2$ and $g(x) = -\frac{1}{2}x + 2$.

a) Represent the functions f and g in the Cartesian plane on the right.

b) Study the functions f and g and complete the following table.



	Function f	Function g
Domain	$\mathbb{R}$	$\mathbb{R}$
Range	$\mathbb{R}$	$\mathbb{R}$
Zero	$\frac{2}{3}$	4
Initial value	−2	2
Sign	$f(x) \geq 0$ if $x \in \left[\frac{2}{3}, +\infty\right[$ $f(x) \leq 0$ if $x \in \left]-\infty, \frac{2}{3}\right]$	$f(x) \geq 0$ if $x \in \left]-\infty, 4\right]$ $f(x) \leq 0$ if $x \in [4, +\infty[$
Variation	f is increasing over $\mathbb{R}$	f is decreasing over $\mathbb{R}$

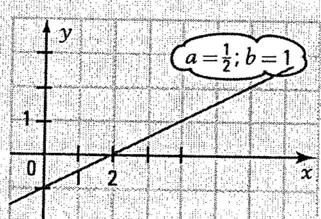
LINEAR FUNCTION – FUNCTIONAL FORM

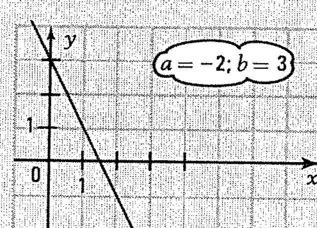
- A **linear function** is a 1st degree polynomial function. It is described by a rule of the form:

  $f(x) = ax + b$ $\quad a \in \mathbb{R}^*$ (function form)

  A linear function represents a situation where the rate of change is **constant**. a represents the rate of change and b represents the initial value (y-intercept).

  Ex.: $f(x) = \frac{1}{2}x - 1$ **Ex.:** $g(x) = -2x + 3$

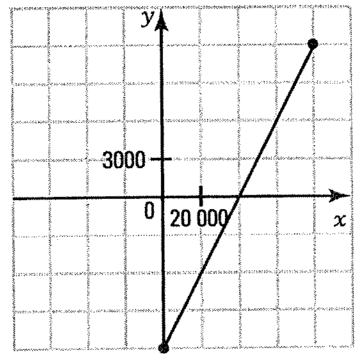




	Function f	Function g
Domain	$\mathbb{R}$	$\mathbb{R}$
Range	$\mathbb{R}$	$\mathbb{R}$
Zero	2	1.5
Initial value	−1	3
Sign	$f(x) \leq 0$ if $x \leq 2$ $f(x) \geq 0$ if $x \geq 2$	$f(x) \geq 0$ if $x \leq 1.5$ $f(x) \leq 0$ if $x \geq 1.5$
Rate of change	$\frac{1}{2}$	−2
Variation	increasing function	decreasing function

- – The function is **increasing** when the rate of change is **positive**.
  – The function is **decreasing** when the rate of change is **negative**.

2. A video game software company establishes that its monthly revenue corresponds to 30% of the amount of sales. The company's fixed monthly operating costs are \$12 000 and the company cannot sell for more than \$80 000 in one month.



a) What is the rule which gives the net revenue y as a function of the amount x of sales? ***y = 0.3 x – 12 000***

b) Represent the function in the Cartesian plane.

c) Determine for this function

1. the domain. ***[0, 80 000]*** 2. the range. ***[–12 000, 12 000]***

d) Determine and interpret

1. the zero of the function. ***\$40 000. Amount of sales to have a zero net revenue.***
2. the initial value of the function. ***\$–12 000. For zero sales, the loss is \$12 000.***

e) 1. What is the rule of the inverse function, which gives the amount x of sales as a function of the net revenue y?

$x = \frac{10}{3}y + 40\ 000$

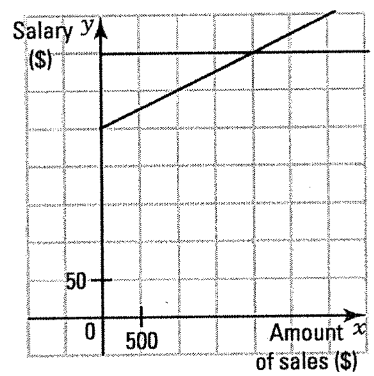
2. What is the amount of sales made by the company if it records a net revenue of \$6000? ***\$60 000***

3. The manager of a store offers two modes of weekly remuneration to the employees.

Mode 1: a fixed salary of \$350.

Mode 2: a base salary of \$250 plus a 5% commission on the amount of sales made.

Let x represent the amount of sales made by an employee during the week and let y represent the weekly salary.



a) Find the rule of the function which gives an employee's weekly salary y as a function of the amount of sales x according to

1. mode 1 ***y = 350*** 2. mode 2 ***y = 0.05x + 250***

b) Represent the two functions in the Cartesian plane.

c) For what amount of weekly sales is the salary the same for both modes of remuneration?
\$2000

d) An employee must choose between modes of remuneration. What mode should he choose?
Mode 1 if he foresees selling less than \$2000, and mode 2 if he foresees selling more than \$2000.

4. A salesman receives a weekly base salary and a commission on his sales. For \$1000 in sales, he receives a salary of \$400 whereas for \$3000 in sales, he receives a salary of \$480.

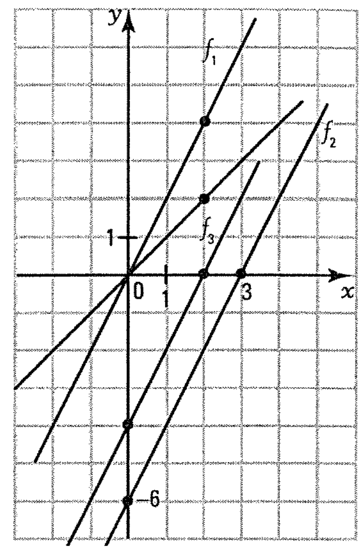
a) What is the rule of the function f which gives the salesman's salary y as a function of the amount of sales x? ***y = 0.04x + 360***

b) Determine

1. the base salary. ***\$360*** 2. the commission. ***4%***

c) Find the rule of the inverse f^{-1}. ***x = 25y – 9000***

ACTIVITY 4 Linear function – Standard form



Consider the basic linear function $y = x$.

a) 1. Draw the graph of the function f_1, image of the graph of the function $y = x$, by the scale change: $(x, y) \to (x, 2y)$.

2. What is the rule of f_1? ***y* = 2*x***

b) 1. Draw the graph of the function f_2, image of the graph of the function f_1, by the horizontal translation: $(x, y) \to (x + 3, y)$.

2. What is the rule of f_2? ***y* = 2(*x* – 3)**

c) 1. Draw the graph of the function f_3, image of the graph of the function f_2, by the vertical translation: $(x, y) \to (x, y + 2)$.

2. What is the rule of f_3? ***y* = 2(*x* – 3) + 2**

d) 1. Explain why f_3 is a linear function

***f_3* is represented by an oblique line.**

2. Write the rule of f_3 in functional form $y = ax + b$.

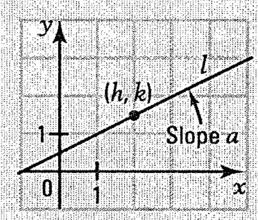
***y* = 2*x* – 4**

3. Identify the rate of change and the initial value of the function f_3.

Rate of change: 2; initial value: –4.

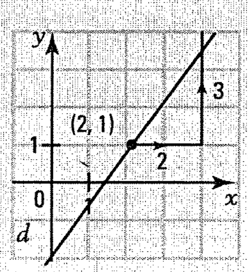
LINEAR FUNCTION – STANDARD FORM

The linear function f represented by a line l with slope a and passing through the point (h, k) is given by the rule:



$$f(x) = a\,(x - h) + k$$ (standard form)

Ex.:

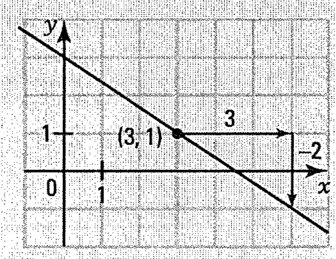


Line passing through (2, 1), with slope $\frac{3}{2}$.

d: $y = \frac{3}{2}(x - 2) + 1$ (standard form)

d: $y = \frac{3}{2}x - 2$ (functional form)

Ex.:



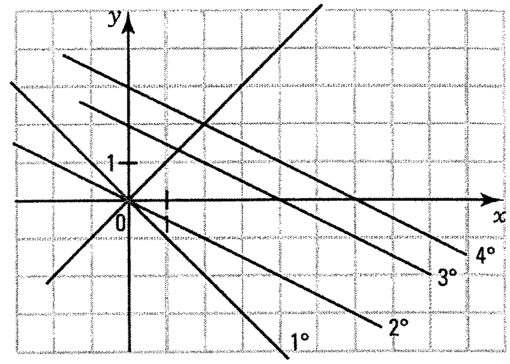
Line passing through (3, 1), with slope $\frac{-2}{3}$.

d: $y = \frac{-2}{3}(x - 3) + 1$ (standard form)

d: $y = \frac{-2}{3}x + 3$ (functional form)

5. Consider the line $y = x$.
This line undergoes a series of transformations, on each preceding image, in the following order.

1. $(x, y) \rightarrow (x, -y)$
2. $(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$
3. $(x, y) \rightarrow (x + 4, y)$
4. $(x, y) \rightarrow (x, y + 1)$



a) Draw the 4 resulting lines.

b) Find the rule of the 4th line in

1. standard form $y = -\frac{1}{2}(x - 4) + 1$
2. functional form $y = -\frac{1}{2}x + 3$

c) For the 4th line, find the

1. x-intercept 6
2. y-intercept 3
3. slope. $-\frac{1}{2}$

6. For each of the following lines, find the equation in
1. standard form
2. functional form

a) The line passing through (–2, 3) with a slope of $\frac{3}{2}$. 1. $y = \frac{3}{2}(x + 2) + 3$ 2. $y = \frac{3}{2}x + 6$

b) The line passing through (3, –1) with a slope of $-\frac{2}{3}$. 1. $y = -\frac{2}{3}(x - 3) - 1$ 2. $y = -\frac{2}{3}x + 1$

7. A line l is represented by the equation $2x + 3y - 6 = 0$.

a) What is the slope of this line? $-\frac{2}{3}$

b) Using a random point on this line, find the equation of this line in standard form.
$A(6, -2) \in l;\ l: y = -\frac{2}{3}(x - 6) - 2$

8. A taxi driver charges a base fee plus \$0.50 per kilometer. For a 10 km ride, the driver charges a total of \$8.
Consider the linear function f which gives the cost y, in dollars, as a function of the distance of the ride x, in kilometers.

a) Find the rule of f in

1. standard form. $y = 0.5(x - 10) + 8$
2. functional form. $y = 0.5x + 3$

b) What is the cost of a 16 km ride? \$11

9. A car consumes 12 l of gas per 100 km. After a 150 km trip, there are 54 litres of gas remaining in the tank.
Consider the linear function f which gives the quantity y of gas remaining in the tank as a function of the distance traveled x since the start of the trip.

a) Find the rule of f in

1. standard form. $y = -0.12(x - 150) + 54$
2. functional form. $y = -0.12x + 72$

b) What is the quantity of gas in the tank at the start of the trip? 72 litres

4.3 Quadratic function – Standard form

ACTIVITY 1 Quadratic function – Standard form

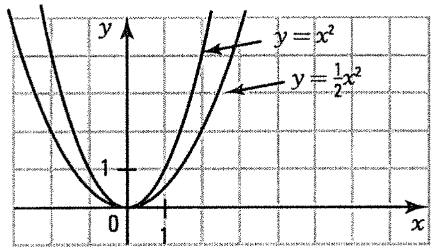
The basic quadratic function $y = x^2$ can be transformed into a quadratic function defined by the rule $y = a(x - h)^2 + k$.

Consider the basic quadratic function $y = x^2$ and the quadratic function $y = \frac{1}{2}(x - 3)^2 - 2$.

a) Identify the parameters a, h and k. ***$a = \frac{1}{2}$, $h = 3$; $k = -2$***

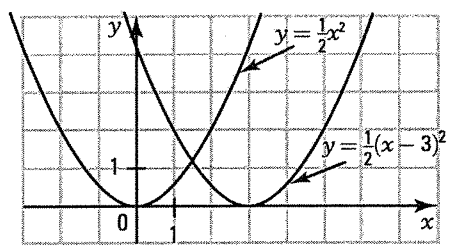
b) Inserting the parameters a, h and k into the rule of the basic quadratic function causes a series of transformations on the graph.

Complete the description of this process.



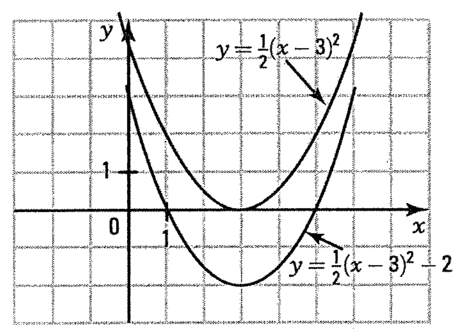
1. From the graph of $y = x^2$, we obtain the graph of $y = \frac{1}{2}x^2$ by the transformation $(x, y) \rightarrow$ ***$\left(x, \frac{1}{2}y\right)$***

   The parameter a causes a ***vertical reduction of factor $\frac{1}{2}$.***



2. From the graph of $y = \frac{1}{2}x^2$, we obtain the graph of $y = \frac{1}{2}(x - 3)^2$ by the transformation $(x, y) \rightarrow$ ***$(x + 3, y)$***

   The parameter h causes a ***horizontal translation of 3 units to the right.***



3. From the graph of $y = \frac{1}{2}(x - 3)^2$, we obtain the graph of $y = \frac{1}{2}(x - 3)^2 - 2$ by the transformation $(x, y) \rightarrow$ ***$(x, y - 2)$.***

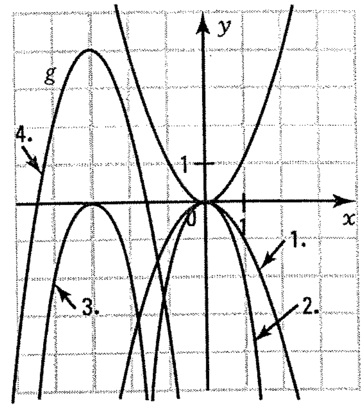
   The parameter k causes a ***vertical translation of 2 units downward.***

c) Verify, using the table of values, that the transformation $(x, y) \rightarrow (x + h, ay + k)$ directly applies the graph of $y = x^2$ onto the graph of $y = \frac{1}{2}(x - 3)^2 - 2$.

x	y	x	y
−1	1	2	$-\frac{3}{2}$
0	0	3	−2
1	1	4	$-\frac{3}{2}$

Indeed, (−1, 1) becomes $\left(2, -\frac{3}{2}\right)$; (0, 0) becomes (3, −2) and (1, 1) becomes $\left(4, -\frac{3}{2}\right)$ by the transformation $(x, y) \rightarrow \left(x + 3, \frac{1}{2}y - 2\right)$.

d) Consider the function $g(x) = -2(x + 3)^2 + 4$.



1. Draw the graph of this function from the graph of the basic function. Explain your procedure.
   1. We draw the curve $y = -x^2$ by symmetry over the x-axis.
   2. We draw the curve $y = -2x^2$ by a vertical stretch of factor 2.
   3. We translate the curve horizontally 3 units to the left.
   4. We translate the curve vertically 4 units upward.
2. Complete the table of values on the right to verify the graph of g.
3. What are the coordinates of the vertex?
   V(−3, 4)

x	−5	−4	−3	−2	−1
y	***−4***	***2***	***4***	***2***	***−4***

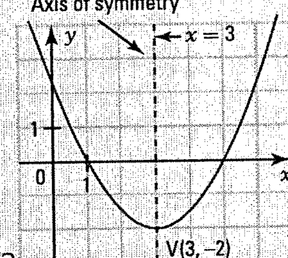
e) Given the parabola with equation $y = a(x - h)^2 + k$, answer true or false.
1. The parabola is open upward if $a > 0$ and downward if $a < 0$. ***True***
2. The vertex of the parabola is $V(h, k)$. ***True***
3. The parabola's axis of symmetry is the vertical line $x = h$. ***True***

QUADRATIC FUNCTION – STANDARD FORM

- The standard form of the rule of a quadratic function is:

$$f(x) = a(x - h)^2 + k$$

- If $\boldsymbol{a} > 0$, the parabola is open upward. ∪
  If $\boldsymbol{a} < 0$, the parabola is open downward. ∩
- The **vertex** of the parabola is: $V(\boldsymbol{h}, \boldsymbol{k})$.
- The parabola's **axis of symmetry** is the vertical line passing through the parabola's vertex. Its equation is: $\boldsymbol{x} = \boldsymbol{h}$.

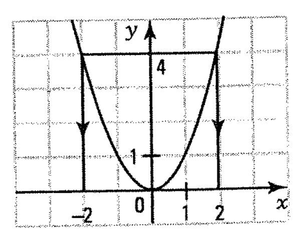


Ex.: $y = \frac{1}{2}(x - 3)^2 - 2$

x	1	2	3	4	5
y	0	−1.5	−2	−1.5	0

ACTIVITY 2 Finding the zeros – Standard form

a) Use the graph of the function $y = x^2$ to solve the following equations.



1. $x^2 = 4$ ***S = {−2, 2}***
2. $x^2 = 1$ ***S = {−1, 1}***
3. $x^2 = 0$ ***S = {0}***
4. $x^2 = -1$ ***S = ∅***

b) Find the solution set of the equation $x^2 = k$ if

1. $k > 0$. ***S = $\{-\sqrt{k}, \sqrt{k}\}$*** 2. $k = 0$. ***S = {0}*** 3. $k < 0$. ***S = ∅***

c) Justify the steps of the solution to the equation $2(x+1)^2 - 18 = 0$.

$2(x+1)^2 - 18 = 0$	
$\Leftrightarrow 2(x+1)^2 = 18$	***Add 18 to each side.***
$\Leftrightarrow (x+1)^2 = 9$	***Divide each side by 2.***
$\Leftrightarrow x+1 = -3$ or $x+1 = 3$	***Two opposite numbers have the same square.***
$\Leftrightarrow x = -4$ or $x = 2$	***Solve each 1st degree equation.***
Thus, $S = \{-4, 2\}$.	***Establish the solution set.***

d) 1. Justify the steps to finding the zeros of the quadratic function $f(x) = a(x-h)^2 + k$.

$a(x-h)^2 + k = 0$	
$\Leftrightarrow a(x-h)^2 = -k$	***Subtract k from each side.***
$\Leftrightarrow (x-h)^2 = -\frac{k}{a}$	***Divide each side by a***
$\Leftrightarrow x - h = -\sqrt{\frac{-k}{a}}$ or $x - h = \sqrt{\frac{-k}{a}}$ if $-\frac{k}{a} > 0$	***Two opposite numbers have the same square.***
$\Leftrightarrow x = h - \sqrt{\frac{-k}{a}}$ or $x = h + \sqrt{\frac{-k}{a}}$	***Isolate x in each equation.***

2. Indicate the number of zeros when

1) $-\frac{k}{a} > 0$. ***2 zeros*** 2) $-\frac{k}{a} = 0$. ***1 zero*** 3) $-\frac{k}{a} < 0$. ***No zeros***

3. What are the zeros when

1) $-\frac{k}{a} > 0$. ***$h - \sqrt{-\frac{k}{a}}$ and $h + \sqrt{\frac{-k}{a}}$*** 2) $-\frac{k}{a} = 0$. ***h***

e) Find the zeros of the following quadratic functions.

1. $f(x) = 2\left(x - \frac{1}{2}\right)^2 - 8$ ***$-\frac{3}{2}$ and $\frac{5}{2}$***
2. $f(x) = 2(x-1)^2 - 10$ ***$1 - \sqrt{5}$ and $1 + \sqrt{5}$***
3. $f(x) = -2(x+3)^2$ ***−3***
4. $f(x) = 3(x-1)^2 + 6$ ***No zero***

FINDING THE ZEROS – STANDARD FORM

- The number of zeros of the quadratic function $f(x) = a(x-h)^2 + k$ depends on the sign of $-\frac{k}{a}$.
- $-\frac{k}{a} > 0$: There are two zeros x_1 and x_2.

$$x_1 = h - \sqrt{-\frac{k}{a}} \text{ and } x_2 = h + \sqrt{-\frac{k}{a}}$$

Ex.: $f(x) = 2(x-1)^2 - 8$
$a = 2;\ h = 1;\ k = -8;\ -\frac{k}{a} = 4$
$x_1 = 1 - \sqrt{4} = -1$ and $x_2 = 1 + \sqrt{4} = 3$
The zeros are −1 and 3.

- $-\frac{k}{a} = 0$: There is only one zero or the two zeros x_1 and x_2 are equal.

$$x_1 = x_2 = h$$

Ex.: $f(x) = 2(x-1)^2$
$a = 2;\ h = 1;\ k = 0;\ \frac{-k}{a} = 0$
$x_1 = x_2 = 1$
The only zero is 1.

- $-\frac{k}{a} < 0$: There are no zeros.

Ex.: $f(x) = 2(x-1)^2 + 8$
$a = 2;\ h = 1;\ k = 8;\ -\frac{k}{a} = -4$
There is no zero since $-\frac{k}{a} < 0$.

- Note that the function has no zeros when a and k have the same sign.

1. Find the zeros of the following quadratic functions.

a) $f(x) = -4(x+2)^2 + 16$ ***-4 and 0***

b) $f(x) = \frac{1}{2}(x+3)^2 - 2$ ***-5 and -1***

c) $f(x) = 2(x+1)^2 - 10$ ***$-1 - \sqrt{5}$ and $-1 + \sqrt{5}$***

d) $f(x) = (x-1)^2 - 7$ ***$1 - \sqrt{7}$ and $-1 + \sqrt{7}$***

e) $f(x) = -2(x+3)^2$ ***-3***

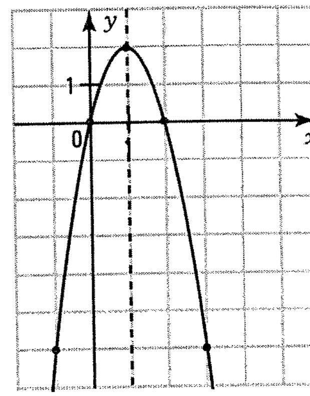
f) $f(x) = 3(x-2)^2 - 27$ ***-1 and 5***

g) $f(x) = 3(x-1)^2 + 6$ ***None***

h) $f(x) = -(x+1)^2$ ***-1***

ACTIVITY 3 Graphing a parabola – Standard form

Consider the function $f(x) = -2(x-1)^2 + 2$.



a) Use the following procedure to graph the quadratic function.

1. Identify the parameters a, h and k. ***$a = -2$; $h = 1$; $k = 2$***
2. Is the parabola open upward or downward? Justify your answer. ***downward since $a < 0$***
3. What are the coordinates of the parabola's vertex? ***V(1, 2)***
4. Find, if they exist, the points where the parabola intersects the x-axis, in other words the zeros of the function.
   $-\frac{k}{a} > 0$. There are two zeros: $x_1 = 0$ and $x_2 = 2$.
5. Complete the following table of values.
6. Graph the parabola.

x	−1	0	1	2	3
y	***-6***	***0***	***2***	***0***	***-6***

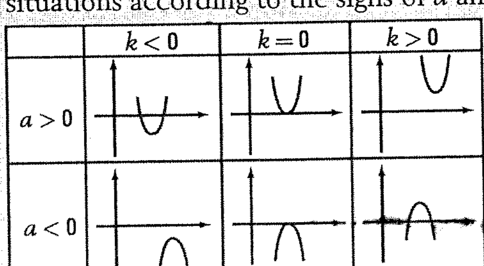
b) Find the y-intercept, in other words the initial value of the function. ***0***

c) Draw the parabola's axis of symmetry and find its equation. ***x = 1***

GRAPHING A PARABOLA – STANDARD FORM

Procedure

1. Identify the parameters a, h and k.
2. Determine the opening according to
3. Determine the coordinates of the vertex. $V(h, k)$.
4. Find, if they exist, the zeros.
   $x_1 = h - \sqrt{\frac{-k}{a}}$, $x_2 = h + \sqrt{\frac{-k}{a}}$
5. Find the y-intercept.
6. Complete a table of values.
7. Graph the parabola. We observe 6 possible situations according to the signs of a and k.

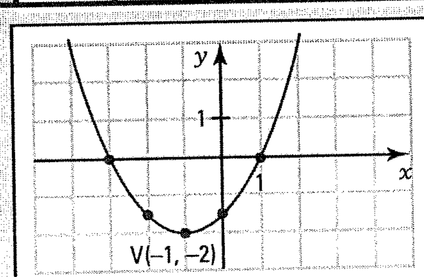


Ex.: $f(x) = \frac{1}{2}(x+1)^2 - 2$

1. $a = \frac{1}{2}$; $h = -1$; $k = -2$
2. Open upward since the sign of $a > 0$.
3. $V(-1, -2)$
4. $x_1 = -3$ and $x_2 = 1$ are the zeros of f.
5. $f(0) = -1.5$. The y-intercept is -1.5.
6.

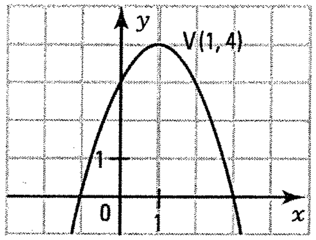
x	−3	−2	−1	0	1
y	0	−1.5	−2	−1.5	0

7.

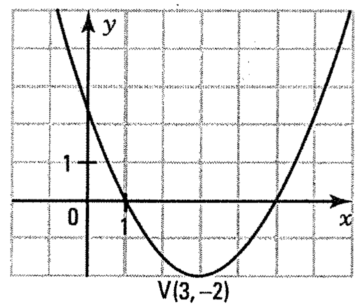


2. Graph the following parabolas.

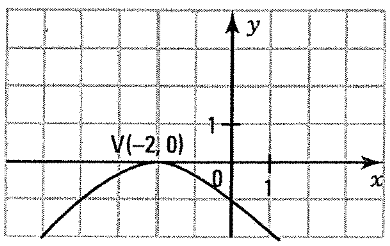
a) $y = -(x - 1)^2 + 4$



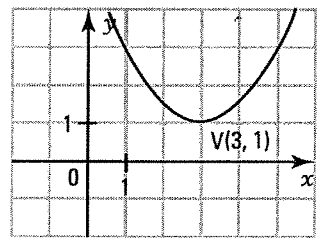
b) $y = \frac{1}{2}(x - 3)^2 - 2$



c) $y = -\frac{1}{4}(x + 2)^2$



d) $y = \frac{1}{2}(x - 3)^2 + 1$



4.4 Quadratic function – General form

ACTIVITY 1 Quadratic function – General form

a) Consider the quadratic function with the rule (standard form) $f(x) = 2(x-1)^2 - 3$.

1. What are the coordinates of the parabola's vertex? ***V(1, -3)***
2. Expand the rule to obtain the form $f(x) = ax^2 + bx + c$ called **general form.**
   $f(x) = 2x^2 - 4x - 1$
3. Identify the coefficients a, b and c. ***$a = 2$, $b = -4$, $c = -1$***
4. Verify that the x-coordinate h of the vertex is equal to $-\frac{b}{2a}$ and that the y-coordinate k of the vertex is equal to $\frac{4ac-b^2}{4a}$.
   $h = 1$ and $-\frac{b}{2a} = 1$; $k = -3$ and $\frac{4ac-b^2}{4a} = -3$

b) Given $f(x) = a(x-h)^2 + k$ (standard form). When this rule is expanded we get:
$f(x) = ax^2 - 2ahx + ah^2 + k$.
By letting $b = -2ah$ and $c = ah^2 + k$, we get the general form: $f(x) = ax^2 + bx + c$.

1. Show that $h = -\frac{b}{2a}$. ***Since $b = -2ah$ then $h = -\frac{b}{2a}$***
2. Show that $k = \frac{4ac-b^2}{4a}$. ***Since $c = ah^2 + k$ then $k = c - ah^2 = c - a\left(\frac{-b}{2a}\right)^2 = c - \frac{ab^2}{4a^2}$***
   $= \frac{4a^2c - ab^2}{4a^2} = \frac{4ac - b^2}{4a}$ $(a \neq 0)$

c) Consider the quadratic function $f(x) = -2x^2 + 4x + 6$ (general form).

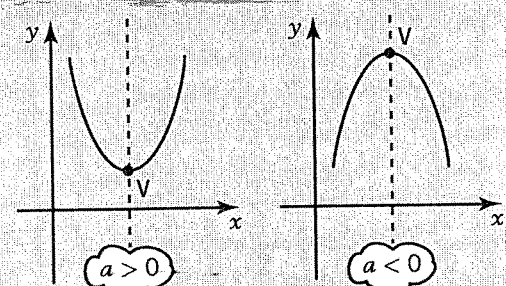
1. Identify the coefficients a, b and c. ***$a = -2$; $b = 4$; $c = 6$***
2. What are the coordinates of the parabola's vertex? ***$-\frac{b}{2a} = 1$; $k = \frac{4ac-b^2}{4a} = 8$; V(1, 8)***

QUADRATIC FUNCTION – GENERAL FORM

- The general form of a quadratic function's rule is:

$$f(x) = ax^2 + bx + c$$

- The vertex of the parabola is $V\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$ or $V\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$ where $\Delta = b^2 - 4ac$.
- The parabola's axis of symmetry has the equation: $x = -\frac{b}{2a}$.



1. For each of the following parabolas, find the coordinates of the vertex and the equation of the axis of symmetry.

a) $y = 2x^2 + 8x + 2$ ***V(-2, -6); x = -2***

b) $y = -3x^2 + 6x - 6$ ***V(1, -3); x = 1***

c) $y = 2x^2 - 3x$ ***$V\left(\frac{3}{4}, \frac{-9}{8}\right)$; $x = \frac{3}{4}$***

d) $y = -2x^2 + 6$ ***V(0, 6); x = 0***

ACTIVITY 2 Finding the zeros – General form

Consider the quadratic function $f(x) = 2x^2 - 7x + 3$ (general form).

a) What equation must we solve to determine the zeros of the function? ***$2x^2 - 7x + 3 = 0$***

b) Solve this equation to determine the zeros. ***$a = 2$; $b = -7$; $c = 3$; $\Delta = 25$; $x_1 = \frac{1}{2}$ and $x_2 = 3$***

FINDING THE ZEROS – GENERAL FORM

Consider the function $f(x) = ax^2 + bx + c$.
To determine the zeros of function f, solve the quadratic equation $ax^2 + bx + c = 0$.

Ex.: To find the zeros of $f(x) = 2x^2 - 5x - 12$.
Solve $2x^2 - 5x - 12 = 0$.
We get the zeros $\frac{-3}{2}$ and 4.

ACTIVITY 3 Graphing a parabola – General form

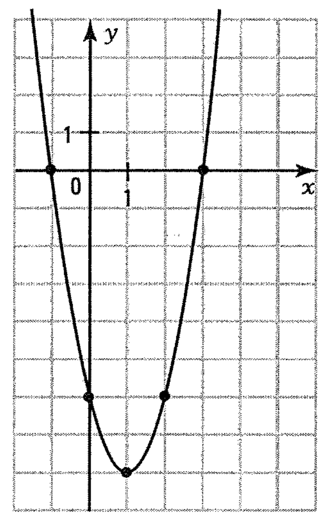
Consider the function $f(x) = 2x^2 - 4x - 6$.

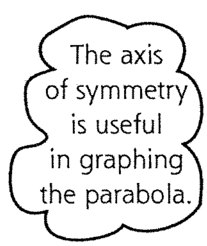
Use the following procedure to graph the function.

1. Identify the parameters a, b and c. ***$a = 2$; $b = -4$; $c = -6$***
2. Is the parabola open upward or downward? ***Upward since > 0***
3. What are the coordinates of the vertex? ***V(1, −8)***
4. Find, if they exist, the zeros of the function.
   $\Delta = 64$. There are two zeros: $x_1 = -1$ and $x_2 = 3$.
5. What is the y-intercept? ***−6***
6. Complete the following table of values.

x	−1	0	1	2	3
y	***0***	***−6***	***−8***	***−6***	***0***

7. Graph the parabola.



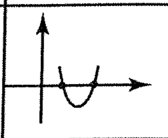
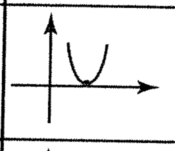
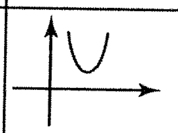
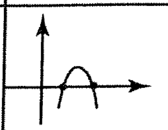
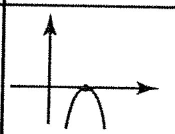
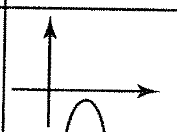


GRAPHING A PARABOLA — GENERAL FORM

Procedure

1. Identify the parameters a, b and c.
2. Determine the opening according to the sign of a.
3. Determine the coordinates of the vertex V.
   $V = \left(-\frac{b}{2a}, -\frac{\Delta}{2a}\right)$ where $\Delta = b^2 - 4ac$.
4. Find the zeros.
   $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$; $x_2 = \frac{-b+\sqrt{\Delta}}{2a}$
5. Find the y-intercept.
6. Complete a table of values.
7. Graph the parabola.

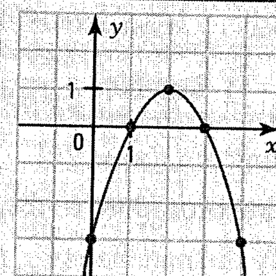
We observe 6 possible situations according to the signs of a and Δ.

	$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
$a > 0$			
$a < 0$			

Ex.: $f(x) = -x^2 + 4x - 3$

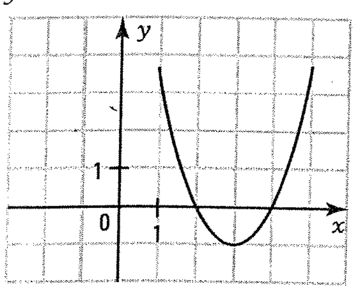
1. $a = -1$, $b = 4$, $c = -3$
2. Open downward since $a < 0$
3. V(2, 1).
4. $\Delta = 4$. There are two zeros: $x_1 = 1$ and $x_2 = 3$.
5. $f(0) = -3$.
6.

x	0	1	2	3	4
y	−3	0	1	0	−3

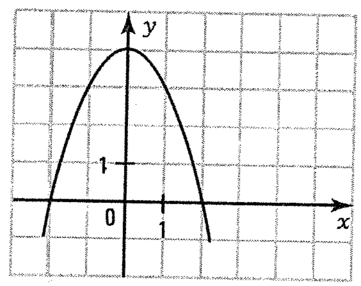
7. 

2. Graph the following parabolas.

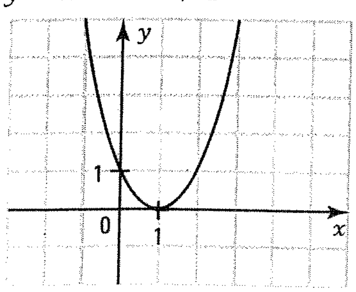
a) $y = x^2 - 6x + 8$



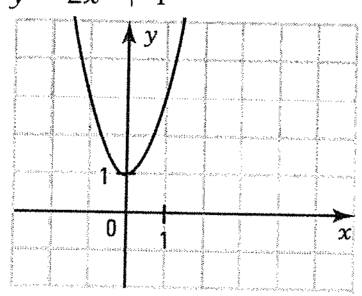
b) $y = -x^2 + 4$



c) $y = x^2 - 2x + 1$



d) $y = 2x^2 + 1$



4.5 Quadratic function – Factored form

ACTIVITY 1 Quadratic functions – Factored form

Consider the quadratic function $f(x) = 2x^2 - 7x + 3$.

a) What is the value of parameter a? $a = 2$

b) Determine the zeros x_1 and x_2 of the function. $x_1 = \frac{1}{2}$ and $x_2 = 3$

c) The factored form of the quadratic function is $f(x) = a\,(x - x_1)\,(x - x_2)$.

Determine the factored form of $f(x) = 2x^2 - 7x + 3$. $f(x) = 2\left(x - \frac{1}{2}\right)(x - 3)$

d) Expand the factored form to get back to the general form.

$2\left(x - \frac{1}{2}\right)(x - 3) = (2x - 1)(x - 3) = 2x^2 - 7x + 3$

QUADRATIC FUNCTION – FACTORED FORM

- Given the general form of a quadratic function $f(x) = ax^2 + bx + c$ with zeros x_1 and x_2. The factored form of the quadratic function is:

$$f(x) = a(x - x_1)(x - x_2)$$

Ex.: – $f(x) = -2x^2 + 5x - 3$ yields the zeros: $x_1 = \frac{3}{2}$ and $x_2 = 1$.

The factored form of f is $f(x) = -2\left(x - \frac{3}{2}\right)(x - 1)$.

– $f(x) = 4x^2 - 12x + 9$ yields only one zero or two equal zeros: $x_1 = x_2 = \frac{3}{2}$.

The factored form of f is $f(x) = 4\left(x - \frac{3}{2}\right)\left(x - \frac{3}{2}\right) = 4\left(x - \frac{3}{2}\right)^2$.

1. In each of the following cases, determine the factored form of the function.

a) $f(x) = 3x^2 - 5x - 2$ $f(x) = 3\left(x + \frac{1}{3}\right)(x - 2)$

b) $f(x) = 2x^2 + 7x + 6$ $f(x) = 2\left(x + \frac{3}{2}\right)(x + 2)$

c) $f(x) = x^2 - 8x + 15$ $f(x) = (x - 3)(x - 5)$

d) $f(x) = -2x^2 + x + 3$ $f(x) = -2\left(x - \frac{3}{2}\right)(x + 1)$

e) $f(x) = 4x^2 - 4x + 1$ $f(x) = 4\left(x - \frac{1}{2}\right)^2$

2. Consider the three forms of a quadratic function: $f(x) = a(x - h)^2 + k$ (standard form), $f(x) = ax^2 + bx + c$ (general form) and $f(x) = a(x - x_1)(x - x_2)$ (factored form). For each given form, find the other two forms.

a) $f(x) = 2(x - 1)^2 - 8$

$f(x) = 2x^2 - 4x - 6$ *(general form)*

$f(x) = 2(x + 1)(x - 3)$ *(factored form)*

b) $f(x) = x^2 - 10x + 16$

$f(x) = (x - 5)^2 - 9$ *(standard form)*

$f(x) = (x - 2)(x - 8)$ *(factored form)*

4.6 Study of a quadratic function

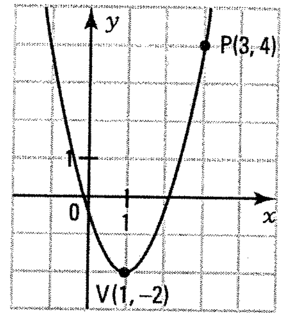
ACTIVITY 1 Finding the rule of a quadratic function

a) Situation: Given the vertex and a point.

The parabola on the right has the vertex V(1, –2) and passes through the point P(3, 4).

The quadratic function represented by this parabola has the rule: $y = a(x - h)^2 + k$ (standard form).

Use the following procedure to determine the rule.

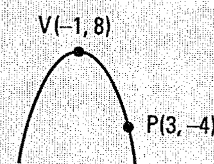


1. Identify h and k. ***h = 1; k = –2***
2. Determine a knowing that the point P(3, 4) verifies the function's rule.
   We have: $y = a(x - 1)^2 - 2;\ 4 = a(3 - 1)^2 - 2;\ a = \frac{3}{2}$
3. What is the rule of this function? $y = \frac{3}{2}(x - 1)^2 - 2$

FINDING THE RULE – GIVEN THE VERTEX AND A POINT

$$y = a(x - h)^2 + k$$

1. Identify h and k.	1. $h = -1, k = 8$ $y = a(x + 1)^2 + 8$
2. Find a after replacing x and y in the rule by the coordinates of the given point P.	2. $-4 = a(3 + 1)^2 + 8$ $-4 = 16a + 8$ $a = \frac{-3}{4}$
3. Deduce the rule.	3. $y = \frac{-3}{4}(x + 1)^2 + 8$

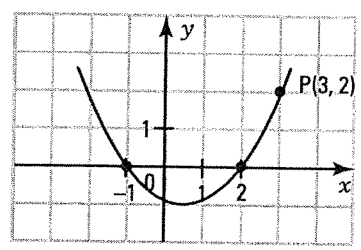


b) Situation: Given the zeros and a point

The parabola on the right has two zeros: –1 and 2 and passes through the point P(3, 2).

The quadratic function represented by this parabola has the rule: $y = a(x - x_1)(x - x_2)$ (factored form).

Use the following procedure to determine the factored form of the rule.



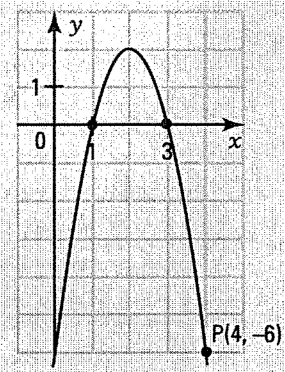
1. Identify x_1 and x_2 $x_1 = -1$ ***and*** $x_2 = 2$
2. Determine a knowing the coordinates of the point (3, 2) verify the rule.
   We have: $y = a(x + 1)(x - 2);\ 2 = a(3 + 1)(3 - 2);\ a = \frac{1}{2}$
3. What is therefore the factored form of the rule? $y = \frac{1}{2}(x + 1)(x - 2)$
4. What is the general form? $y = \frac{1}{2}x^2 - \frac{1}{2}x - 1$

FINDING THE RULE – GIVEN THE ZEROS AND A POINT

$y = a(x - x_1)(x - x_2)$

1. Identify the zeros x_1 and x_2.
2. Determine a after replacing x and y in the rule by the coordinates of the point P.
3. Deduce the rule of the function.

1. $x_1 = 1; x_2 = 3$
   $y = a(x - 1)(x - 3)$
2. $-6 = a(4 - 1)(4 - 3)$
   $-6 = 3a$
   $a = -2$
3. $y = -2(x - 1)(x - 3)$ (factored form)
   $y = -2x^2 + 8x - 6$ (general form)



1. Find the equation of the parabola with the vertex V and passing through the given point P.

a) V(–1, 4) and P(2, –2) $y = -\frac{2}{3}(x + 1)^2 + 4$

b) V(0, 0) and P(–1, 2) $y = 2x^2$

c) V(2, 0) and P(1, 4) $y = 4(x - 2)^2$

d) V(0, –1) and P(2, 1) $y = \frac{1}{2}x^2 - 1$

2. Find the rule, in general form, of each of the following quadratic functions.

a) A function with –5 and 2 as zeros and passing through the point P(3, 16).
$y = 2x^2 + 6x - 20$

b) A function with –3 and –1 as zeros and an initial value of –6.
$y = -2x^2 - 8x - 6$

c) A function with the unique zero –2 and passing through the point P(–1, 3).
$y = 3x^2 + 12x + 12$

d) A function with the vertex V(–1, 4) and passing through the point P(2, –5).
$y = -x^2 - 2x + 3$

e) A function with the vertex V(1, –8) and one of the zeros equal to 3.
$y = 2x^2 - 4x - 6$

3. What is the vertex of the parabola that has –2 and 4 for zeros and passes through the point A(5, 21)?
$y = 3(x + 2)(x - 4)$; V(1, –27)

4. A parabola with zeros –1 and 3 passes through the point A(2, 6). What is the y-coordinate of the point B on the parabola that has an x-coordinate of 4?
$y = -2(x + 1)(x - 3)$; B(4, –10). ***The y-coordinate of point B is –10.***

5. A parabola with zeros –3 and 4 passes through the point A(2, –20). What are the points on this parabola that have a y-coordinate equal to 16?
$y = 2(x + 3)(x - 4)$; $P_1(-4, 16)$ and $P_2(5, 16)$

6. What is the y-intercept of the parabola with zeros –3 and –1 and passing through the point A(–2, 2)?

$y = -2(x + 3)(x + 1)$; the y-intercept is equal to –6.

7. What is the equation of the axis of symmetry of a parabola with zeros –3 and 4?

$x = \frac{1}{2}$

8. The table of values on the right gives the coordinates of different points on a parabola. What is the equation of this parabola?

x	y
0	5
1	8
3	8
5	0

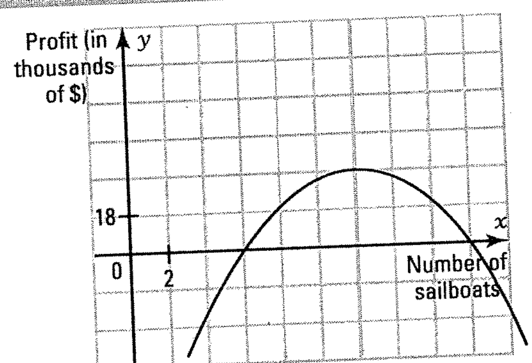
Axis of symmetry: $x = 2$; The zeros are –1 and 5.

$y = -(x + 1)(x - 5)$; $y = -x^2 + 4x + 5$

ACTIVITY 2 Sign of a quadratic function

The sales manager of a company making sailboats has established that the profit $f(x)$, in thousands of dollars, resulting from the sale of x sailboats in a month is represented by the quadratic function defined by the rule $f(x) = -(x - 12)^2 + 36$.

This function f is represented on the right.



1. Determine and interpret the zeros of f. ***6 and 18***

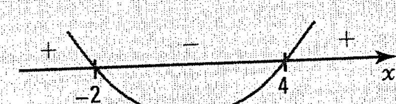
   The profit is zero when the company sells 6 or 18 sailboats during the month.

2. What must the number x of sailboats sold be during the month for the profit to be strictly
   1) positive? ***$6 < x < 18$***
   2) negative? ***$x \leqslant 6$ or $x \geqslant 18$***

SIGN OF A QUADRATIC FUNCTION

- To determine the sign of a quadratic function,
  1. find the zeros (if they exist) of the function.
  2. sketch a graph of the function.
  3. deduce the sign of the function.
- The number line method (page 21) also enables you to determine the sign of a quadratic function.

Ex.: $f(x) = 4(x - 1)^2 - 36$



$f(x) \geqslant 0$ if $x \in]-\infty, -2] \cup [4, +\infty[$

$f(x) \leqslant 0$ if $x \in [-2, 4]$

9. Determine the sign of the following quadratic functions.

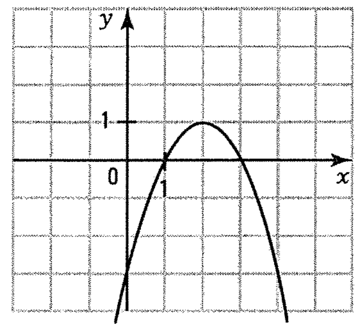
a) $f(x) = (x + 1)^2 - 16$ $f(x) \geq 0$ if $x \in]-\infty, -5] \cup [3, +\infty[$; $f(x) \leq 0$, if $x \in [-5, 3]$

b) $f(x) = -2x^2 + 7x - 6$ $f(x) \geq 0$ if $x \in \left[\frac{3}{2}, 2\right]$; $f(x) \leq 0$ if $x \in \left]-\infty, \frac{3}{2}\right] \cup [2, +\infty[$

c) $f(x) = x^2 - 2x + 1$ $f(x) \geq 0, \forall x \in \mathbb{R}$

d) $f(x) = -4x^2 + 4x - 1$ $f(x) \leq 0, \forall x \in \mathbb{R}$

10. The function $f(x) = -x^2 + 4x - 3$ is represented on the right.



a) What is the domain of f? $\text{dom } f = \mathbb{R}$

b) What is the range of f? $\text{ran } f =]-\infty, 1]$

c) What are the zeros of f? 1 and 3

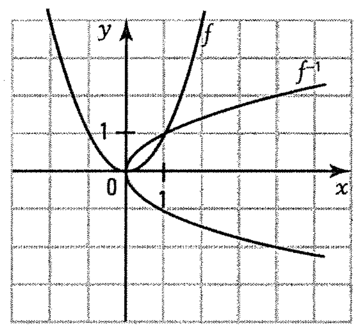
d) What is the initial value of f? -3

e) Study the sign of f.
$f(x) \leq 0$ if $x \in]-\infty, 1] \cup [3, +\infty[$; $f(x) \geq 0$ if $x \in [1, 3]$

f) Study the variation of f. $f \nearrow$ if $x \in]-\infty, 2]$; $f \searrow$ if $x \in [2, +\infty[$

g) Find the extrema of f. $\max f = 1$

11. The function $f(x) = x^2$ is represented on the right.



a) Explain why the inverse f^{-1} is not a function.
There exists a horizontal line, (ex.: the line $y = 4$) that intersects the graph of f at 2 points (the points (−2, 4) and (2, 4))

b) Graph the inverse f^{-1}.

12. Determine the domain and range of the following functions.

a) $f(x) = -x^2 + 4x + 5$ $\text{Dom } f = \mathbb{R}$; $\text{ran } f =]-\infty, 9]$

b) $f(x) = x^2 + 2x - 15$ $\text{Dom } f = \mathbb{R}$; $\text{ran } f = [-16, +\infty[$

13. Study the variation of the following functions.

a) $f(x) = x^2 - x - 6$ $f \searrow$ if $x \in \left]-\infty, \frac{1}{2}\right]$ and $f \nearrow$ if $x \in \left[\frac{1}{2}, +\infty\right[$

b) $f(x) = -2x^2 + 3x - 1$ $f \nearrow$ if $x \in \left]-\infty, \frac{3}{4}\right]$ and $f \searrow$ if $x \in \left[\frac{3}{4}, +\infty\right[$

14. What are the zeros of the function $y = -3x^2 + 11x - 6$? $\frac{2}{3}$ and 3

15. Find the values of x for which $f(x) = x^2 + 5x - 14$ is positive. $]-\infty, -7] \cup [2, +\infty[$

16. What is the range of the function $f(x) = -x^2 + 2x + 15$? $\text{Ran } f =]-\infty, 16]$

17. What is the y-intercept of $y = 3x^2 - 2x + 5$? 5

18. Find the extrema and its nature (maximum or minimum) for $y = -x^2 - 2x + 3$.

A maximum; 4

19. What is the equation of the axis of symmetry for the parabola $y = -2x^2 + 5x - 3$?

The line with equation $x = \frac{5}{4}$

20. For what values of x is the function $f(x) = 2x^2 - x - 6$ decreasing? $x \in \left]-\infty, \frac{1}{4}\right]$

21. A stone is thrown upward from the top of a bridge located above the St. Laurence river. The polynomial $h(t) = -5t^2 + 20t + 60$ gives the height h (in m) of the stone as a function of time t (in sec) since it was thrown.
The polynomial $v(t) = -10t + 20$ gives the velocity v(in m/s) of the stone at the instant t.

a) Determine

1. the height of the bridge. ***h(0) = 60 m***
2. the initial velocity of the stone. ***v(0) = 20 m/s***

b) Determine the velocity of the stone at the instant when it hits the water.

h(t) = 0 ⇔ −5t² + 20t + 60 = 0 ⇔ t = −2 or t = 6. At the instant t = 6 s the stone hits the water.

v(6) = −40 m/s. The velocity is equal to 40 m/s.

c) Find the maximum height reached by the stone and verify that the stone's velocity is zero at the instant when it reaches its maximum height.

The maximum height is 80 m and is reached at the instant 2 s. v(2) = 0 m/s

22. The value of a share, in dollars, x weeks after its purchase is given by the rule $y = -0.1x^2 + x + 4.5$. Do you make a profit or a loss if the share is sold two weeks after reaching its maximum value?

Value at purchase: \$4.50; V(5, 7); f(7) = \$6.60. A profit of \$2.10 per share is made.

23. The position $f(t)$, in metres, of a diver relative to the surface is described by the rule $f(t) = 0.5t^2 - 6t + 10$ where t represents elapsed time, in seconds. How long was the diver under water?

f(t) ≤ 0 ⇔ 2 ≤ t ≤ 10. The diver was under water during 8 seconds.

24. The trajectory of a stone thrown from a seaside cliff is a partial parabola. The position $f(t)$, in metres, of the stone relative to sea level is given by $f(t) = -t^2 + 8t + 20$ where t represents elapsed time in seconds since it was thrown. How many seconds after reaching its maximum height will the stone hit the water?

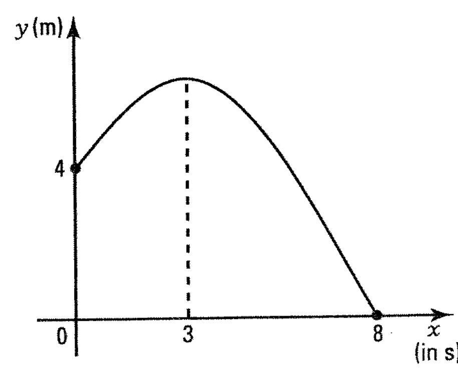
After 6 seconds.

25. A stone is thrown upward from a height of 4 m. After 3 s, it reaches its maximum height and after 8 s, it hits the ground. Its trajectory is parabolic.

1. What is the maximum height reached by the stone?

6.25 m

2. Determine the elapsed time from the moment the stone was at a height of 2.25 m during its descent to the moment it hit the ground. ***1 second***



Evaluation 4

1. Find the rule of the following functions.

a) f is a constant function such that $f(1) = 2$. ***f(x) = 2***

b) f is a linear function such that $f(1) = 2$ and $f(3) = 5$. $f(x) = \frac{3}{2}x + \frac{1}{2}$

c) f is a quadratic function with zeros –3 and 2 and an initial value of –12.
$f(x) = 2x^2 + 2x - 12$

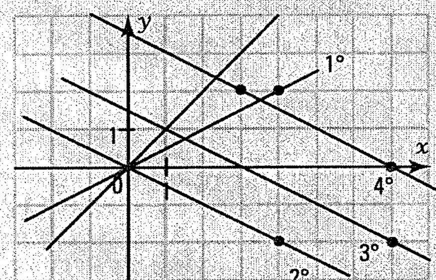
d) f is represented by a parabola with the vertex V(2, 1) and passing through the point P(4, –7).
$f(x) = -2(x - 2)^2 + 1$

2. At the start of a car ride, the gas tank contains 66 litres. After traveling 50 km, the gas tank contains 60 litres. What is the rule of the linear function which gives the remaining quantity y of gas in the tank as a function of the distance traveled x in km? $y = 66 - 0.12x$

3. The line $y = x$ undergoes the following series of transformations, each on the previous image, in the given order:

1. $(x, y) \to \left(x, \frac{1}{2}y\right)$ 2. $(x, y) \to (x, -y)$

3. $(x, y) \to (x + 3, y)$ 4. $(x, y) \to (x, y + 2)$



a) Draw the 4 successive lines.

b) Find the equation of the 4th line in

1. standard form. $y = -\frac{1}{2}(x - 3) + 2$ 2. functional form. $y = -\frac{1}{2}x + \frac{7}{2}$

4. A line l with slope $\frac{3}{2}$ passes through the point A(4, 1). Find the equation of the line l in

a) standard form. $y = \frac{3}{2}(x - 4) + 1$ **b)** functional form. $y = \frac{3}{2}x - 5$

5. A parabola with the vertex V(3, 16) passes through the point A(5, 12). What is its y-intercept?
7

6. A parabola intersects the x-axis at –2 and 4 and passes through the point A(2, –24). Find the coordinates of its vertex. ***V(1, –27)***

7. Determine the domain and range of the following functions.

a) $f(x) = -3(x - 2)^2 + 5$
dom $f = \mathbb{R}$
ran $f =]-\infty, 5]$

b) $f(x) = 2x^2 + 4x - 9$
dom $f = \mathbb{R}$
ran $f = [-11, +\infty[$

8. Find the zeros of the function $f(x) = -3(x + 1)^2 + 12$. $x_1 = -3$ and $x_2 = 1$

9. Determine the initial value of $f(x) = -\frac{1}{2}(x + 4)^2 + 9$. $y = 1$

10. Determine over which interval the function $f(x) = 2x^2 - 5x - 3$ is positive.
$f(x) \geq 0$ over $\left]-\infty, -\frac{1}{2}\right] \cup [3, +\infty[$

11. Determine over which interval the function $f(x) = 3x^2 + 6x - 5$ is increasing. ***[-1, +∞[***

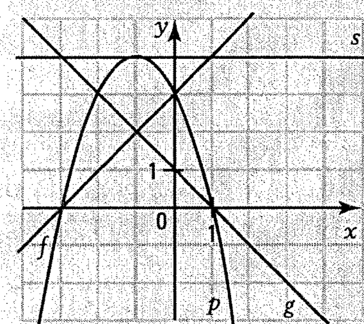
12. Determine the extrema of the function $f(x) = -2x^2 + 12x - 7$. ***max f = 11***

13. What is the axis of symmetry of the function $f(x) = -\frac{1}{4}x^2 + 3x + 1$? ***x = 6***

14. Determine the values of x for which the function $f(x) = -3(x + 4)^2 + 5$ is equal to -7.

x = -6 or x = -2

15. The functions $f(x) = x + 3$ and $g(x) = -x + 1$ are represented on the right.



a) Graph the function s given that $s(x) = f(x) + g(x)$.

s(x) = 4

b) Graph the function p given that $p(x) = f(x) \cdot g(x)$.

p(x) = -x² - 2x + 3

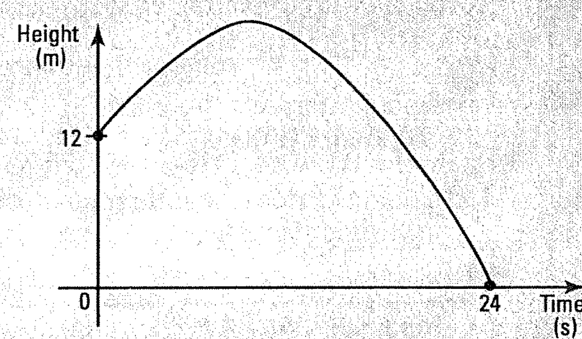
16. A stone is thrown upward from the top of a seaside cliff. The function which gives the stone's height h (in m) above sea level as a function of time t (in sec.) since it was thrown has the rule: $h = -t^2 + 12t + 160$.
Find the interval of time over which the height of the stone is at least 180 m above sea level.

Between the instants t = 2 and t = 10 seconds after it was thrown.

17. The height h, in metres, of a diver relative to the water level is described by the rule $h = \frac{1}{2}t^2 - 6t + 10$ where t represents the elapsed time, in seconds, since the start of the dive. How long did the diver remain underwater?

During 8 seconds.

18. A projectile is thrown upward from a height of 12 m. After 10 seconds, it reaches its maximum height and after 24 seconds, it hits the ground.
Knowing that its trajectory follows the rule of a quadratic function, find the elapsed time between the moment it reaches a height of 6.5 m, on its descent, and the time when it hits the ground



$y = -\frac{1}{8}(x + 4)(x - 24)$.

It reaches, on its descent, a height of 6.5 m at the instant t = 22 sec. The elapsed time is therefore 2 sec.

Chapter 5

Real functions

CHALLENGE 5

- **5.1** Absolute value function
- **5.2** Square root function
- **5.3** Greatest integer function
- **5.4** Rational function

EVALUATION 5