CHALLENGE 5

1. Determine the domain and range of the following functions.

a) $f(x) = -2|x-3| + 1$

dom f = ℝ

ran f =]−∞, 1]

b) $f(x) = -\sqrt{-x+1} + 1$

dom f =]−∞, 1]

ran f =]−∞, 1]

c) $f(x) = \frac{2}{3(x-1)} - 1$

dom f = ℝ \ {1}

ran f = R \ {−1}

2. Determine the set of all zeros of the function $y = 3\left[\frac{1}{2}(x-1)\right] + 6$. [−3, −1[

3. Determine the zeros of the following functions.

a) $f(x) = -2|x-1| + 6$

−2 and 4

b) $f(x) = -2\sqrt{x-3} + 6$

12

c) $f(x) = \frac{-3}{2(x+1)} + 1$

$\frac{1}{2}$

4. What are the equations of the asymptotes of the function $f(x) = \frac{2}{5(x-1)} - 4$?

The lines defined by the equations x = 1 and y = −4.

5. Study the sign of the following functions.

a) $f(x) = 4\left|-\frac{1}{2}(x-1)\right| - 4$ *f(x)* ⩽ 0 if *x* ∈ [−1, 3]; *f(x)* ⩾ 0 if *x* ∈]−∞, −1] ∪ [3, +∞[

b) $f(x) = -2\sqrt{x+3} + 4$ *f(x)* ⩽ 0 if *x* ∈ [1, +∞[; *f(x)* ⩾ 0 if *x* ∈ [−3, 1]

c) $f(x) = \frac{4}{x-3} + 2$ *f(x)* ⩽ 0 if *x* ∈ [1, 3[; *f(x)* ⩾ 0 if *x* ∈]−∞, 1] ∪]3, +∞[

6. Describe the variation of the following functions.

a) $f(x) = -\frac{2}{3}|x-2| + 4$ *f* ↗ over]−∞, 2]; *f* ↘ over [2, +∞[

b) $f(x) = -\frac{1}{2}\sqrt{-2(x-1)} + 1$ *f* ↗ over]−∞, 1]

c) $f(x) = \frac{2}{x-1} + 1$ *f* ↘ over ℝ \ {1}

7. Find the rule of

a) an absolute value function whose graph has a vertex at V(−2, 6) and passes through the point A(1, −3). $y = -3|x+2| + 6$

b) a rational function passing through the point A(3, 4) with asymptotes defined by the lines $x = 1$ and $y = 2$.

$y = \frac{4}{x-1} + 2$

c) a square root function whose graph has a vertex at V(−4, −2) and passes through the point A(5, 4). $y = 2\sqrt{x+4} - 2$

5.1 Absolute value function

ACTIVITY 1 Absolute value of a real number

On a winter day, the temperature (in °C), recorded at noon, has an absolute value of 5.

a) What is the recorded temperature that day if the temperature is:

1. above 0°C? *5 °C*
2. below 0°C? *-5 °C*

b) We represent the absolute value of a number x by $|x|$. Determine:

1. $|+10| =$ *10*
2. $|-10| =$ *10*
3. $|0| =$ *0*

c) Is it true to say that two opposite numbers have the same absolute value? *Yes*

ABSOLUTE VALUE OF A REAL NUMBER

The absolute value of a real number a, written $|a|$, is defined by:

$$|a| = \begin{cases} a & \text{if } a \geqslant 0 \\ -a & \text{if } a < 0 \end{cases}$$

Ex.: $|+4| = 4$; $|-3| = 3$; $|0| = 0$.

Note that the absolute value of a real number is never negative.

1. Determine the following absolute values.

a) $|+8| =$ *8* **b)** $|-4.7| =$ *4.7* **c)** $|0| =$ *0* **d)** $|\pi| =$ π

ACTIVITY 2 Properties

Consider a real number a and a non-zero real number b. Answer true or false.

a) $|a| \geqslant 0$ *True*

b) $|a| = |-a|$ *True*

c) $|a + b| = |a| + |b|$ *False*

d) $|a - b| = |a| - |b|$ *False*

e) $|a \cdot b| = |a| \cdot |b|$ *True*

f) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ *True*

PROPERTIES

For any real number a and any real number b, we have the following properties.

• $|a| \geqslant 0$	**Ex.:** $|+5| \geqslant 0$; $|-4| \geqslant 0$
• $|a| = |-a|$	$|4| = |-4|$
• $|a + b| \leqslant |a| + |b|$	$|7 + (-2)| \leqslant |7| + |-2|$
• $|a - b| \geqslant |a| - |b|$	$|5 - (-3)| \geqslant |5| - |-3|$
• $|ab| = |a| \cdot |b|$	$|8 \times (-3)| = |8| \times |-3|$
• $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ $(b \neq 0)$	$\left|\frac{-8}{2}\right| = \frac{|-8|}{|2|}$

2. Complete the following using the appropriate symbol =, >, <.

a) $|x + 5|$ __>__ 0 **b)** $|x - 3|$ __=__ $|3 - x|$ **c)** $|2(x - 1)|$ __=__ $2|x - 1|$

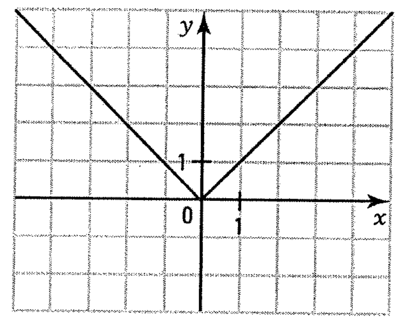
d) $|7 - 12|$ __>__ $|7| - |12|$ **e)** $\left|\frac{x+2}{x-1}\right|$ __=__ $\frac{|x+2|}{|x-1|}$ **f)** $|-6 + 9|$ __<__ $|-6| + |9|$

ACTIVITY 3 Basic absolute value function

Consider the function f defined by the rule $y = |x|$.

a) Complete the following table of values

x	−3	−2	−1	0	1	2	3
y	***3***	***2***	***1***	***0***	***1***	***2***	***3***



b) Represent the function f in the Cartesian plane.

c) Determine

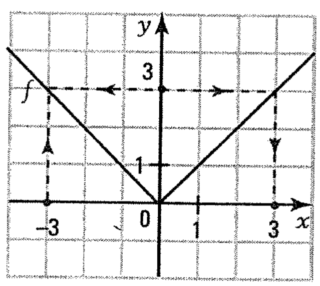
1. dom f. ___$\mathbb{R}$___
2. ran f. ___$\mathbb{R}_+$___
3. the zero of f. ___***0***___
4. the initial value of f. ___***0***___
5. the sign of f. ___***$f(x) \geq 0$ over $\mathbb{R}$.***___
6. the variation of f. ___***$f \nearrow$ over $[0, +\infty[$, $f \searrow$ over $]-\infty, 0]$***___
7. the extrema of f. ___***min $f = 0$***___

d) Using the graph of the function $y = |x|$, solve the equation

1. $|x| = 2$ ___**$S = \{-2, 2\}$**___ 2. $|x| = 0$ ___**$S = \{0\}$**___ 3. $|x| = -1$ ___**$S = \varnothing$**___

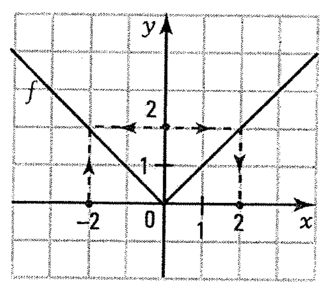
e) Using the graph of the function $y = |x|$, solve the inequality

1. $|x| \leq 3$



___**$S = [-3, 3]$**___

2. $|x| \geq 2$



___**$S =]-\infty, -2] \cup [2, +\infty[$**___

f) What is the solution set to the inequality

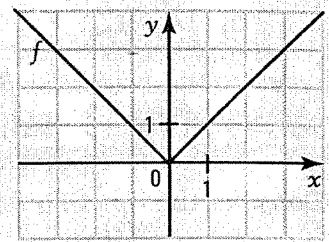
1. $|x| \geq k$ when $k < 0$. ___**$S = \mathbb{R}$**___
2. $|x| \leq k$ when $k < 0$. ___**$S = \varnothing$**___

BASIC ABSOLUTE VALUE FUNCTION

- The function f defined by the rule:

$$f(x) = |x|$$

is called the basic **absolute value** function.



- We have:

dom $f = \mathbb{R}$ **ran** $f = \mathbb{R}_+$

The **zero** of f is 0.   The **initial value** of f is 0.

Sign of f: $f(x) \geqslant 0$ over $\mathbb{R}$.

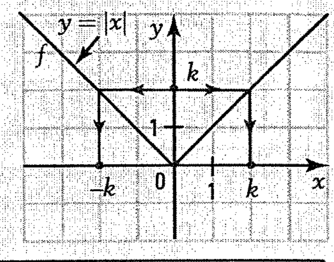
Variation of f: f is increasing over $\mathbb{R}_+$; f is decreasing over $\mathbb{R}_-$.

The function f has a **minimum** of 0.

- The number of solutions to the equation:

$$\boxed{|x| = k}$$

depends on the sign of k.

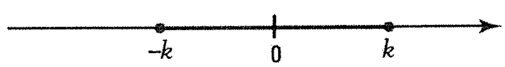


If $k > 0$	If $k = 0$	If $k < 0$
The equation has 2 solutions. $x = -k$ or $x = k$ **Ex.:** $\|x\| = 3$ $S = \{-3, 3\}$	The equation has 1 solution. $x = 0$ **Ex.:** $\|x\| = 0$ $S = \{0\}$	The equation has no solution. **Ex.:** $\|x\| = -5$ $S = \varnothing$

- Given a positive real number k, we have:

$|x| \leqslant k$

$\Leftrightarrow x \geqslant -k$ **and** $x \leqslant k$

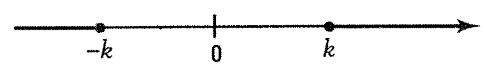


$S = [-k, k]$

Ex.: The inequality $|x| \leqslant 5$ has the solution set: $S = [-5, 5]$.

$|x| \geqslant k$

$\Leftrightarrow x \leqslant -k$ **or** $x \geqslant k$

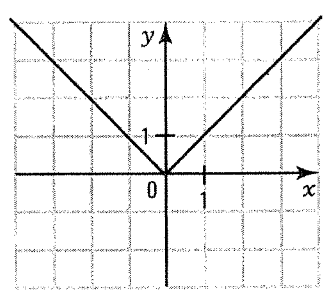


$S =]-\infty, -k] \cup [k, +\infty[$

Ex.: The inequality $|x| \geqslant 5$ has the solution set: $S =]-\infty, -5] \cup [5, +\infty[$.

3. Consider the basic absolute value function $f(x) = |x|$ represented on the right.

Using the graph, solve the following equations or inequalities:



1. $|x| = 1$ **$S = \{-1, 1\}$**
2. $|x| = 0$ **$S = \{0\}$**
3. $|x| = -1$ **$S = \varnothing$**
4. $|x| \leqslant 1$ **$S = [-1, 1]$**
5. $|x| > 3$ **$S =]-\infty, -3[\cup]3, +\infty[$**
6. $|x| > -1$ **$S = \mathbb{R}$**

4. For each of the following inequalities, determine the solution set and represent it on the real number line.

a) $|x| > 10$

$S =]-\infty, -10[\cup]10, +\infty[$

b) $|x| \leqslant 4$

$S = [-4, 4]$

c) $|x| > -3$

$S = \mathbb{R}$

d) $|x| \leq -2$

$S = \varnothing$

e) $|x| \geq 0$

$S = \mathbb{R}$

f) $|x| \leq 0$

$S = \{0\}$

5. Solve the following equations.

a) $|x| = 12$

$S = \{-12, 12\}$

b) $|x| = -8$

$S = \varnothing$

c) $|x + 5| = 0$

$S = \{-5\}$

d) $|2x + 1| = 7$

$S = \{-4, 3\}$

e) $\left|\frac{1}{2}x - 5\right| = 4$

$S = \{2, 18\}$

f) $|6 - x| = -3$

$S = \varnothing$

6. Solve the following equations.

a) $2|x - 5| - 4 = 0$

$S = \{3, 7\}$

b) $-2|3x - 1| + 4 = -6$

$S = \left\{-\frac{4}{3}, 2\right\}$

c) $12 - |6 - 2x| = 3$

$S = \left\{-\frac{3}{2}, \frac{15}{2}\right\}$

d) $|x - 5| + 8 = 2$

$S = \varnothing$

e) $-3|2x + 5| + 6 = 6$

$S = \left\{-\frac{5}{2}\right\}$

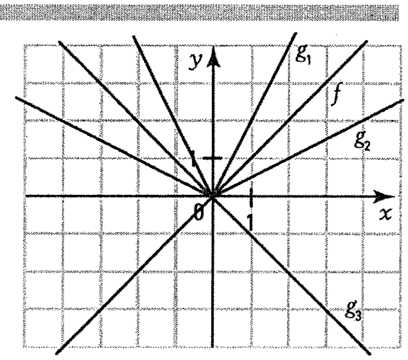
f) $|4x - 5| + 6 = 9$

$S = \left\{\frac{1}{2}, 2\right\}$

ACTIVITY 4 Absolute value function $y = a|b(x - h)| + k$

The basic absolute value function $f(x) = |x|$ can be transformed into an absolute value function with the rule

$$g(x) = a|b(x - h)| + k$$



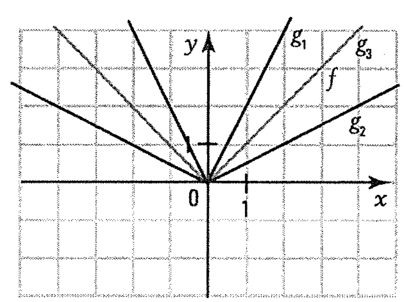
a) Consider the basic absolute value function $f(x) = |x|$ and the absolute value function $g(x) = a|x|$.

Represent, in the same Cartesian plane, the functions $g_1(x) = 2|x|$, $g_2(x) = \frac{1}{2}|x|$ and $g_3(x) = -|x|$ and explain how to deduce the graph of g from the graph of f when

1. $a > 1$: ***by a horizontal reduction.***
2. $0 < a < 1$: ***by a horizontal stretch.***
3. $a = -1$: ***by a reflection about the y-axis.***
4. Complete: From the graph of $f(x) = |x|$, we obtain the graph of $g(x) = a|x|$ by the transformation $(x, y) \rightarrow$ ***(x, ay)***.
5. Is the graph of $g(x) = a|x|$ open upward or downward when

   1) $a > 0$? ***Upward***
   2) $a < 0$? ***Downward***

b) Consider the basic absolute value function $f(x) = |x|$ and the absolute value function $g(x) = |bx|$.



Represent, in the same Cartesian plane, the functions $g_1(x) = |2x|$, $g_2(x) = \left|\frac{1}{2}x\right|$ and $g_3(x) = |-x|$ and explain how to deduce the graph of g from the graph of f when

1. $b > 1$: ***by a horizontal reduction.***

2. $0 < b < 1$: ***by a horizontal stretch.***

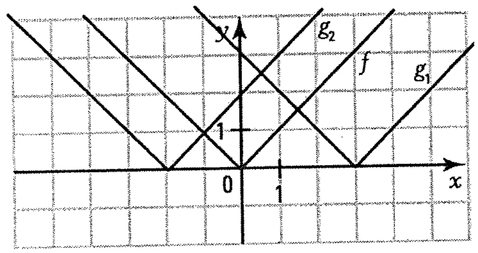
3. $b = -1$: ***by a reflection about the y-axis.***

4. Complete: From the graph of $f(x) = |x|$, we obtain the graph of $g(x) = |bx|$ by the transformation $(x, y) \rightarrow$ ***$\left(\frac{x}{b}, y\right)$***

5. Compare the graphs of the functions $y = 2|x|$ and $y = |2x|$ obtained in a) and b). Justify your answer. ***They are the same. In fact, $|2x| = |2| \cdot |x| = 2|x|$.***

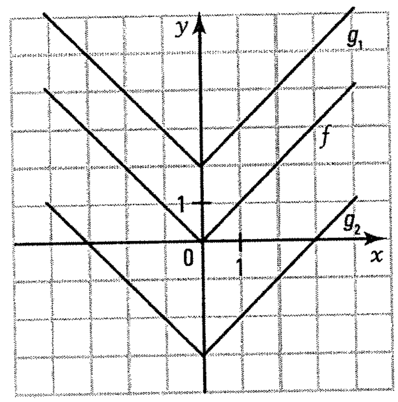
6. Compare the graphs $f(x) = |x|$ and $f(x) = |-x|$. Justify your answer.
They are the same. In fact, $|x| = |-x|$.

c) Consider the basic absolute value function $f(x) = |x|$ and the absolute value function $g(x) = |x - h|$.
Represent, in the same Cartesian plane, the functions $g_1(x) = |x - 3|$ and $g_2(x) = |x + 2|$ and explain how to deduce the graph of g from the graph of f when



1. $h > 0$: ***by a horizontal translation to the right.***
2. $h < 0$: ***by a horizontal translation to the left.***
3. Complete: From the graph of $f(x) = |x|$, we obtain the graph of $g(x) = |x - h|$ by the transformation $(x, y) \rightarrow$ ***$(x + h, y)$***.

d) Consider the basic absolute value function $f(x) = |x|$ and the absolute value function $g(x) = |x| + k$.
Represent, in the same Cartesian plane, the functions $g_1(x) = |x| + 2$ and $g_2(x) = |x| - 3$ and explain how to deduce the graph of g from the graph of f when



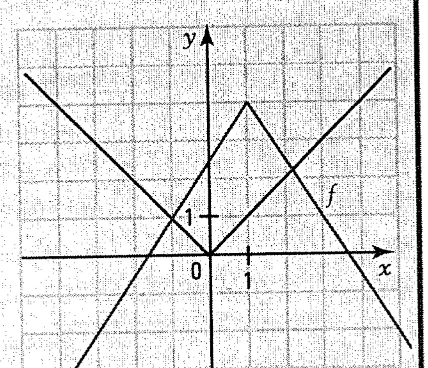
1. $k > 0$: ***by a vertical translation upward.***
2. $k < 0$: ***by a vertical translation downward.***
3. Complete: From the graph of $f(x) = |x|$, we obtain the graph of $g(x) = |x| + k$ by the transformation $(x, y) \rightarrow$ ***$(x, y + k)$.***

ABSOLUTE VALUE FUNCTION $f(x) = a|b(x - h)| + k$

The graph of the function $f(x) = a|b(x - h)| + k$ is deduced from the graph of the basic absolute value function $y = |x|$ by the transformation

$$(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$$

Ex.: The graph of the function $f(x) = -3\left|\frac{1}{2}(x - 1)\right| + 4$ is deduced from the graph of the basic absolute value function $y = |x|$ by the transformation: $(x, y) \rightarrow (2x + 1, -3y + 4)$



7. The following functions have a rule of the form $f(x) = a|b(x-h)| + k$.
$f_1(x) = 3|x|$, $f_2(x) = |2x|$, $f_3(x) = |x+4|$, $f_4(x) = |x| + 1$ and $f_5(x) = 2|3(x-1)| - 4$.

Complete the table on the right by determining, for each function, the parameters a, b, h and k and by giving the rule of the transformation which enables you to obtain the function from the basic absolute value function $g(x) = |x|$.

	a	b	h	k	Rule
$f_1(x) = 3\|x\|$	3	1	0	0	$(x, y) \rightarrow (x, 3y)$
$f_2(x) = \|2x\|$	1	2	0	0	$(x, y) \rightarrow \left(\frac{x}{2}, y\right)$
$f_3(x) = \|x+4\|$	1	1	-4	0	$(x, y) \rightarrow (x-4, y)$
$f_4(x) = \|x\| + 1$	1	1	0	1	$(x, y) \rightarrow (x, y+1)$
$f_5(x) = 2\|3(x-1)\| - 4$	2	3	1	-4	$(x, y) \rightarrow \left(\frac{x}{3}+1, 2y-4\right)$

8. In each of the following cases, we apply a transformation to the basic absolute value function $y = |x|$. Find the rule of the function obtained by applying the given transformation.

a) $(x, y) \rightarrow (x, -y)$ $y = -|x|$

b) $(x, y) \rightarrow (x-2, y+4)$ $y = |x+2| + 4$

c) $(x, y) \rightarrow \left(\frac{x}{2}, y\right)$ $y = |2x|$

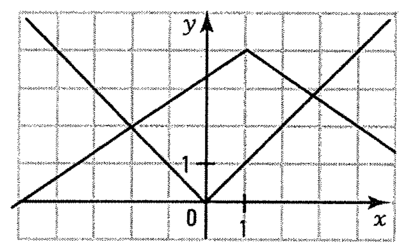
d) $(x, y) \rightarrow (5x, y)$ $y = \left|\frac{x}{5}\right|$

e) $(x, y) \rightarrow (3x, -7y)$ $y = -7\left|\frac{1}{3}x\right|$

f) $(x, y) \rightarrow \left(\frac{x}{3} + 1, 2y - 4\right)$ $y = 2|3(x-1)| - 4$

9. From the basic absolute value function and using the transformation $(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$, represent the function $y = -2\left|\frac{1}{3}(x-1)\right| + 4$ in the Cartesian plane.

For example, (1, 1) → (4, 2)



Activity 5 Graphing the function $f(x) = a|b(x-h)| + k$

Consider the function $f(x) = 4\left|-\frac{1}{2}(x-1)\right| - 4$.

a) Identify the parameters a, b, h and k.

$a = 4$, $b = -\frac{1}{2}$, $h = 1$ and $k = -4$

b) Is the graph open upward or downward? Justify your answer.

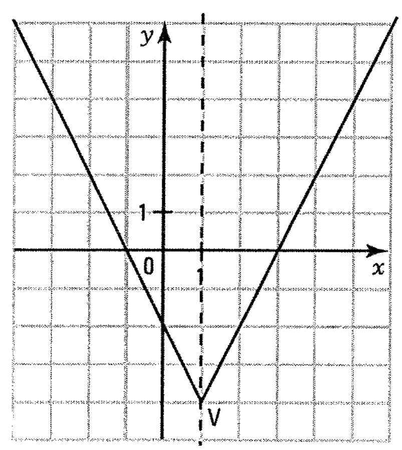
Upward, $a > 0$.

c) What are the coordinates of the vertex? $V(1, -4)$

d) Find the zeros of the function. -1 and 3

e) Represent the function f in the Cartesian plane after completing the following table of values.

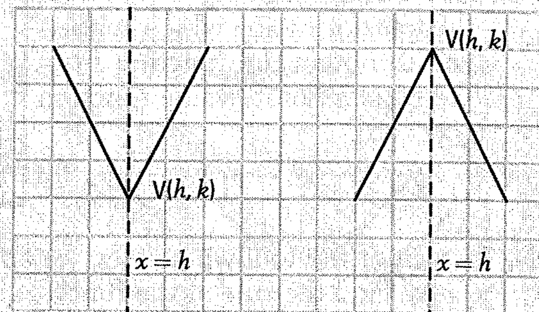
x	−2	1	4
y	2	-4	2



GRAPH OF AN ABSOLUTE VALUE FUNCTION

Consider the **absolute value** function defined by the rule:

$$f(x) = a|b(x - h)| + k$$



- The graph is open:
  - upward if $a > 0$.
  - downward if $a < 0$.
- The graph has the vertex: $V(h, k)$
- The graph has the following line as an axis of symmetry:

$$x = h$$

10. Write the rules of the following functions in the form $y = a|x - h| + k$ and identify the parameters a, b, h and k.

a) $y = -2|3x + 3| + 5$
$y = -6|x + 1| + 5$; $a = -6$, $h = -1$, $k = 5$

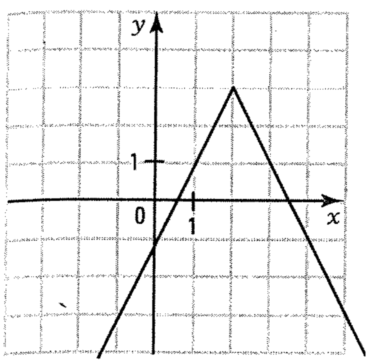
b) $y = 4|6 - 3x| + 5$
$y = 12|x - 2| + 5$; $a = 12$, $h = 2$, $k = 5$

c) $y = -\frac{1}{2}|8x - 4| + 3$
$y = -4\left|x - \frac{1}{2}\right| + 3$; $a = -4$, $h = \frac{1}{2}$, $k = 3$

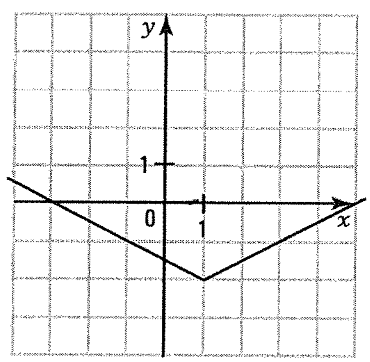
d) $y = -\frac{5}{6}\left|4 - \frac{1}{5}x\right| + 3$
$y = -\frac{1}{6}|x - 20| + 3$; $a = -\frac{1}{6}$, $h = 20$, $k = 3$

11. Graph the following functions.

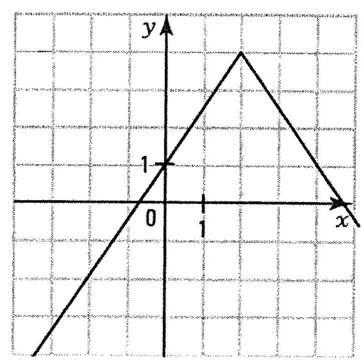
a) $y = -2|x - 2| + 3$



b) $y = \frac{1}{8}|4 - 4x| - 2$



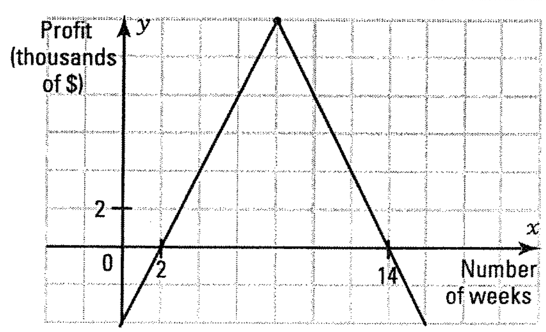
c) $y = -\frac{1}{2}|3x - 6| + 4$



ACTIVITY 6 Sign of an absolute value function

The manager of a clothing store has established that the profit y, in thousands of dollars, made each week is given by $y = -2|x - 12| + 20$, where x represents the number of weeks elapsed since the opening of the store.

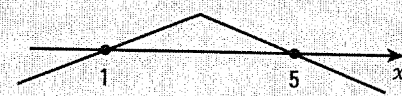
1. Determine the zeros of this function. ***2 and 14***
2. Draw a sketch of the graph.
3. Determine over what interval the profit is positive.
The profit is positive if $x \in [2, 14]$.



SIGN OF AN ABSOLUTE VALUE FUNCTION

To determine the sign of an absolute value function,
1. find the zeros (if they exist) of the function;
2. draw a sketch of the graph;
3. deduce the sign of the function.

Ex.: $f(x) = -2|x - 3| + 4$



$f(x) \leqslant 0$ if $x \in]-\infty, 1] \cup [5, +\infty[$
$f(x) \geqslant 0$ if $x \in [1, 5]$

12. Determine the sign of the following functions.

a) $f(x) = -|x + 1| + 3$

f(x) ⩾ 0 if x ∈ [−4, 2]

f(x) ⩽ 0 if x ∈]−∞, −4] ∪ [2, +∞[

b) $f(x) = \frac{1}{2}|x - 1| - 2$

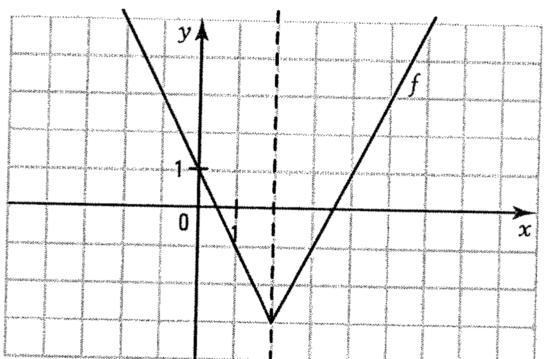
f(x) ⩾ 0 if x ∈]−∞, −3] ∪ [5, +∞[

f(x) ⩽ 0 if x ∈ [−3, 5]

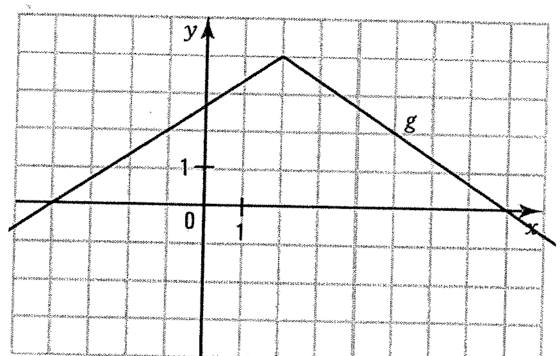
ACTIVITY 7 Study of an absolute value function

Consider the functions $f(x) = \frac{1}{2}|8 - 4x| - 3$ and $g(x) = -\frac{1}{3}|2x - 4| + 4$.

a) Write each of the rules in the form $y = a|x - h| + k$ and represent the functions in the Cartesian plane.



f(x) = 2 |x − 2| − 3



g(x) = −$\frac{2}{3}$ |x − 2| + 4

b) Do a study of each of the preceding functions and complete the table below.

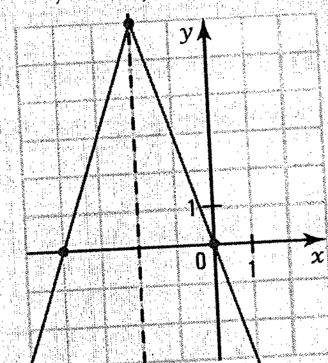
Properties	Function f	Function g
Domain	ℝ	ℝ
Range	*[−3, +∞[*	*]−∞, 4]*
Zeros	*$\frac{1}{2}$ and $\frac{7}{2}$*	*−4 and 8*
Initial value	*1*	*$\frac{8}{3}$*
Sign	*f(x) ⩾ 0 over $\left]-\infty, \frac{1}{2}\right] \cup \left[\frac{7}{2}, +\infty\right[$* *f(x) ⩽ 0 over $\left[\frac{1}{2}, \frac{7}{2}\right]$*	*f(x) ⩾ 0 over [−4, 8]* *f(x) ⩽ 0 over]−∞, −4] ∪ [8, +∞[*
Variation	*f ↘ over]−∞, 2]; f ↗ over [2, +∞[*	*f ↗ over]−∞, 2]; f ↘ over [2, +∞[*
Extrema	*min f = −3*	*max f = 4*

STUDY OF AN ABSOLUTE VALUE FUNCTION

Given the absolute value function: $f(x) = a|b(x - h)| + k$, we have:

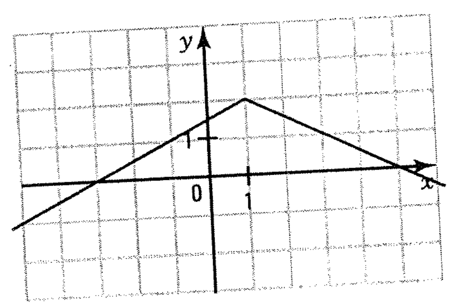
- **dom** $f = \mathbb{R}$.
- **ran** $f = [k, +\infty[$ if $a > 0$; $[-\infty, k[$ if $a < 0$.
- **The zero(s)** of f exist if a and k are opposite signs or if $k = 0$.
- **To study the sign of** f,
  - we find the zero(s) if they exist;
  - we establish the sign of f from a sketch of the graph.
- **Variation**
  If $a > 0$, f is decreasing over $]-\infty, h]$. If $a < 0$, f is increasing over $]-\infty, h]$.
  f is increasing over $[h, +\infty[$. f is decreasing over $[h, +\infty[$.
- **Extrema**
  If $a > 0$, f has a minimum. min $f = k$.
  If $a < 0$, f has a maximum. max $f = k$.

Ex.: Consider the function $f(x) = -3|x + 2| + 6$. ($a = -3$, $b = 1$, $h = -2$, $k = 6$)



- Open downward, $a < 0$.
- Vertex: V(–2, 6).
- Axis of symmetry: $x = -2$.
- Zeros: $-3|x + 2| + 6 = 0$
  $|x + 2| = 2$
  $\Leftrightarrow \quad x + 2 = -2 \quad$ or $x + 2 = 2$
  $x = -4 \quad$ or $\quad x = 0$
- Initial value: $y = 0$.
- dom $f = \mathbb{R}$; ran $f =]-\infty, 6]$
- Sign of f: $f(x) \geqslant 0$ over $[-4, 0]$; $f(x) \leqslant 0$ over $]-\infty, -4] \cup [0, +\infty[$.
- Variation of f: f is increasing over $]-\infty, -2]$; f is decreasing over $[-2, +\infty[$.
- max $f = 6$.

13. Represent the graph and do a study of the function
$f(x) = -\frac{1}{4}|2(x - 1)| + 2$.



***dom** = $\mathbb{R}$; **ran** = $]-\infty, 2]$.*
***Zeros:** –3 and 5.*
***Initial value:** 1.5.*
***Sign:** $f(x) \geqslant 0$ over $[-3, 5]$.*
$f(x) < 0$ over $]-\infty, 2[\cup]5, +\infty[$.
***Variation:** $f \nearrow$ over $]-\infty, 1]$; $f \searrow$ over $[1, +\infty[$*
***Extrema:** max = 2*

14. Determine the domain and range of each of the following functions.

a) $y = -2|x + 5| - 1$

dom = ℝ, ran =]–∞, –1]

b) $y = \frac{1}{4}|-2(x - 1)| + 5$

dom = ℝ, ran = [5, +∞[

15. Determine the zeros of the following functions.

a) $y = 3|x - 5| - 6$ **3 and 7**

b) $y = -\frac{1}{2}|6 - 3x| + 4$ $-\frac{2}{3}$ **and** $\frac{14}{3}$

c) $y = 4|2x + 1| + 8$ **No zero**

d) $y = -5|6 - x|$ **6**

16. Consider the linear function $f(x) = 2x - 3$ and the absolute value function $g(x) = 3|3x + 5| - 4$. Determine the initial value of the composite

a) $g \circ f$: **8**

b) $f \circ g$: **19**

17. Determine the interval over which each of the following functions is positive.

a) $y = -\frac{1}{3}|x - 5| + 2$

f(x) ⩾ 0 over [–1, 11]

b) $y = 2|3 - 2x| - 4$

f(x) ⩾ 0 over $\left]-\infty, \frac{1}{2}\right] \cup \left[\frac{5}{2}, +\infty\right[$

c) $y = \frac{3}{4}|-2x + 4| - 3$

f(x) ⩾ 0 over]–∞, 0] ∪ [4, +∞[

d) $y = 3|x - 5| + 6$

f(x) ⩾ 0 over ℝ

18. Determine the interval over which each of the following functions is increasing.

a) $y = 5|6 - 4x| + 2$

f ↗ over $\left[\frac{3}{2}, +\infty\right[$

b) $y = -3|2x + 4| + 5$

f ↗ over]–∞, –2]

19. Determine the solution set to each of the following inequalities.

a) $|x - 5| > 3$

S =]–∞, 2[∪]8, +∞[

b) $|6 - x| \leqslant 1$

S = [5, 7]

c) $|3x - 2| \geqslant 4$

S = $\left]-\infty, -\frac{2}{3}\right] \cup [2, +\infty[$

d) $|2x + 5| \leqslant 0$

S = $\left\{-\frac{5}{2}\right\}$

e) $-2|x + 1| + 5 > -5$

S =]–6, 4[

f) $3|2 - x| + 4 > 1$

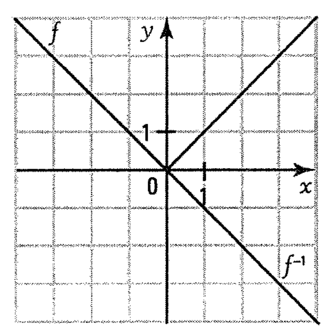
S = ℝ

20. The function $f(x) = |x|$ is represented on the right.

a) Explain why the inverse f^{-1} is not a function.

There exists a horizontal line (ex.: the line y = 2) that intersects the graph at 2 points (the points (–2, 2) and (2, 2)).

b) Graph the inverse f^{-1}.



21. Study each of the following functions and complete the following table.

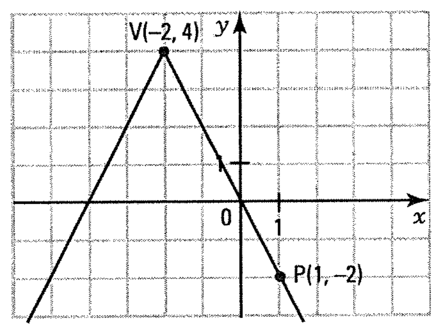
	$f(x) = -2\|x - 1\| + 4$	$f(x) = 3\|x + 2\| - 6$	$f(x) = \frac{1}{2}\|x - 4\| + 3$	$f(x) = -3\|5 - x\|$
Dom f	ℝ	ℝ	ℝ	ℝ
Ran f	]–∞, 4]	[–6, +∞[	[3, +∞[	]–∞, 0]
Zero(s) if they exist	–1 and 3	–and 0	None	5
Initial value	2	0	5	–15
Sign	$f(x) \geqslant 0$ over [–1, 3] $f(x) < 0$ over]–∞, –1[∪]3, +∞[	$f(x) \geqslant 0$ over]–∞, –4] ∪ [0, +∞[$f(x) < 0$ over]–4, 0[	$f(x) \geqslant 0$ over ℝ $f(x) < 0$ never	$f(x) \geqslant 0$ over {5} $f(x) < 0$ over ℝ\{5}
Variation	f ↗ over]–∞, 1] f ↘ over [1, +∞[	f ↗ over [–2, +∞[f ↘ over]–∞, –2]	f ↗ over [4, +∞[f ↘ over]–∞, 4]	f ↗ over]–∞, 5] f ↘ over [5, +∞[
Extrema	max = 4	min = –6	min = 3	max = 0

ACTIVITY 8 Finding the rule of an absolute value function

The rule of any absolute value function can be written in the form $f(x) = a|x - h| + k$.

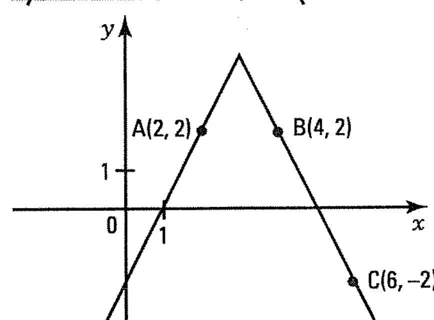
a) Consider the function $f(x) = 3|-2(x - 5)| + 7$.
Write the rule of this function in the form $f(x) = a|x - h| + k$. $y = 6|x - 5| + 7$

b) Consider the absolute value function with the vertex V(–2, 4) and passing through the point P(1, –2).



1. Identify h and k. $h = -2, k = 4$
2. Determine a knowing that the coordinates of the point P(1, –2) verify the rule of the function.
   We have $y = a|x + 2| + 4; -2 = a|1 + 2| + 4; a = -2$
3. What is the rule of the function? $f(x) = -2|x + 2| + 4$

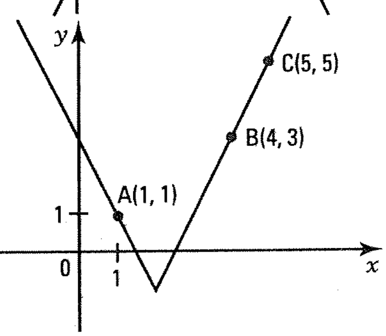
c) The graph of the absolute value function on the right passes through 2 points A(2, 2) and B(4, 2) which have the same y-coordinate and through the point C(6, –2).
Find the rule of this function.



$h = \frac{2 + 4}{2} = 3$; slope of BC = –2; BC: $y = -2x + 10$

$k = 4; y = -2|x - 3| + 4$

d) The graph of the absolute value function on the right passes through the points A(1, 1); B(4, 3) and C(5, 5).
Find the rule of this function.



BC: $y = 2x - 5$; SA: $y = -2x + 3$; V(2, –1)

$y = 2|x - 2| - 1$

FINDING THE RULE OF AN ABSOLUTE VALUE FUNCTION

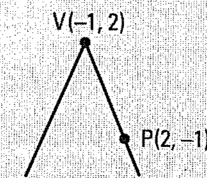
The rule of any absolute value function can be written in the form:

$$f(x) = a|x - h| + k$$

1st case: The vertex V and a point P are given.

1. Identify the parameters h and k.
2. Find a after replacing x and y in the rule by the coordinates of the given point P.
3. Deduce the rule.

1. $h = -1$ and $k = 2$
   $y = a|x + 1| + 2$
2. $-1 = a|2 + 1| + 2$
   $-1 = 3a + 2$
   $a = -1$
3. $y = -|x + 1| + 2$

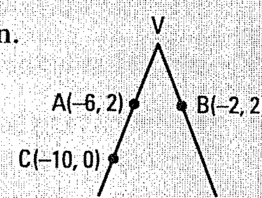


2nd case: Three points, of which two have the same y-coordinate, are given.

1. Identify h as half the sum of the x-coordinates of the points with the same y-coordinates.
2. Find the slope of the ray passing through two given points, and establish parameter a according to the opening of the graph.
3. Find k after replacing x and y in the rule by the coordinates of one of the given . points
4. Deduce the rule.

$h = \frac{(-6) + (-2)}{2} = -4$

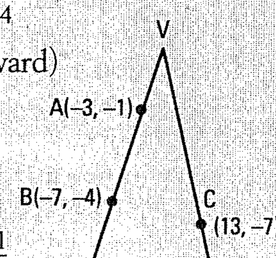
2. Slope $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{2}$
   $a = -\frac{1}{2}$ (open downward)
3. $y = -\frac{1}{2}|x + 4| + k$
   $2 = -\frac{1}{2}|-2 + 4| + k$
   $k = 3$
4. $y = -\frac{1}{2}|x + 4| + 3$



3th case: Any three points are given.

1. Find the slope of the ray passing through two given points, and establish parameter a according to the opening of the graph.
2. Find the equation of each ray knowing that their slopes are opposite.
3. Find the coordinates (h, k) of the vertex V, which is the intersection of the two rays.
4. Deduce the rule.

1. Slope $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{3}{4}$
   $a = \frac{-3}{4}$(open downward)
2. $y_1 = \frac{3}{4}x + \frac{5}{4}$
   $y_2 = -\frac{3}{4}x + \frac{11}{4}$
3. $\frac{3}{4}x + \frac{5}{4} = -\frac{3}{4}x + \frac{11}{4}$
   $6x = 6$
   $x = 1 \Rightarrow y = 2$
   Thus, V(1, 2)
4. $y = -\frac{3}{4}|x - 1| + 2$



22. Find the rule of an absolute value function whose graph

a) has the vertex V(3, 4) and passes through the point P(7, 6). $y = \frac{1}{2}|x - 3| + 4$

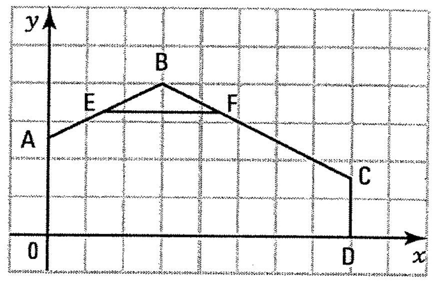
b) passes through the points A(2, –6), B(5, –8) and C(–4, –6). $y = -\frac{2}{3}|x + 1| - 4$

c) passes through the points A(1, –1), B(3, –5) and C(–4, –3). $y = -2|x + 1| + 3$

23. In order to draw the simulated trajectory of a toy airplane, Ethan uses the rule of an absolute value function that gives the airplane's height y, in metres, as a function of elapsed time x, in seconds. The rule of the function is $y = -\frac{5}{4}|x - 8| + 10$.

For how many seconds is the height of the airplane above 7 m? ***4.8 seconds***

24. In the Cartesian plane on the right, a view of an airplane hangar is represented with the roof of the hangar corresponding to an absolute value function given by the rule $y = -\frac{1}{2}|x - 6| + 8$.

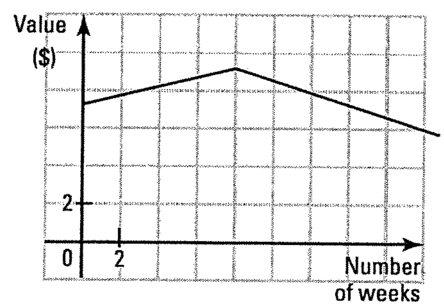


a) What is the height of the wall A0? ***5 m***

b) What is the height of the wall CD if the width of the hangar is equal to 16 m? ***3 m***

c) The ceiling EF is built at a height of 6.5 m. What is the width of the ceiling? ***5.6 m***

25. The graph on the right represents the evolution of a share's value on the stock market. Eight weeks after its purchase, the share reaches its maximum value of \$9. If it initially was worth \$7, what will it be worth after 13 weeks?



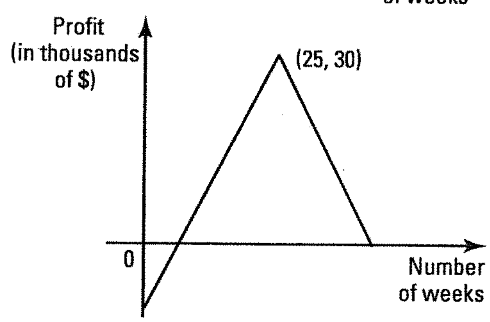
$y = a|x - 8| + 9;\ 7 = 8a + 9;\ a = -\frac{1}{4}$

$y = -\frac{1}{4}|x - 8| + 9.$

It will be worth \$7.75.

26. The graph on the right represents the profit of a recycling company during its first 40 weeks of operation.

During how many weeks was the profit greater than \$15 000?



$y = -2|x - 25| + 30$

$-2|x - 25| + 30 = 15;\ x = 17.5$ or $x = 32.5.$

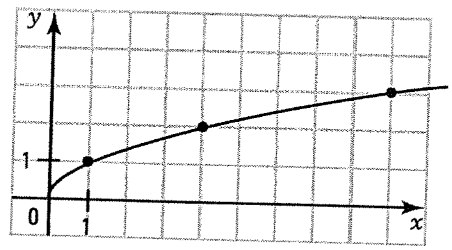
During 15 weeks.

27. The air conditioning system in an office building has been programmed so that it turns on when the outside temperature reaches 23 °C and turns off when it reaches 20 °C. The outside temperature varies according to the rule of the absolute value function given by $y = -3|x - 6| + 35$ where x represents the elapsed number of hours since 6 a.m. and y represents the outside temperature in °C. How many hours was the system on?

It turns on at 8 a.m. and turns off at 5 p.m. The system is on during 9 hours.

5.2 Square root function

ACTIVITY 1 Square root functions



Consider the function f defined by the rule $f(x) = \sqrt{x}$.

a) What condition must be placed on x for $\sqrt{x}$ to exist in $\mathbb{R}$?

x must be positive or zero

b) Complete the following table of values.

x	0	1	4	9
y	*0*	*1*	*2*	*3*

c) Represent the function f in the Cartesian plane.

d) Determine

1. dom $f =$ $\mathbb{R}_+$
2. ran $f =$ $\mathbb{R}_+$
3. the zero of f. ***0***
4. the initial value of f. ***0***
5. the sign of f ***$f(x) \geqslant 0$ over $\mathbb{R}_+$.***
6. the variation of f. ***$f \nearrow$ over $\mathbb{R}_+$.***
7. the extrema of f. ***min $f = 0$***

BASIC SQUARE ROOT FUNCTION

- The function f defined by the rule:

$$f(x) = \sqrt{x}$$

is called the basic square root function.

- We have:

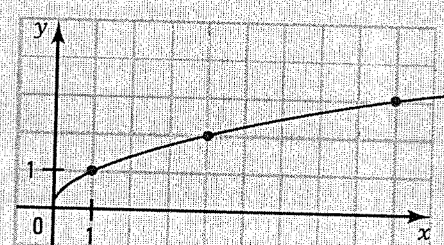
dom $f = \mathbb{R}_+$. ran $f = \mathbb{R}_+$.

The zero of f is 0. The **initial value** is 0.

Sign of f: $f(x) \geqslant 0, \forall\, x \in$ dom f.

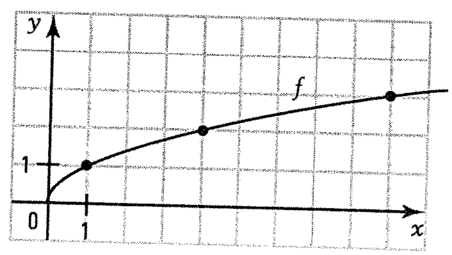
Variation of f: f is increasing, $\forall\, x \in$ dom f.

The function f has a **minimum** equal to 0.



The point $O(0, 0)$ is the **vertex** of the function.

1. Consider the basic square root function $f(x) = \sqrt{x}$ represented on the right.



Using the graph, find the values of x for which

a) $f(x) = 3$. ***$x = 9$***

b) $f(x) \geqslant 1$. ***$x \in [1, +\infty[$***

c) $0 \leqslant f(x) < 2$. ***$x \in [0, 4[$***

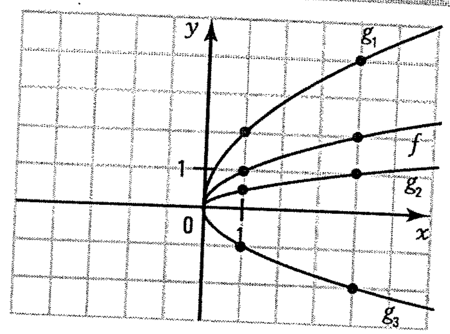
d) $f(x) < 0$. ***None since the function is never strictly negative.***

e) $1 < f(x) < 3$. ***$x \in]1, 9[$***

ACTIVITY 2 Square root function $y = a\sqrt{b(x-h)} + k$

The basic square root function $f(x) = \sqrt{x}$ can be transformed into a square root function defined by the rule

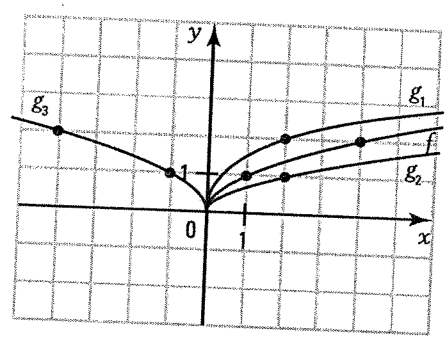
$$g(x) = a\sqrt{b(x-h)} + k.$$



a) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = a\sqrt{x}$.

Represent, in the same Cartesian plane, the functions $g_1(x) = 2\sqrt{x}$, $g_2(x) = \frac{1}{2}\sqrt{x}$ and $g_3(x) = -\sqrt{x}$ and explain how to deduce the graph of g from the graph of f when

1. $a > 1$: ***by a vertical stretch.***
2. $0 < a < 1$: ***by a vertical reduction.***
3. $a = -1$: ***by a reflection over the x axis.***
4. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = a\sqrt{x}$ by the transformation $(x, y) \rightarrow$ ***(x, ay)*** .
5. Is the graph of $g(x) = a\sqrt{x}$ located in the 1st or 4th quadrant when
   1) $a > 0$? ***1st quadrant.*** 2) $a < 0$? ***4th quadrant.***



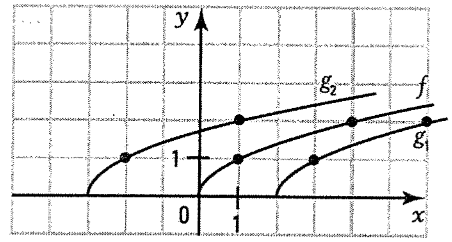
b) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = \sqrt{bx}$.

Represent, in the same Cartesian plane, the functions $g_1(x) = \sqrt{2x}$, $g_2(x) = \sqrt{\frac{1}{2}x}$ and $g_3(x) = \sqrt{-x}$ and explain how to deduce the graph of g from the graph of f when

1. $b > 1$: ***by a horizontal reduction.***
2. $0 < b < 1$: ***by a horizontal stretch.***
3. $b = -1$: ***by a reflection over the y axis.***
4. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = \sqrt{bx}$ by the transformation $(x, y) \rightarrow$ $\left(\frac{x}{b}, y\right)$
5. In which quadrant is the graph of $g(x) = \sqrt{bx}$ when
   1) $b > 0$? ***1st quadrant.*** 2) $b < 0$? ***2nd quadrant.***
6. What can you say about the graph of the function $y = 2\sqrt{x}$ and that of the function $y = \sqrt{4x}$? Justify your answer.
   They are the same. In fact, $\sqrt{4x} = \sqrt{4} \cdot \sqrt{x} = 2\sqrt{x}$ (property of radicals).

c) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = \sqrt{x-h}$.

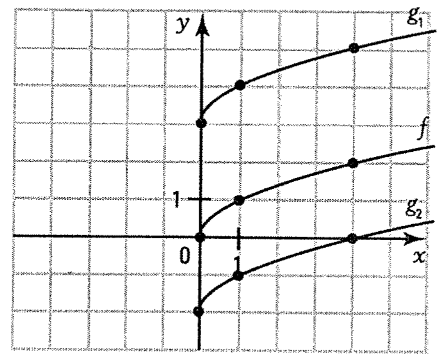
Represent, in the same Cartesian plane, the functions $g_1(x) = \sqrt{x-2}$ and $g_2(x) = \sqrt{x+3}$ and explain how to deduce the graph of g from the graph of f when



1. $h > 0$: ***by a horizontal translation to the right.***
2. $h < 0$: ***by a horizontal translation to the left.***
3. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = \sqrt{x-h}$ by the transformation $(x, y) \rightarrow$ ***(x + h, y)***

d) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = \sqrt{x} + k$.

Represent, in the same Cartesian plane, the functions $g_1(x) = \sqrt{x} + 3$ and $g_2(x) = \sqrt{x} - 2$ and explain how to deduce the graph of g from the graph of f when



1. $k > 0$: ***by a vertical translation upward.***
2. $k < 0$: ***by a vertical translation downward.***
3. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = \sqrt{x} + k$ by the transformation $(x, y) \rightarrow$ ***(x, y + k)***

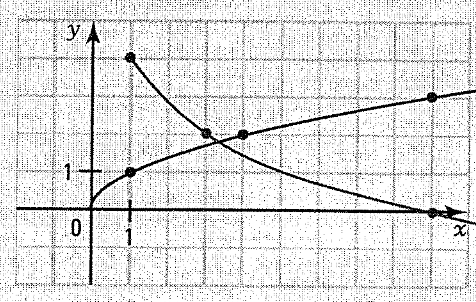
SQUARE ROOT FUNCTION $f(x) = a\sqrt{b(x-h)} + k$

The graph of the function $f(x) = a\sqrt{b(x-h)} + k$ is deduced from the graph of the basic square root function $y = \sqrt{x}$ by the transformation

$$(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$$

Ex.: The basic square root function $y = \sqrt{x}$ and the square root function

$y = -2\sqrt{\frac{1}{2}(x-1)} + 4$ are represented on the right.



The rule of the transformation applied to the graph of the basic square root function is:
$(x, y) \rightarrow (2x + 1, -2y + 4)$.

2. The following functions have a rule of the form $f(x) = a\sqrt{b(x-h)} + k$.

$f_1(x) = 3\sqrt{x}$, $f_2(x) = \sqrt{2x}$, $f_3(x) = \sqrt{x+4}$, $f_4(x) = \sqrt{x} + 1$ and $f_5(x) = 2\sqrt{3(x-1)} - 4$.

Complete the table on the right by determining, for each function, the parameters a, b, h and k and by giving the rule of the transformation which enables you to obtain the function from the basic function $g(x) = \sqrt{x}$.

	a	b	h	k	Rule
$f_1(x) = 3\sqrt{x}$	***3***	***1***	***0***	***0***	$(x, y) \rightarrow (x, 3y)$
$f_2(x) = \sqrt{2x}$	***1***	***2***	***0***	***0***	$(x, y) \rightarrow \left(\frac{x}{2}, y\right)$
$f_3(x) = \sqrt{x+4}$	***1***	***1***	***-4***	***0***	$(x, y) \rightarrow (x - 4, y)$
$f_4(x) = \sqrt{x} + 1$	***1***	***1***	***0***	***1***	$(x, y) \rightarrow (x, y + 1)$
$f_5(x) = 2\sqrt{3(x-1)} - 4$	***2***	***3***	***1***	***-4***	$(x, y) \rightarrow \left(\frac{x}{3} + 1, 2y - 4\right)$

3. In each of the following cases, we apply a transformation to the basic square root function $f(x) = \sqrt{x}$ Find the rule of the function obtained by applying the given transformation.

a) $(x, y) \rightarrow (-x, y)$ $y = \sqrt{-x}$

b) $(x, y) \rightarrow (x - 5, y + 2)$ $y = \sqrt{x + 5} + 2$

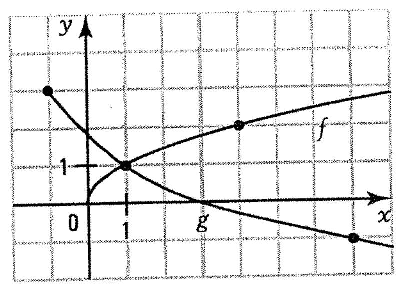
c) $(x, y) \rightarrow \left(\frac{x}{5}, y\right)$ $y = \sqrt{5x}$

d) $(x, y) \rightarrow (x, -4y)$ $y = -4\sqrt{x}$

e) $(x, y) \rightarrow (2x, -6y)$ $y = -6\sqrt{\frac{1}{2}x}$

f) $(x, y) \rightarrow \left(\frac{x}{4} + 3, 3y - 5\right)$ $y = 3\sqrt{4(x - 3)} - 5$

4. Consider the functions $f(x) = \sqrt{x}$ and $g(x) = -2\sqrt{\frac{1}{2}(x+1)} + 3$.



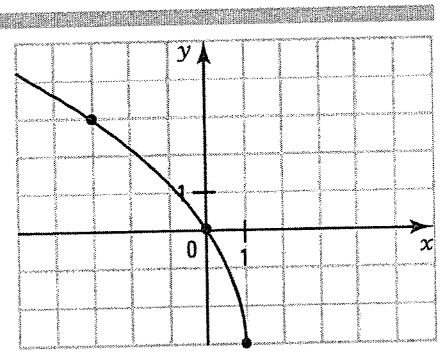
a) Give the rule of the transformation which enables you to obtain the graph of g from the graph of f.

$(x, y) \rightarrow (2x - 1, -2y + 3)$

b) Draw the graph of g from the graph of f.

ACTIVITY 3 Graph of a square root function

Consider the function $f(x) = 6\sqrt{-\frac{1}{4}(x-1)} - 3$.



a) Identify the parameters a, b, h and k.

$a = 6$, $b = -\frac{1}{4}$, $h = 1$ and $k = -3$

b) Write the rule of the function in the form $f(x) = a\sqrt{-(x-h)} + k$.

$f(x) = 6\sqrt{-\frac{1}{4}(x-1)} - 3 = 6\sqrt{\frac{1}{4}}\sqrt{-(x-1)} - 3 = 3\sqrt{-(x-1)} - 3$

c) What are the coordinates of the vertex? $V(1, -3)$

d) Represent the function f in the Cartesian plane after completing the following table of values.

x	-3	0	1
y	***3***	***0***	***-3***

e) What is the zero of f? ***0***

ACTIVITY 4 Finding the zero of a square root function

a) Consider the function with the rule: $y = -2\sqrt{3(x+1)} + 6$.

Justify the steps in finding the zero of this function.

$-2\sqrt{3(x+1)} + 6 = 0$ ***Replace y by 0.***

$\sqrt{3(x+1)} = 3$ ***Isolate the square root.***

$3(x+1) = 9$ ***Square each side of the equality.***

$x + 1 = 3$ ***Divide each side by 3.***

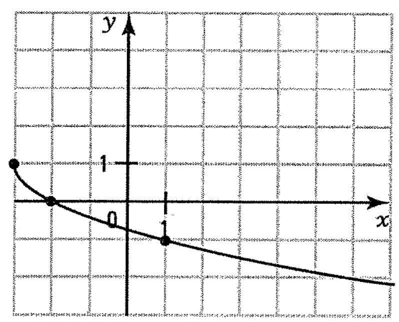
$x = 2$ ***Subtract 1 from each side.***

b) Under what conditions does the zero of a function $y = a\sqrt{b(x-h)} + k$ exist?

If a and k are opposite signs or if k = 0.

ACTIVITY 5 Study of a square root function

Consider the function f with rule $y = -\frac{1}{2}\sqrt{4x+12} + 1$.



a) Write the rule in the form $y = a\sqrt{x-h} + k$.

$y = \frac{-1}{2}\sqrt{4(x+3)} + 1 = -\sqrt{x+3} + 1$

b) Graph the function f.

c) Determine

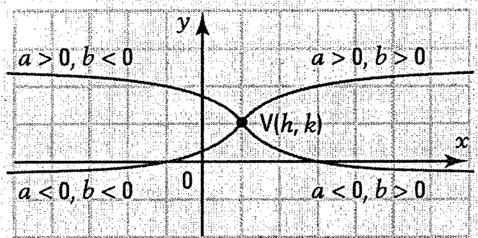
1. dom f = ***[−3, +∞[***
2. ran f = ***]−∞, 1]***
3. the zero of f (if it exists) ***−2***
4. the initial value of f. ***$-\sqrt{3} + 1$***
5. the sign of f. ***f(x) ≥ 0 over [−3, −2]   f(x) ≤ 0 over [−2, +∞[***
6. the variation of f. ***f ↗ never;   f ↘ over [−3, +∞[***
7. the extrema of f. ***max f = 1***

SQUARE ROOT FUNCTION

Consider the **square root function**

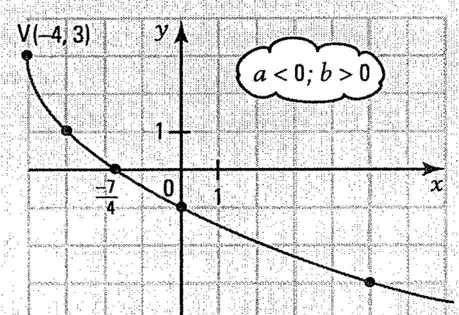
$$f(x) = a\sqrt{b(x-h)} + k$$

We have the following four cases:



- **dom** $f = [h, +\infty[$ if $b > 0$; **ran** $f = [k, +\infty[$ if $a > 0$;
  dom $f =]-\infty, h]$ if $b < 0$. **ran** $f =]-\infty, k]$ if $a < 0$.
- The **zero** of f exists if a and k are opposite signs or if $k = 0$.
- To study the **sign** of f,
  - we find the zero (if it exists);
  - we establish the sign of f from a sketch of the graph.
- **Variation**
  If $ab > 0$, f is increasing over the domain.
  If $ab < 0$, f is decreasing over the domain.
- **Extrema**
  If $a > 0$, f has a minimum. min $f = k$.
  If $a < 0$, f has a maximum. max $f = k$.

  Ex.: Consider the function $f(x) = -2\sqrt{x+4} + 3$ $(a = -2, b = 1, h = -4, k = 3)$.



- Vertex: V(−4, 3)
- dom $f = [-4, +\infty[$; ran $f =]-\infty, 3]$.
- Zero: $-2\sqrt{x+4} + 3 = 0$

$$\sqrt{x+4} = \frac{3}{2}$$

$$x + 4 = \frac{9}{4}$$

$$x = -\frac{7}{4}$$

- Initial value: $y = -1$
- Sign of f: $f(x) \geqslant 0$ over $\left[-4, -\frac{7}{4}\right]$; $f(x) \leqslant 0$ over $\left[-\frac{7}{4}, +\infty\right[$.
- Variation of f: f is decreasing, $\forall\, x \in$ dom f.
- max $f = 3$.

5. Write the rules of the square root functions in the form $y = a\sqrt{x-h}+k$ or $y = a\sqrt{-(x-h)}+k$.

a) $y = -2\sqrt{4x+8}+3$

$y = -4\sqrt{x+2}+3$

b) $y = 2\sqrt{9x-36}+4$

$y = 6\sqrt{x-4}+4$

c) $y = -\frac{1}{2}\sqrt{18-9x}+1$

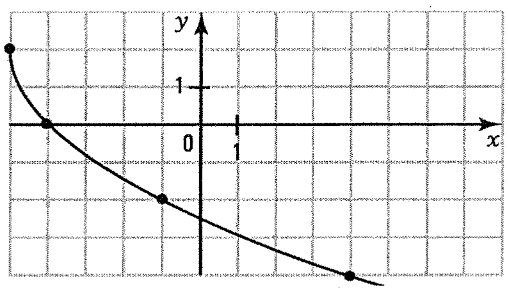
$y = -\frac{3}{2}\sqrt{-(x-2)}+1$

d) $y = -\frac{3}{4}\sqrt{2-4x}+7$

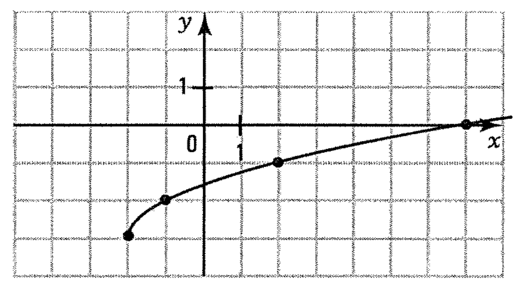
$y = -\frac{3}{2}\sqrt{-\left(x-\frac{1}{2}\right)}+7$

6. Represent the following square root functions in the Cartesian plane.

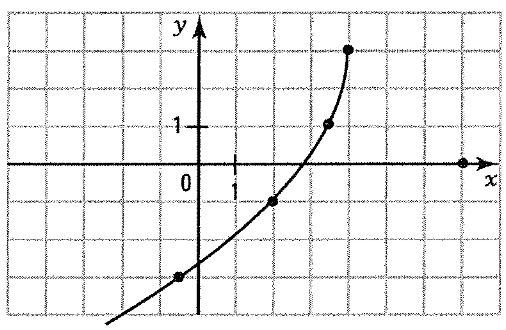
a) $y = -2\sqrt{x+5}+2$



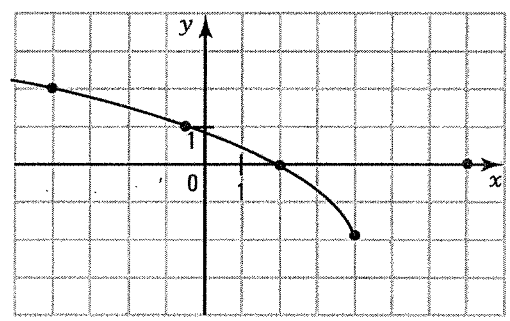
b) $y = \frac{1}{2}\sqrt{4x+8}-3$



c) $y = -2\sqrt{-2(x-4)}+3$

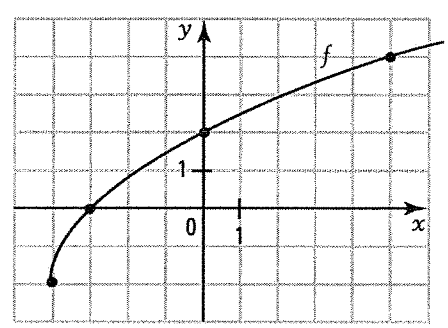


d) $y = \sqrt{-2(x-4)}-2$



7. Consider the function $f(x) = 2\sqrt{x+4}-2$.

a) Graph the function f.



b) Study the function f.

dom = [–4, +∞[ran = [–2, +∞[

zero: –3; initial value: 2

$f(x) \geqslant 0$ over [–3, +∞[; $f(x) \leqslant 0$ over [–4, –3]

$f \nearrow, \forall x \in$ dom f

min f = –2

c) Using the graph of f, solve the inequality

1. $f(x) \geqslant 2$ **[0, +∞[**

2. $f(x) \leqslant 4$ **[–4, 5]**

8. Determine the domain and range of the following functions.

a) $y = -2\sqrt{6 - 3x} + 4$

dom =]−∞, 2]: ran =]−∞, 4]

b) $y = 3\sqrt{4x + 2} - 1$

dom** = $\left[-\frac{1}{2}, +\infty\right[$ **; ran = [−1, +∞[

9. Determine the zero and initial value of the following functions.

a) $y = -3\sqrt{6 - 4x} + 9$

zero: $-\frac{3}{4}$, ***i.v.:*** $-3\sqrt{6} + 9$

b) $y = 2\sqrt{4x - 1} - 1$

zero: $\frac{5}{16}$, ***i.v.: does not exist***

c) $y = 2\sqrt{x - 5} + 4$

No zero, i.v.: does not exist

d) $y = -2\sqrt{3x + 1}$

zero: $-\frac{1}{3}$, ***i.v.: −2***

10. Consider the absolute value function $f(x) = -2|6 - 2x| + 8$ and the square root function $g(x) = 3\sqrt{\frac{1}{2}(x + 4)} - 5$. Determine

a) $g \circ f(4) =$ ***1***

b) $f \circ g(-2) =$ ***−12***

11. Determine the interval over which each of these functions is positive.

a) $f(x) = 3\sqrt{x + 5} - 6$

f(x) ≥ 0 over [−1, +∞[

b) $f(x) = -2\sqrt{6 + 4x} + 4$

f(x) ≥ 0 over $\left[-\frac{3}{2}, -\frac{1}{2}\right]$

c) $f(x) = \frac{1}{2}\sqrt{4 - x} + 5$

f(x) ≥ 0 over]−∞, 4]

d) $f(x) = -3\sqrt{-2x + 8} - 1$

f(x) is never positive

12. Solve the following inequalities.

a) $-2\sqrt{x + 3} + 2 \geq 0$

S = [−3, −2]

b) $\sqrt{3x + 4} < -1$

S = ∅

c) $5\sqrt{2 - x} > 4$

S = $\left]-\infty, \frac{34}{25}\right[$

d) $\sqrt{\frac{1}{2}x + 8} > 0$

S =]−16, +∞[

13. Determine the interval over which each of these functions is increasing.

a) $f(x) = 3\sqrt{-2(x - 1)} + 5$

f is never increasing.

b) $f(x) = -2\sqrt{-3(x + 4)}$

f ↗ over]−∞, −4]

14. Study each of the following functions and complete the following table.

	$f_1(x) = 3\sqrt{x-2} - 1$	$f_2(x) = -2\sqrt{\frac{1}{2}(x+4)} + 6$	$f_3(x) = \sqrt{2-x} + 1$	$f_4(x) = -2\sqrt{-x} + 4$
Domain	*[2, +∞[*	*[–4, +∞[*	*]–∞, 2]*	*]–∞, 0]*
Range	*[–1, +∞[*	*]–∞, 6]*	*[1, +∞[*	*]–∞, 4]*
Zero	$\frac{19}{9}$	*14*	*does not exist*	*–4*
Initial value	*does not exist*	$-2\sqrt{2} + 6$	$\sqrt{2} + 1$	*4*
Sign	$f(x) \geqslant 0$ over $\left[\frac{19}{9}, +\infty\right[$ $f(x) < 0$ over $\left[2, \frac{19}{9}\right[$	$f(x) \geqslant 0$ over $[-4, 14]$ $f(x) < 0$ over $]14, +\infty[$	$f(x) \geqslant 0$ over $]-\infty, 2]$ $f(x) < 0$ never	$f(x) \geqslant 0$ over $[-4, 0]$ $f(x) < 0$ over $]-\infty, -4[$
Variation	*f ↗ over [2, +∞[* *f ↘ never*	*f ↗ never* *f ↘ over [–4, +∞[*	*f ↗ never* *f ↘ over]–∞, 2]*	*f ↗ over]–∞, 0]* *f ↘ never*
Extrema	*min = –1*	*max = 6*	*min = 1*	*max = 4*

ACTIVITY 6 Finding the rule of a square root function

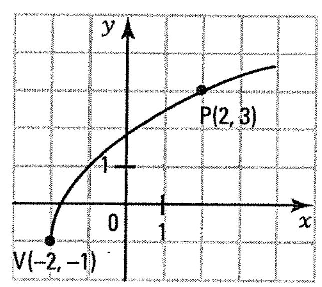
Any square root function can be written in the form $f(x) = a\sqrt{x-h} + k$ *or* $f(x) = a\sqrt{-(x-h)} + k$.

Consider the functions $f(x) = 3\sqrt{2x+4} - 5$ and $g(x) = 5\sqrt{-4x+8} - 1$.

Write the rule of each function in the form $y = a\sqrt{x-h} + k$ or $y = a\sqrt{-(x-h)} + k$.

$f(x) = 3\sqrt{2(x+2)} - 5 = 3\sqrt{2} \cdot \sqrt{x+2} - 5$ and $g(x) = 5\sqrt{-4(x-2)} - 1 = 10\sqrt{-(x-2)} - 1$

b) We consider the function represented on the right.



1. Which of the two rules $y = a\sqrt{x-h} + k$ or $y = a\sqrt{-(x-h)} + k$ corresponds to the graph of this function?

   $y = a\sqrt{-(x-h)} + k$

2. Identify h and k. $h = -2, k = -1$

3. Determine a knowing that the coordinates of the point P(2, 3) verify the rule of the function.

   $y = a\sqrt{x+2} - 1$

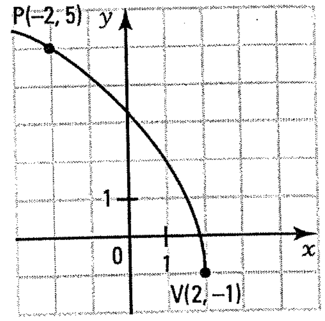
   $3 = a\sqrt{2+2} - 1$

   $4 = 2a$

   $a = 2$

4. What is the rule of the function? $y = 2\sqrt{x+2} - 1$

c) Consider the square root function whose graph has a vertex at V(2, –1) and passes through the point P(–2, 5).



1. Which of the two rules $y = a\sqrt{x - h} + k$ or $y = a\sqrt{-(x - h)} + k$ corresponds to the graph of this function?
   $y = a\sqrt{-(x - h)} + k$
2. Identify h and k. $h = 2, k = -1$
3. Determine a knowing that the coordinates of the point P(–2, 5) verify the rule of the function.
   $y = a\sqrt{-(x - 2)} - 1;\ 5 = a\sqrt{-(-2 - 2)} - 1;\ 6 = 2a;\ a = 3$
4. What is the rule of the function? $y = 3\sqrt{-(x - 2)} - 1$

d) What is the domain of a square root function if its rule is of the form

1. $f(x) = a\sqrt{x - h} + k$. $\text{dom } f = [h, +\infty[$
2. $f(x) = a\sqrt{-(x - h)} + k$. $\text{dom } f =]-\infty, h]$

FINDING THE RULE OF A SQUARE ROOT FUNCTION

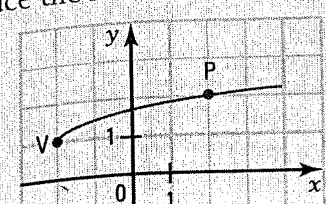
Any square root function can be written, depending on its domain, in the form:

$$f(x) = a\sqrt{x - h} + k \quad \text{or} \quad f(x) = a\sqrt{-(x - h)} + k$$

The vertex V and a point P are given.

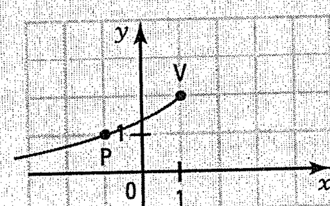
1. Determine the form of the rule, $y = a\sqrt{x - h} + k$ or $y = a\sqrt{-(x - h)} + k$.
2. Identify parameters h and k.
3. Determine a after replacing, in the rule, x and y by the coordinates of the point P.
4. Deduce the rule.

Ex.: **a)**



1. $y = a\sqrt{x - h} + k$
2. $y = a\sqrt{x + 2} + 1$
3. $2 = a\sqrt{2 + 2} + 1$
   $1 = 2a$
   $a = \frac{1}{2}$
4. rule: $y = \frac{1}{2}\sqrt{x + 2} + 1$

b)



1. $y = a\sqrt{-(x - h)} + k$
2. $y = a\sqrt{-(x - 1)} + 2$
3. $1 = a\sqrt{-(-1 - 1)} + 2$
   $-1 = a\sqrt{2}$
   $a = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$
4. rule: $y = -\frac{\sqrt{2}}{2}\sqrt{-(x - 1)} + 2$
   or $y = -\frac{1}{2}\sqrt{-2(x - 1)} + 2$

15. Find the rule of each of the square root functions given its vertex V and a point P on its graph.

a) V(5, 3) and P(9, 3.5)

$y = \frac{1}{4}\sqrt{x - 5} + 3$

b) V(−2, −1) and P(−6, −4)

$y = -\frac{3}{2}\sqrt{-(x + 2)} - 1$

c) V(−2, 4) and P(23, 2)

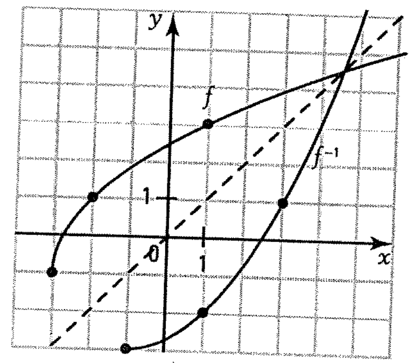
$y = -\frac{2}{5}\sqrt{x + 2} + 4$

d) V(5, 3) and P(−13, 5)

$y = \frac{1}{3}\sqrt{-2(x - 5)} + 3$

ACTIVITY 7 Inverse of a square root function

Consider the function $f(x) = 2\sqrt{x + 3} - 1$.



a) In the same Cartesian plane,

1. graph the function f.
2. deduce the graph of f^{-1}.

b) Complete: The graphs of the function f and its inverse f^{-1} are symmetrical about the ***bisector of the 1st quadrant***

c) Is the inverse f^{-1} a function? Justify your answer.

Yes, because any vertical line only intersects the graph of f^{-1} in at most one point.

d) 1. Determine

1) dom f ***[−3, +∞[*** 2) ran f ***[−1, +∞[*** 3) dom f^{-1} ***[−1, +∞[*** 4) ran f^{-1} ***[−3, +∞[***

2. Verify that

1) dom f^{-1} = ran f ***True*** 2) ran f^{-1} = dom f ***True***

e) Justify the steps in finding the rule of the inverse function f^{-1}.

1. Isolate x in the equation $y = 2\sqrt{x + 3} - 1$.

$y + 1 = 2\sqrt{x + 3}$ ***Add 1 to each side.***

$\frac{1}{2}(y + 1) = \sqrt{x + 3}$ ***Divide each side by 2.***

$\frac{1}{4}(y + 1)^2 = x + 3$ ***Square both sides.***

$\frac{1}{4}(y + 1)^2 - 3 = x$ ***Subtract 3 from each side.***

2. Interchange the letters x and y to obtain the rule of the inverse.

You get: $y = \frac{1}{4}(x + 1)^2 - 3$.

3. What restriction must be set on the variable x? Justify your answer.

$x \geqslant -1$ since dom f^{-1} = ran f = [−1, +∞[

The inverse of the square root function $y = 2\sqrt{x + 3} - 1$ is therefore the function $y = \frac{1}{4}(x + 1)^2 - 3 \quad (x \geqslant -1)$

The graphic representation of the inverse corresponds to a semi-parabola.

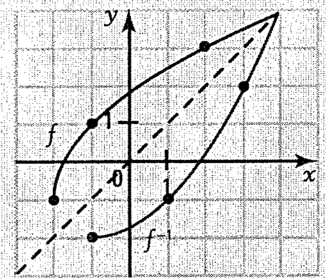
INVERSE OF A SQUARE ROOT FUNCTION

The **inverse** of a square root function is a function whose graph is a **semi-parabola**.

Ex.: $f(x) = 2\sqrt{x+2} - 1$ has the inverse:

$f^{-1}(x) = \frac{1}{4}(x+1)^2 - 2 \qquad (x \geqslant -1)$

Note that dom f^{-1} = ran $f = [-1, +\infty[$.
The graphs of f and f^{-1} are symmetrical about the bisector of the 1st quadrant.



16. Determine the rule of the inverse of the following functions and indicate the domain of the inverse.

a) $y = 4\sqrt{x-1} + 7$

$y = \frac{1}{4}(x-7)^2 + 1$; dom = $[7, +\infty[$

b) $y = -3\sqrt{x+4} - 1$

$y = \frac{1}{9}(x+1)^2 - 4$; dom = $]-\infty, -1]$

c) $y = 4\sqrt{-(x+3)} - 2$

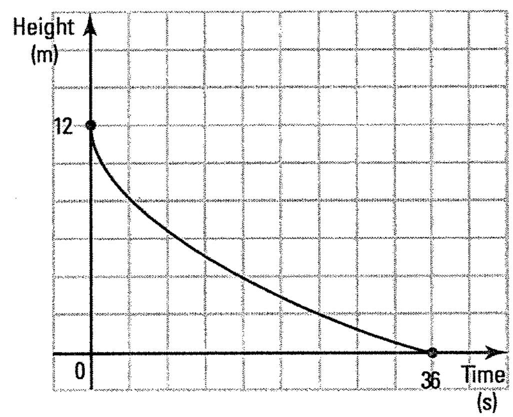
$y = -\frac{1}{16}(x+2)^2 - 3$; dom = $[-2, +\infty[$

d) $y = -2\sqrt{-(x-5)} + 4$

$y = -\frac{1}{4}(x-4)^2 + 5$; dom = $]-\infty, 4]$

17. At a water park, Raphael is getting ready to go down a slide.
The function f represented on the right gives Raphael's height h (in m) as a function of elapsed time t (in s) since his departure.
At what instant will he be at a height of 4 m?

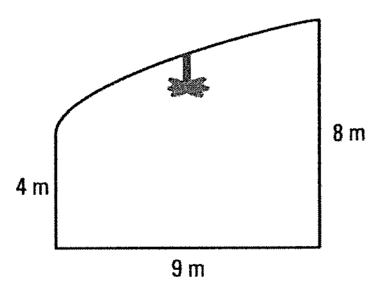
$h(t) = -2\sqrt{t} + 12$; after 16 seconds.



18. The lateral view of a solarium is represented by the graph on the right where the glass ceiling follows the curve of a square root function. A light is located at the centre of the room as indicated in the figure. Determine at what height the base of the light is located. (Round your answer to the nearest tenth.

$V(0, 4)$; $y = a\sqrt{x} + 4$; $P(9, 8)$; $y = \frac{4}{3}\sqrt{x} + 4$.

When x = 4.5 m, the height is y = 6.8 m.



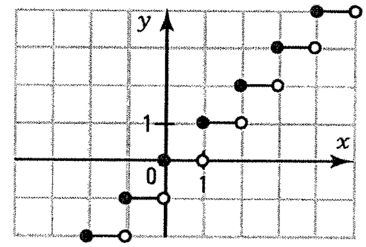
5.3 Greatest integer function

ACTIVITY 1 Basic greatest integer function

a) The greatest integer of a real number x is represented by $[x]$.

1. Give the definition of the greatest integer of a real number x.
   The greatest integer of a real number x is equal to the greatest integer less than or equal to x.
2. Calculate
   1) $[3.76]$ ***3*** 2) $[-1.25]$ ***-2***
3. In what interval is x located if
   1) $[x] = 2$ ***$x \in [2, 3[$*** 2) $[x] = -2$ ***$x \in [-2, -1[$***

b) 1. Graph the basic greatest integer function $f(x) = [x]$ in the Cartesian plane on the right.

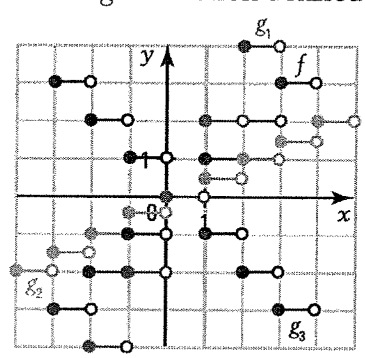


2. Determine
   1) dom f $\mathbb{R}$ 2) ran f $\mathbb{Z}$

ACTIVITY 2 Greatest integer function $y = a[b(x - h)] + k$

The basic greatest integer function $y = [x]$ can be transformed into a greatest integer function defined by the rule $y = a[b(x - h)] + k$.

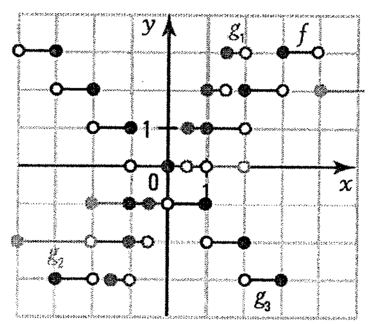
a) Consider the basic greatest integer function $f(x) = [x]$ and the greatest integer function $g(x) = a[x]$.



Represent, in the same Cartesian plane, in different colors, the functions $g_1(x) = 2[x]$, $g_2(x) = \frac{1}{2}[x]$ and $g_3(x) = -[x]$ and explain how to deduce the graph of g from the graph of f when

1. $a > 1$: ***By a vertical stretch.***
2. $0 < a < 1$: ***By a vertical reduction.***
3. $a = -1$: ***By a reflection over the x axis.***
4. Complete: the parameter a affects the step ***height*** .

b) Consider the basic greatest integer function $f(x) = [x]$ and the greatest integer function $g(x) = a[bx]$.



Represent, in the same Cartesian plane, in different colors, the functions $g_1(x) = [2x]$, $g_2(x) = \left[\frac{1}{2}x\right]$ and $g_3(x) = [-x]$ and explain how to deduce the graph of g from the graph of f when

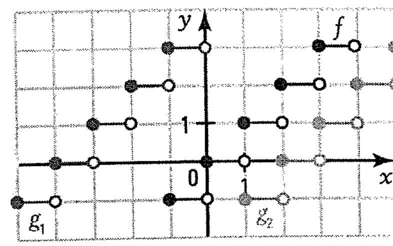
1. $b > 1$: ***By a horizontal reduction.***
2. $0 < b < 1$: ***By a horizontal stretch.***
3. $b = -1$: ***By a reflection over the y axis.***

4. Complete: the parameter b affects the step ***length*** .

5. Represent one step if

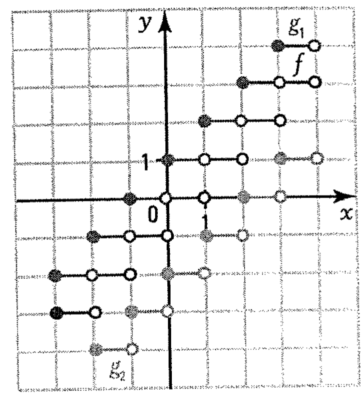
   1) $b > 0$. ●—○   2) $b < 0$. ○—●

c) Consider the basic greatest integer function $f(x) = [x]$ and the greatest integer function $g(x) = [x - h]$.
Represent, in the same Cartesian plane, in different colors, the functions $g_1(x) = [x - 2]$ and $g_2(x) = [x + 4]$ and explain how to deduce the graph of g from the graph of f when



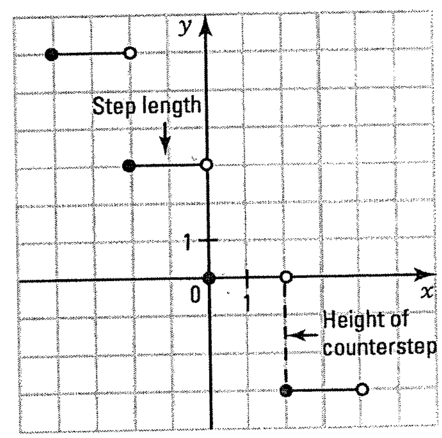
1. $h > 0$: ***By a horizontal translation to the right.***
2. $h < 0$: ***By a horizontal translation to the left.***

d) Consider the basic greatest integer function $f(x) = [x]$ and the greatest integer function $g(x) = [x] + k$.
Represent, in the same Cartesian plane, in different colors, the functions $g_1(x) = [x] + 1$ and $g_2(x) = [x] - 2$ and explain how to deduce the graph of g from the graph of f when



1. $k > 0$: ***By a vertical translation upward.***
2. $k < 0$: ***By a vertical translation downward.***

e) The function $f(x) = -3\left[\frac{1}{2}(x + 4)\right] + 6$ is represented on the right.



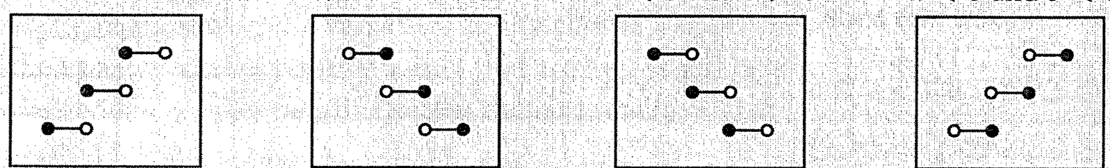
1. Identify the parameters a, b, h and k.
   $a = -3$, $b = \frac{1}{2}$, $h = -4$, $k = 6$
2. Verify that
   1) each step has a length of $\frac{1}{|b|}$. ***$\frac{1}{|b|} = 2$***
   2) the height of the counterstep is $|a|$. ***$|a| = |-3| = 3$***
3. Determine
   1) dom f ***$\mathbb{R}$***   2) ran f ***$\{y \mid y = -3m + 6, m \in \mathbb{Z}\}$***
   3) the zeros of f. ***[0, 2[***   4) the initial value of f. ***0***
   5) the sign of f. ***$f(x) \geq 0$ over $]-\infty, 2[$; $f(x) \leq 0$ over $[2, +\infty[$***
   6) the variation of f. ***$f \searrow$ over $\mathbb{R}$***

GREATEST INTEGER FUNCTION

We consider the **greatest integer function** $f(x) = a[b(x - h)] + k$.

- The Cartesian graph is a step function.
- Each step has a length of $\frac{1}{|b|}$.
  - If $b > 0$, the steps are closed on the left and open on the right (●—○).
  - If $b < 0$, the steps are open on the left and closed on the right (○—●).
- The height of the counter step $|a|$.
- dom $f = \mathbb{R}$, ran $f = \{y | y = am + k, m \in \mathbb{Z}\}$
- – If $ab > 0$, the function is increasing.
  – If $ab < 0$, the function is decreasing.
- The function f has zeros if and only if k is a multiple of a.
- The signs of a and b help us distinguish 4 cases:

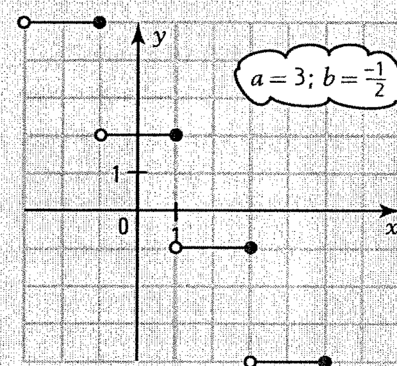
$a > 0$ and $b > 0$ | $a > 0$ and $b < 0$ | $a < 0$ and $b > 0$ | $a < 0$ and $b < 0$



Ex.: Given the function $f(x) = 3\left[-\frac{1}{2}(x - 1)\right] + 2$.

We have: $a = 3$; $b = \frac{-1}{2}$; $h = 1$ and $k = 2$.

- The length of a step is $\frac{1}{|b|} = 2$.
- The height of the counterstep is $|a| = 3$.
- dom $f = \mathbb{R}$
- ran $f = \{y \mid y = 3m + 2, m \in \mathbb{Z}\}$
- zeros of f: f has no zeros, since k is not a multiple of a.
- initial value of f: 2.
- $f(x) > 0$ if $x \leqslant 1$; $f(x) < 0$ if $x > 1$
- f is decreasing over $\mathbb{R}$ since $ab < 0$.
- f has no extrema.



1. Determine the domain and range of the following functions.

a) $y = 4\left[\frac{1}{3}(x - 5)\right] - 2$

dom = ℝ ran = {y | y = 4m − 2, m ∈ ℤ}

b) $y = -2[4(x + 1)] + 4$

dom = ℝ; ran = {y | y = −2m + 4, m ∈ ℤ}

2. Determine the zeros of the following functions.

a) $y = 2\left[\frac{1}{4}(x + 1)\right] - 6$

[11, 15[

b) $y = -3[2(x - 4)] - 12$

[2, 2.5[

c) $y = 4[2x] + 2$

No zero

d) $y = -5[x - 8]$

[8, 9[

3. Determine the initial value of the function $f(x) = -3\left[\frac{1}{5}(x-9)\right] + 10$ ***16***

4. Determine over what interval the function $f(x) = 3\left[\frac{1}{4}(x-1)\right] + 6$ is positive. ***[-7, +∞[***

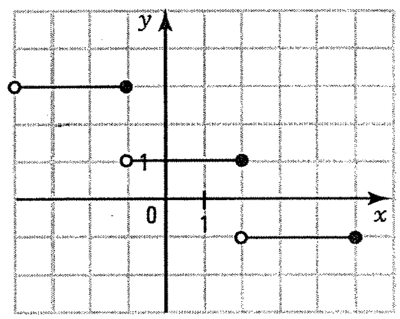
5. Determine over what interval the function $f(x) = 5[x - 3] + 1$ is strictly negative. ***]-∞, 3[***

6. Determine over what interval the function $f(x) = 3\left[\frac{1}{2}(x-7)\right] + 6$ is increasing. ***f ↗ over ℝ***

7. Consider the functions $f(x) = 2\left[\frac{1}{4}(x-1)\right] + 2$ and $g(x) = -3\sqrt{x+5} + 4$.

Determine $g \circ f(7) =$ ***-5***

8. Determine the rule of the greatest integer function represented on the right.



We choose (h, k) = (2, 1)

o—● implies that $b < 0$, $b = -\frac{1}{3}$

f ↘ implies that $a > 0$, $a = 2$

Rule: $y = 2\left[-\frac{1}{3}(x-2)\right] + 1$

9. A salesman in a store receives a weekly base salary of \$300 plus a commission of \$40 for every 10 items he sells during that week.

a) Find the rule of the function which gives the salesman's salary y as a function of the number of items sold x. $y = 40\left[\frac{x}{10}\right] + 300$

b) What is this salesman's salary if he sold 84 items this week? ***\$620***

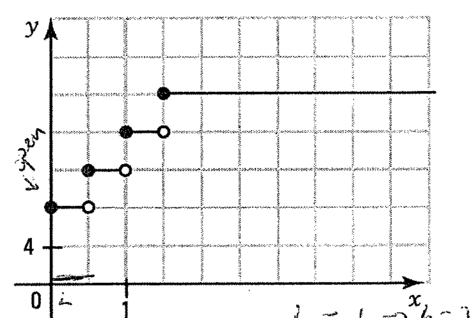
c) In what interval is the number of items sold if the salesman's salary is \$500?
In the interval [50, 60[

d) Can this salesman earn a salary of \$450? Justify your answer.
No, the equation $40\left[\frac{x}{10}\right] + 300 = 450$ ***has no solution since*** $\left[\frac{x}{10}\right] \neq 3.75$.

10. The cost of parking in a lot is \$8 for a duration of less than 30 min. Afterward, the cost increases by \$4 for every 30 minutes or part thereof. The maximum cost is \$20 per day.

a) What is the rule of the function which gives the cost y (in \$) as a function of the parking duration x (in hours).
$y = 4[2x] + 8$

b) Represent this situation in the Cartesian plane on the right.



c) What is the cost for a parking duration of 1 h 40 min? ***\$20***

d) In what interval is the parking duration if the cost is \$12?
In the interval $\left[\frac{1}{2}, 1\right[$

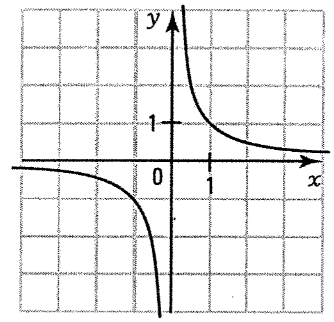
5.4 Rational function

ACTIVITY 1 Basic rational function

Consider the function $f(x) = \frac{1}{x}$.

a) What restriction must be imposed on the variable x?

x must be a non-zero real number.



b) Complete the following table.

x	−4	−2	−1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x)$	$-\frac{1}{4}$	$-\frac{1}{2}$	−1	−2	−4		4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

c) Indicate what number the variable y approaches as

1. the variable x takes positive values that are bigger and bigger. **0**
2. the variable x takes negative values that are smaller and smaller. **0**

d) Indicate the behaviour of the variable y as

1. the variable x takes positive values closer and closer to zero.
   The variable y takes bigger and bigger positive values.
2. the variable x takes negative values closer and closer to zero.
   The variable y takes smaller and smaller negative values.

e) Graph the function in the Cartesian plane above.

f) Observe the branch of the hyperbola located in the 1st quadrant.

1. When x takes positive values that are bigger and bigger, the branch gets closer and closer to the x-axis without ever touching it. *We say that the x-axis is a **horizontal asymptote** to the curve.* What is the equation of this asymptote? ***y = 0***
2. When x takes positive values closer and closer to zero, the branch gets closer and closer to the y-axis without ever touching it. *We say that the y-axis is a **vertical asymptote** to the curve.* What is the equation of this asymptote? ***x = 0***

g) Observe the branch of the hyperbola located in the 3rd quadrant.

1. Do we observe a horizontal asymptote? If yes, what is its equation?
   Yes; y = 0
2. Do we observe a vertical asymptote? If yes, what is its equation?
   Yes; x = 0

The represented curve is called a **hyperbola**. This hyperbola consists of two branches. Place a random point M(x, y) on a branch and verify that the point M′(−x, −y) is located on the other branch. The origin 0, mid-point of the segment MM′, is therefore called the **symmetric centre** of the hyperbola.

h) Determine

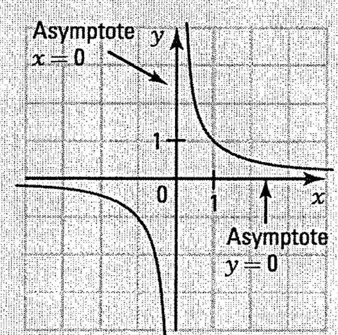
1. dom $f =$ $\mathbb{R}^*$
2. ran $f =$ $\mathbb{R}^*$
3. lthe zero of f. ***does not exist***
4. the initial value of f. ***does not exist***
5. the sign of f. ***$f(x) \geqslant 0$ over $\mathbb{R}^*_+$; $f(x) < 0$ over $\mathbb{R}^*_-$.***
6. the variation of f. ***$f \searrow$ over $\mathbb{R}^*$; f is never increasing.***
7. the extrema of f (if it exists). ***does not exist***

BASIC RATIONAL FUNCTION

- The function f defined by the rule:

$$f(x) = \frac{1}{x}$$

is called the **basic rational function.**



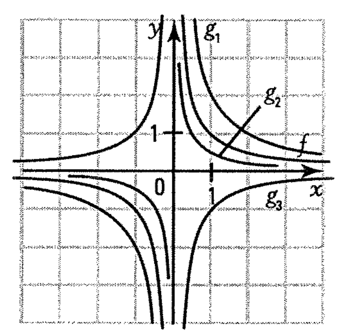
- We have:
  - dom $f = \mathbb{R}^*$, ran $f = \mathbb{R}^*$.
  - f has no zeros.
  - f is decreasing over $\mathbb{R}^*$.
  - The represented curve is called a **hyperbola**. This hyperbola consists of two branches.
  - The origin 0 is the **symmetrical** centre of the hyperbola.
  - The hyperbola has two **asymptotes** which are the x-axis and the y-axis.

ACTIVITY 2 Rational function $y = \dfrac{a}{b(x-h)} + k$

The basic rational function $f(x) = \frac{1}{x}$ can be transformed into a rational function with the rule

$$g(x) = \frac{a}{b(x-h)} + k \quad \text{(standard form)}$$

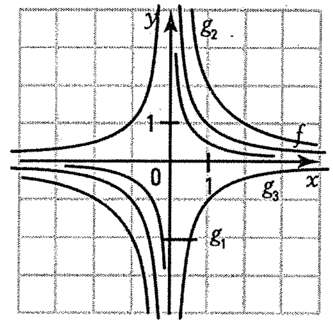
a) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{a}{x}$.



Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{2}{x}$, $g_2(x) = \frac{0.5}{x}$ and $g_3(x) = \frac{-1}{x}$ and explain how to deduce the graph of g from the graph of f when

1. $a > 1$: ***By a vertical stretch.***
2. $0 < a < 1$: ***By a vertical reduction.***
3. $a = -1$: ***By a reflection about the x-axis.***
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{a}{x}$ by the transformation $(x, y) \rightarrow$ ***(x, ay)***.

b) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{1}{bx}$.



Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{1}{2x}$, $g_2(x) = \frac{1}{0.5x}$ and $g_3(x) = \frac{1}{-x}$ and explain how to deduce the graph of g from the graph of f when

1. $b > 1$: ***By a horizontal reduction.***
2. $0 < b < 1$: ***By a horizontal stretch.***
3. $b = -1$: ***By a reflection about the y-axis.***
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{1}{bx}$ by the transformation $(x, y) \rightarrow$ $\left(\frac{x}{b}, y\right)$.
5. Compare the graphs of f and g in each of the following cases and justify your answer.

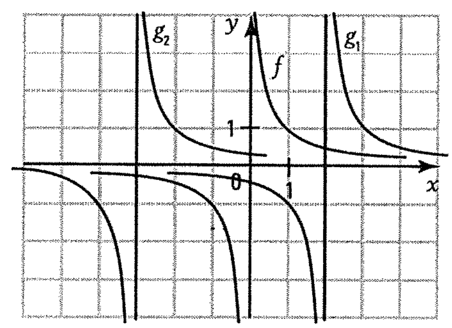
   $f(x) = \frac{2}{x}$ and $g(x) = \frac{1}{0.5x}$: ***They are the same. Indeed,*** $\frac{1}{0.5x} = \frac{1}{\frac{1}{2}x} = \frac{2}{x}$.

   $f(x) = \frac{0.5}{x}$ and $g(x) = \frac{1}{2x}$: ***They are the same. Indeed,*** $\frac{0.5}{x} = \frac{\frac{1}{2}}{x} = \frac{1}{2x}$.

   $f(x) = \frac{-1}{x}$ and $g(x) = \frac{1}{-x}$: ***They are the same. Indeed,*** $\frac{-1}{x} = \frac{1}{-x}$.

   The reflection about the x-axis and the reflection about the y-axis have the same effect on the basic rational function.

c) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{1}{x-h}$.

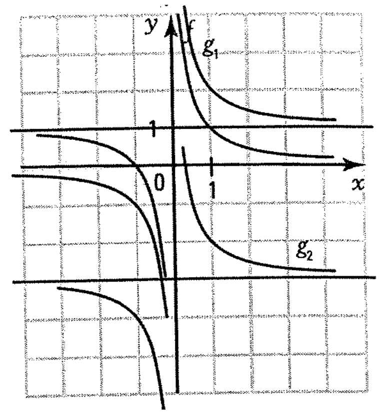


Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{1}{x-2}$, $g_2(x) = \frac{1}{x+3}$ and explain how to deduce the graph of g from the graph of f when

1. $h > 0$: ***By a horizontal translation to the right.***
2. $h < 0$: ***By a horizontal translation to the left.***
3. What is the equation of the vertical asymptote of the function $g(x) = \frac{1}{x-h}$? $x = h$
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{1}{x-h}$ by the transformation $(x, y) \rightarrow$ $(x + h, y)$.

d) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{1}{x} + k$.

Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{1}{x} + 1$, $g_2(x) = \frac{1}{x} - 3$ and explain how to deduce the graph of g from the graph of f when



1. $k > 0$: ***By a vertical translation upward.***
2. $k < 0$: ***By a vertical translation downward.***
3. What is the equation of the horizontal asymptote of the function $g(x) = \frac{1}{x} + k$? ***$y = k$.***
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{1}{x} + k$ by the transformation $(x, y) \to$ ***$(x, y + k)$***.

RATIONAL FUNCTION – STANDARD FORM

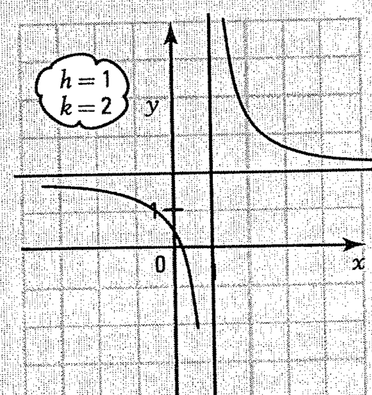
- The **standard form** of a rational function is:

$$f(x) = \frac{a}{b(x - h)} + k$$

- The graph is deduced from the graph of the basic rational function $y = \frac{1}{x}$ by the transformation

$$(x, y) \to \left(\frac{x}{b} + h, ay + k\right)$$

- This hyperbola has two asymptotes, the **vertical** asymptote with equation $x = h$ and the **horizontal** asymptote with equation $y = k$.
- The point (h, k) is the **symmetrical centre** of the hyperbola.



Ex.: To graph the hyperbola $y = \frac{3}{2(x - 1)} + 2$,

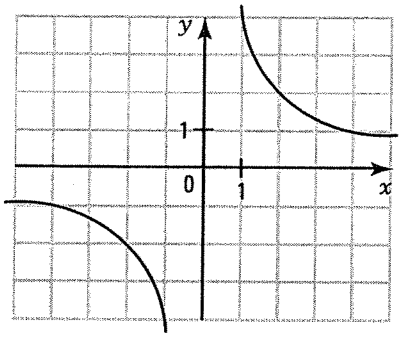
1. we draw the asymptotes.
   - vertical asymptote: $x = 1$.
   - horizontal asymptote: $y = 2$.
2. we complete a table of values.

x	−2	−1	0	1	2	3	4
y	1.5	1.25	0.5		3.5	2.75	2.5

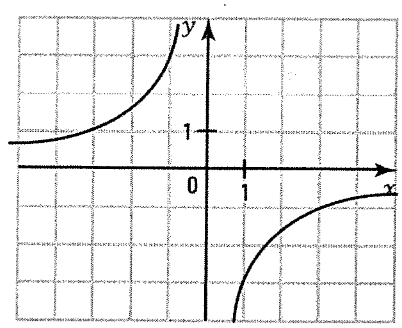
3. we draw the hyperbola using the symmetrical centre (h, k).

1. Graph the following rational functions.

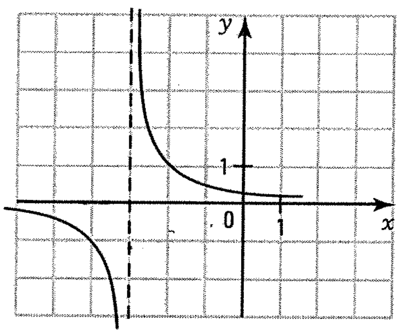
a) $f(x) = \frac{4}{x}$.



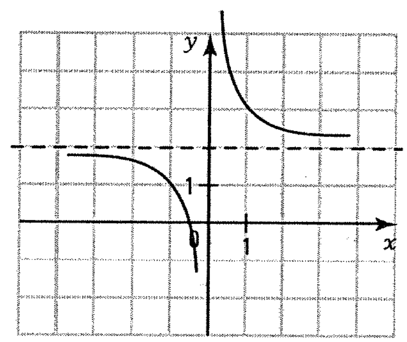
b) $f(x) = -\frac{3}{x}$.



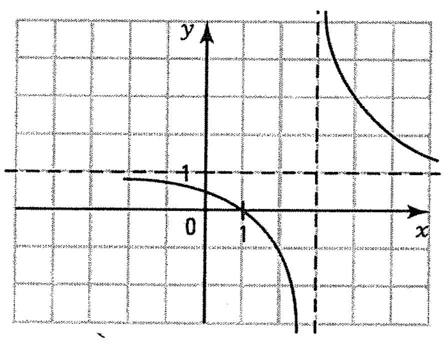
c) $f(x) = \frac{1}{x+3}$.



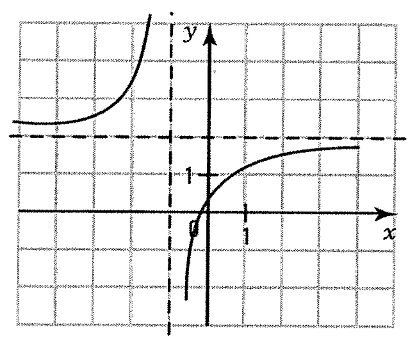
d) $f(x) = \frac{1}{x} + 2$.



e) $f(x) = \frac{2}{x-3} + 1$.

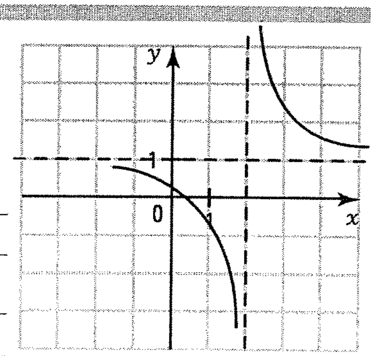


f) $f(x) = \frac{3}{-2(x+1)} + 2$.



ACTIVITY 3 Study of a rational function

Consider the function f defined by the rule $y = \frac{3}{2(x-2)} + 1$.



a) Graph the function in the Cartesian plane on the right.

b) Determine

1. dom f = ***ℝ\{2}***
2. ran f = ***ℝ\{1}***
3. the zero of f (if it exists). ***0.5***
4. the initial value of f. ***0.25***
5. the sign of f. ***$f(x) \geqslant 0$ over $\left]-\infty, \frac{1}{2}\right] \cup \left]2, +\infty\right[$; $f(x) \leqslant 0$ over $\left[\frac{1}{2}, 2\right[$***
6. the variation of f. ***f ↘ over ℝ\{2}; f ↗ never***
7. the extrema of f. ***Does not exist***

ACTIVITY 4 Finding the zero of a rational function

a) Consider the function defined by the rule: $y = \frac{-3}{4(x-2)} + 5$.

Justify the steps which enable you to find the zero of this function.

$\frac{-3}{4(x-2)} + 5 = 0$ *Replace y by zero.*

$\frac{-3}{4(x-2)} = -5$ *Subtract 5 from each side.*

$-20(x-2) = -3$ *The cross products are equal.*

$x - 2 = \frac{3}{20}$ *Divide each side by –20.*

$x = \frac{43}{20}$ *Add 2 to each side.*

b) Under what condition does the zero of a rational function defined by the rule $y = \frac{a}{b(x-h)} + k$ exist? *If $k \neq 0$*

STUDY OF A RATIONAL FUNCTION

Consider the rational function

$$f(x) = \frac{a}{b(x-h)} + k \quad \text{(standard form)}$$

- dom $f = \mathbb{R}\setminus\{h\}$; ran $f = \mathbb{R}\setminus\{k\}$
- The zero of f exists if $k \neq 0$, the initial value of f exists if $h \neq 0$.
- To study the sign of f,
  - we find the zero (if it exists);
  - we establish the sign of f using a sketch of the graph.
- Variation
  - If $ab > 0$, f is decreasing over the domain.
  - If $ab < 0$, f is increasing over the domain.
- The rational function has no extrema.

2. Determine the domain and range of the following functions.

a) $y = \frac{-2}{4(x+5)} - 7$

dom = $\mathbb{R}\setminus\{-5\}$; ran = $\mathbb{R}\setminus\{-7\}$

b) $y = \frac{3}{2(x-1)} + 4$

dom = $\mathbb{R}\setminus\{1\}$; ran = $\mathbb{R}\setminus\{4\}$

3. Determine the zero and initial value of the following functions

a) $y = \frac{3}{x-5} + 4$

Zero: $\frac{17}{4}$, i.v.: $\frac{17}{5}$

b) $y = \frac{-2}{3(x+1)}$

Zero: none, i.v.: $-\frac{2}{3}$

c) $y = \frac{-5}{4x} + 10$

Zero: $\frac{1}{8}$; i.v.: none

4. Determine the interval over which the function $f(x) = \frac{-4}{5(x-1)} + 3$ is positive. $]-\infty, 1[\cup [\frac{19}{15}, +\infty[$

5. Determine the interval over which the function $f(x) = \frac{3}{2(x+2)} - 1$ is strictly positive. $]-2, \frac{-1}{2}[$

6. Study the variation of the function $f(x) = \frac{-2}{5(x-1)} + 4$. ***f* ↗ over** $\mathbb{R}\backslash\{1\}$

7. Consider the functions $f(x) = -2|-x+4| + 5$, $g(x) = 3\sqrt{x-3} + 2$, $h(x) = \frac{6}{5(x-1)} + \frac{22}{5}$ and $i(x) = 3\left[\frac{1}{4}(x-2)\right] + 1$. Determine $f \circ g \circ h \circ i\ (1) =$ **3**

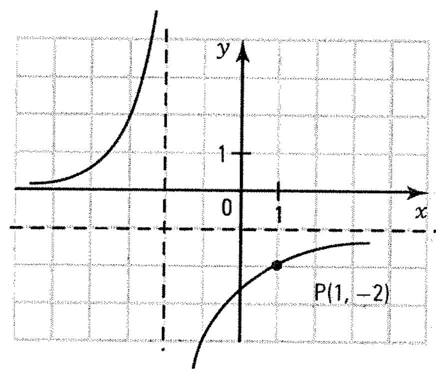
8. Given $f(x) = \frac{3}{2(x-4)} + 1$ and $g(x) = 3x - 1$. Determine, in standard form, the rule of the function $f \circ g$. $\boldsymbol{f \circ g(x) = f(3x-2) = \frac{3}{2(3x-6)} + 1 = \frac{1}{2(x-2)} + 1.}$

ACTIVITY 5 Finding the rule of a rational function

Any rule of a rational function can be written in the form $y = \frac{a}{x-h} + k$.

a) Consider the function $y = \frac{-3}{6(x-2)} + 1$. Write the rule of this function in the form $y = \frac{a}{x-h} + k$.

$\boldsymbol{y = \frac{-0.5}{x-2} + 1}$



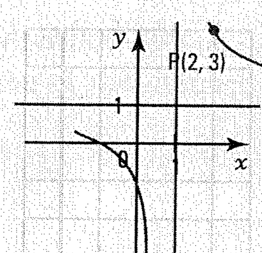
b) Consider a rational function whose graph passes through the point P(1, –2).

1. Identify h and k. $\boldsymbol{h = -2, k = -1}$
2. Determine a knowing that the coordinates of the point P(1, –2) verify the rule of the function.
   We have: $\boldsymbol{y = \frac{a}{x+2} - 1;\ -2 = \frac{a}{1+2} - 1;\ -1 = \frac{a}{3};\ a = -3.}$
3. What is the rule of the function? $\boldsymbol{y = \frac{-3}{x+2} - 1}$

FINDING THE RULE OF A RATIONAL FUNCTION

Any rule of a rational function can be written in the form

$$y = \frac{a}{x-h} + k$$



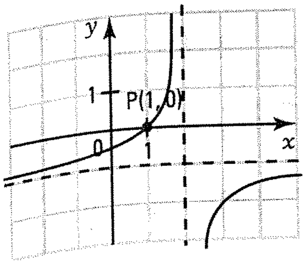
The asymptotes and a point P are known.

1. Identify the parameters h and k.
2. Find a after replacing, in the rule, x and y by the coordinates of the given point P.
3. Deduce the rule.

1. $h = 1$ and $k = 1$
   $y = \frac{a}{x-1} + 1$
2. $3 = \frac{a}{2-1} + 1$
   $a = 2$
3. $y = \frac{2}{x-1} + 1$

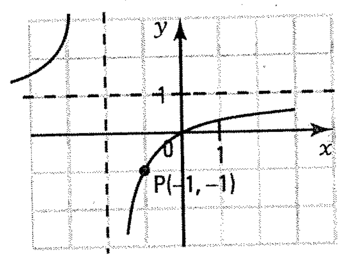
9. Find the rule of the following rational functions.

a)



$y = \frac{-1}{x-2} - 1$

b)



$y = \frac{-2}{x+2} + 1$

ACTIVITY 6 Inverse of a rational function – Standard form

a) Consider the rational function f defined by the rule: $y = \frac{-2}{3(x+1)} - 5$.

Justify the steps which enable you to determine the rule of the inverse function f^{-1}.

1. Isolate x in the equation $y = \frac{-2}{3(x+1)} - 5$

$y + 5 = \frac{-2}{3(x+1)}$ — ***Add 5 to each side.***

$3(x+1) = \frac{-2}{y+5}$ — ***Switch the extremes.***

$x + 1 = \frac{-2}{3(y+5)}$ — ***Divide each side by 3.***

$x = \frac{-2}{3(y+5)} - 1$ — ***Subtract 1 from each side.***

2. Interchange the letters x and y to obtain the rule of the inverse. We get: $y = \frac{-2}{3(x+5)} - 1$

b) Complete: The inverse of a rational function is a ***rational*** function.

c) 1. Determine

1) dom f = $\mathbb{R}\setminus\{-1\}$ 2) ran f = $\mathbb{R}\setminus\{-5\}$

3) dom f^{-1} = $\mathbb{R}\setminus\{-5\}$ 4) ran f^{-1} = $\mathbb{R}\setminus\{-1\}$

2. Verify that dom f^{-1} = ran f and that ran f^{-1} = dom f.

INVERSE OF A RATIONAL FUNCTION

The **inverse** of a rational function is a rational function.

Ex.: Given the rational function defined by the rule $y = \frac{-2}{3(x+1)} - 5$.

The inverse f^{-1} is a rational function defined by the rule $y = \frac{-2}{3(x+5)} - 1$.

(See activity 6 for finding the rule of f^{-1})

Note that dom f = ran $f^{-1} = \mathbb{R}\backslash\{1\}$ and that ran f = dom $f^{-1} = \mathbb{R}\backslash\{5\}$

10. Determine the inverse of the following rational functions.

a) $y = \frac{3}{x+5} - 1$ $y = \frac{3}{x+1} - 5$

b) $y = \frac{-1}{2(x-4)} + 3$ $y = \frac{-1}{2(x-3)} + 4$

11. A train travels a distance of 240 km. We consider the function f which gives the duration t (in h) of the trip as a function of the train's speed v (in km/h).

v (km/h)	40	60	80	120	160
t (h)	6	4	3	2	1.5

a) Complete the table of values on the right.

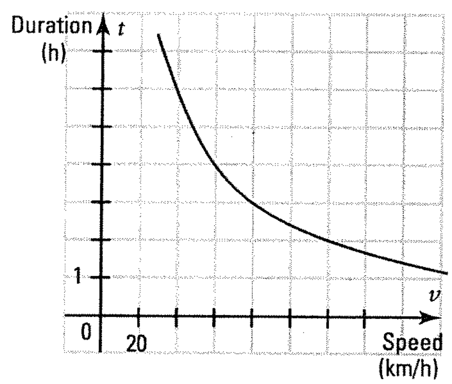
b) Is the rate of change of the function f constant? *No*

c) Verify that the product of the variables vt is constant. $vt = 240$

We say that the duration of the trip is **inversely proportional** to the speed or that the speed is inversely proportional to the duration of the trip.

d) What is the rule of the function? $t = \frac{240}{v}$

e) Graph the function f in the Cartesian plane.



f) Determine

1. dom f. $]0, +\infty[$
2. ran f. $]0, +\infty[$

g) When one variable increases, does the other variable increase or decrease? *It decreases.*

h) Is the function f increasing or decreasing? Justify your answer.

Decreasing, since the duration decreases as the speed increases.

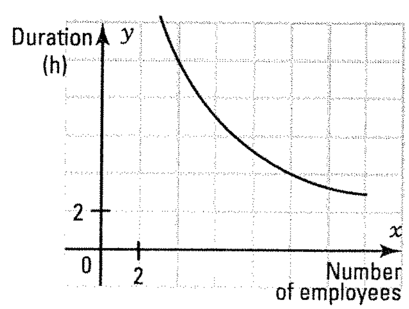
12. Renovations to a home require a total of 40 h of work for one employee. Consider the function f which gives the duration y (in h) of work per employee as a function of the number of employees x hired to do the renovations.

a) Complete the following table of values.

x	1	2	4	5	8	10
y	40	20	10	8	5	4

b) What is the rule of function f? $y = \frac{40}{x}$

c) Graph the function f in the Cartesian plane.



d) Is the function f increasing or decreasing? *Decreasing*

ACTIVITY 7 Rational function – General form

Consider the rational function defined by the rule $y = \frac{3}{2(x-5)} + 4$ (standard form).

a) Justify the steps which enable you to write the rule of this function in the form $y = \frac{ax+b}{cx+d}$.

$y = \frac{3}{2(x-5)} + 4 = \frac{3}{2(x-5)} + \frac{8(x-5)}{2(x-5)}$ ***Finding a common denominator***

$= \frac{3+8(x-5)}{2(x-5)}$ ***Addition of the 2 fractions;*** $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

$= \frac{8x-37}{2x-10}$ ***Simplification***

The form $y = \frac{ax+b}{cx+d}$ *is called the* ***general form*** *of a rational function.*

b) 1. Identify the parameters h and k of the standard form. ***h = 5; k = 4***

2. Identify the parameters a, b, c and d of the general form. ***a = 8, b = −37, c = 2, d = −10***

3. Verify that the vertical asymptote has the equation $x = -\frac{d}{c}$. ***x = h = 2 and x = −*** $\frac{d}{c}$ ***= 2***

4. Verify that the horizontal asymptote has the equation $y = \frac{a}{c}$. ***y = k = 4 and y =*** $\frac{a}{c}$ ***= 4***

c) Consider the rational function $y = \frac{5x-3}{2x+4}$ (general form)

To obtain the standard form from the general form $y = \frac{A(x)}{B(x)}$ where $A(x) = 5x - 3$ and $B(x) = 2x + 4$, we proceed in the following manner:

1° Determine the quotient $Q(x)$ and the remainder $R(x)$ from Euclidean division (i.e. long division) of $A(x)$ by $B(x)$.

A(x)	B(x)
R(x)	Q(x)

2° From the Euclidean relation $A(x) = B(x) \cdot Q(x) + R(x)$, we deduce the standard form of the rule.

$$\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{Q(x)}$$

1. Perform the Euclidean division of $A(x) = 5x - 3$ by $B(x) = 2x + 4$ and determine the quotient $Q(x)$ and the remainder $R(x)$.

5x − 3	2x + 4
− 5x + 10	$\frac{5}{2}$
−13	

Q(x) = $\frac{5}{2}$***; R(x) = −13***

2. Deduce the standard form of the rule of the function $y = \frac{5x-3}{2x+4}$. $y = \frac{5}{2} + \frac{-13}{2(x+2)}$

RATIONAL FUNCTION – GENERAL FORM

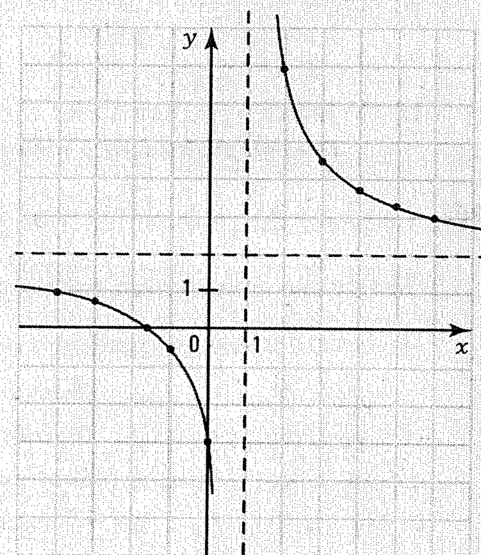
- The general form of the rule of a rational function is:

$$f(x) = \frac{ax+b}{cx+d}$$

- $\operatorname{dom} f = \mathbb{R}\setminus\left\{-\frac{d}{c}\right\}$; $\operatorname{ran} f = \mathbb{R}\setminus\left\{\frac{a}{c}\right\}$.
- Vertical asymptote: $x = -\frac{d}{c}$; horizontal asymptote: $y = \frac{a}{c}$.

Ex.: Given the rational function $f(x) = \frac{2x+3}{x-1}$.

- $\operatorname{dom} f = \mathbb{R}\setminus\{1\}$; $\operatorname{ran} f = \mathbb{R}\setminus\{2\}$.
- Vertical asymptote: $x = 1$;
  horizontal asymptote: $y = 2$.
- Zero of f: $f(x) = 0 \Leftrightarrow 2x + 3 = 0 \Leftrightarrow x = -\frac{3}{2}$.
- Sign of f: $f(x) \geqslant 0 \Leftrightarrow x \in \left]-\infty, -\frac{3}{2}\right] \cup \left]1, +\infty\right[$
  $f(x) \leqslant 0 \Leftrightarrow x \in \left[-\frac{3}{2}, 1\right[$.
- Variation of f: f is decreasing over $\mathbb{R}\setminus\{1\}$.
- f has no extrema.



13. Determine the domain and range of the following rational functions.

a) $y = \frac{3x+2}{x-5}$

***dom** = $\mathbb{R}\setminus\{5\}$, **ran f** = $\mathbb{R}\setminus\{3\}$*

b) $y = \frac{-2x+4}{3x-6}$

***dom** = $\mathbb{R}\setminus\{2\}$, **ran f** = $\mathbb{R}\setminus\left\{-\frac{2}{3}\right\}$*

c) $y = \frac{5x+4}{2x-3}$

***dom** = $\mathbb{R}\setminus\left\{\frac{3}{2}\right\}$, **ran f** = $\mathbb{R}\setminus\left\{\frac{5}{2}\right\}$*

14. Determine the zero (if it exists) and the initial value (if it exists) of the following functions.

a) $y = \frac{3x-2}{x-4}$

Zero:** $\frac{2}{3}$, **i.v.: 3

b) $y = \frac{-5x+10}{2x-5}$

Zero: 2, i.v.: -2

c) $y = \frac{-2x-6}{4x}$

Zéro: -3, i.v.: does not exist

15. Determine over which interval the following functions are positive.

a) $y = \frac{4x+2}{x-3}$

***f(x) ⩾ 0 over** $\left]-\infty, \frac{-1}{2}\right] \cup \left]3, +\infty\right[$*

b) $y = \frac{-2x+8}{4x-2}$

***f(x) ⩾ 0 over** $\left]\frac{1}{2}, 4\right]$*

16. Study the variation of the following functions.

a) $y = \frac{-4x+9}{x-3}$

***f** ↗ **over** $\mathbb{R}\setminus\{3\}$*

b) $y = \frac{2x+5}{3x-2}$

***f** ↘ **over** $\mathbb{R}\setminus\left\{\frac{2}{3}\right\}$*

17. Write the rule of the following rational functions in general form.

a) $y=\frac{3}{2(x-1)}+4$ $y=\frac{8x-5}{2x-2}$

b) $y=\frac{-2}{5(x-3)}-1$ $y=\frac{-5x+13}{5x-15}$

18. Write the rule of the following rational functions in standard form.

a) $y=\frac{3x+2}{x-3}$

$y=\frac{11}{x-3}+3$

b) $y=\frac{4x+3}{2x-6}$

$y=\frac{15}{2(x-3)}+2$

c) $y=\frac{-2x+5}{3x+4}$

$y=\frac{23}{9\left(x+\frac{4}{3}\right)}-\frac{2}{3}$

19. Consider the rational functions $f(x)=\frac{2x+3}{x-4}$ and $g(x)=\frac{3x+5}{x+3}$.

a) Determine the rule of the composite

1. $g\circ f(x)=$ $y=\frac{11x-11}{5x-9}$

2. $f\circ g(x)=$ $y=\frac{9x+19}{-x-7}$

b) What can you say about the composition of a rational function with a rational function?

The composition of a rational function with a rational function is also a rational function.

20. Consider the rational function $y=\frac{5x+4}{x-3}$ (general form).

Justify the steps which enable you to determine the rule of the inverse f^{-1}.

1. Isolate x in the equation $y=\frac{5x+4}{x-3}$.

$y(x-3)=5x+4$ *Cross products are equal.*

$xy-3y=5x+4$ *Distributive property of multiplication over subtraction.*

$xy-5x=3y+4$ *Subtract 5x and add 3y to each side.*

$x(y-5)=3y+4$ *Factor out x on the left side.*

$x=\frac{3y+4}{y-5}$ *Isolate the variable x.*

2. Switch the letters x and y to obtain the rule of the inverse.

We get: $y=\frac{3x+4}{x-5}$.

21. Consider the rational function $f(x)=\frac{3x-2}{2x+5}$.

a) Determine the rule of the inverse f^{-1}. $f^{-1}(x)=\frac{-5x-2}{2x-3}$

b) Verify that

1. $f\circ f^{-1}(x)=x$

2. $f^{-1}\circ f(x)=x$

22. Consider the rational function $f(x)=\frac{-2x+3}{4x+1}$.

a) Determine the domain and range of f. $\text{dom } f=\mathbb{R}\setminus\left\{-\frac{1}{4}\right\}$, $\text{ran } f=\mathbb{R}\setminus\left\{-\frac{1}{2}\right\}$

b) Determine the rule of the inverse f^{-1}. $f^{-1}(x)=\frac{-x+3}{4x+2}$

c) Determine the domain and range of the inverse f^{-1} and verify that dom f^{-1} = ran f and ran f^{-1} = dom f.

$\text{dom } f^{-1}=\mathbb{R}\setminus\left\{-\frac{1}{2}\right\}$, $\text{ran } f^{-1}=\mathbb{R}\setminus\left\{-\frac{1}{4}\right\}$

Evaluation 5

1. Determine the domain and range of each of the following functions.

a) $y = 4|x - 5| + 8$

dom = ℝ, ran = [8, +∞[

b) $y = \frac{1}{2}\sqrt{-(x - 4)} + 3$

dom =]−∞, 4], ran = [3, +∞[

c) $y = -2\left[\frac{1}{3}(x - 5)\right] + 4$

dom = ℝ,
ran = {y | y = −2m + 4, m ∈ ℤ}

d) $y = \frac{4}{3(x - 1)} + 2$

dom = ℝ\{1}, ran = ℝ\{2}

2. Determine the zero (s) and the initial value of each of the following functions.

a) $y = \frac{3}{4}\sqrt{x + 1} - 3$

Zero: 15, i.v.: $\frac{-9}{4}$

b) $y = 3\left[\frac{1}{2}(x - 5)\right] + 6$

Zeros: [1, 3[, i.v.: −3

c) $y = \frac{-2}{5(x - 1)} + 4$

Zero: $\frac{11}{10}$, ***i.v.:*** $\frac{22}{5}$

d) $y = 3|2x - 1| - 6$

Zeros: $\frac{-1}{2}$ ***and*** $\frac{3}{2}$, ***i.v.: −3***

3. Determine over what interval each of the following functions is negative.

a) $y = 2|8 - x| - 12$

[2, 14]

b) $y = \frac{2}{x - 5} + 4$

$\left]\frac{9}{2}, 5\right[$

c) $y = -2\sqrt{6 - x} + 4$

]−∞, 2]

d) $y = -\left[\frac{x}{2}\right] - 3$

[−6, +∞[

4. Determine over what interval each of the following functions is increasing.

a) $y = -[6 - 3x] + 1$

ℝ

b) $y = -3\sqrt{-(x - 1)} + 4$

]−∞, 1]

c) $y = 3|x - 5| + 2$

[5, +∞[

d) $y = \frac{3}{2(x - 1)} + 5$

∅

5. Determine, if it exists, the extrema of each of the following functions.

a) $y = -2|3 - 2x| + 5$

max = 5

b) $y = -2\sqrt{x} + 7$

max = 7

6. Find the rule of the inverse of each of the following functions.

a) $y = \frac{2x + 1}{x - 3}$

$y = \frac{3x + 1}{x - 2}$

b) $y = 3\sqrt{2 - x} + 4$

$y = -\frac{1}{9}(x - 4)^2 + 2, x \geqslant 4$

c) $y = \frac{3}{2(x - 1)} + 8$

$y = \frac{3}{2(x - 8)} + 1$

7. Consider the following real functions.

$f(x) = 3x - 8$ $\qquad g(x) = 3\sqrt{2x+1} - 5$ $\qquad h(x) = -2|x - 4| + 12$

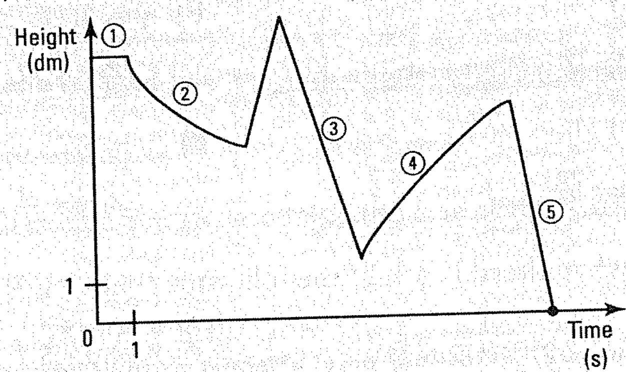
$i(x) = 3(x - 2)^2 + 4$ $\qquad k(x) = \frac{2}{x-5} + 1$ $\qquad l(x) = 3\left[\frac{1}{5}(x + 4)\right] - 6$

Determine

a) $f \circ g(4) =$ ***4*** **b)** $l \circ h(3) =$ ***0*** **c)** $k \circ i(5) =$ $\boldsymbol{\frac{14}{13}}$

d) $f \circ l(0) =$ ***−26*** **e)** $k \circ f \circ h(2) =$ $\boldsymbol{\frac{13}{11}}$ **f)** $l \circ h(-6) =$ ***−9***

8. The path of a marble in a child's game can be represented by the graph in the Cartesian plane below. Initially, the marble is at a height of 7 dm from the ground.



$$f(x) = \begin{cases} 7 & \text{if } 0 \leqslant x < 1 \\ \frac{3}{x} + 4 & \text{if } 1 \leqslant x \leqslant 4 \\ \boldsymbol{a}|x - 5| + 8 & \text{if } 4 \leqslant x \leqslant 7 \\ 2\sqrt{x - 7} + \boldsymbol{k} & \text{if } 7 \leqslant x \leqslant 11 \\ -5.5x + \boldsymbol{b} & \text{if } 11 \leqslant x \leqslant \boldsymbol{t} \end{cases}$$

Determine the duration t of the marble's path.

f(4) = 4.75: $\quad$ ***a |4 − 5| + 8 = 4.75;*** $\quad$ ***a = −3.25***

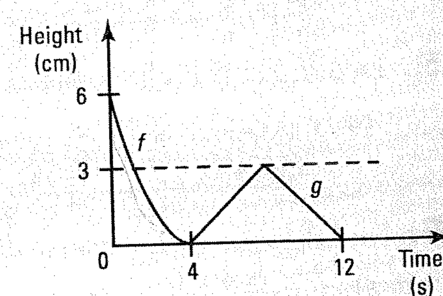
f(7) = 1.5; k = 1.5; $\quad$ $\boldsymbol{f(11) = 2\sqrt{11 - 7} + 1.5 = 5.5}$

−5.5(11) + b = 5.5; b = 66; −5.5 t + 66 = 0 ⇒ t = 12 s.

9. Aaron is playing an electronic game. The height of a flashing dot on the screen can be modeled by a square root function f from 0 to 4 seconds and by an absolute value function g from 4 to 12 seconds as indicated by the graph on the right.

The starting point of the flashing dot is the vertex of the function f.

Determine at what times the flashing dot is at a height of 1.5 cm.



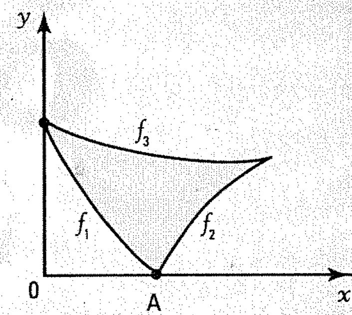
$\boldsymbol{f(x) = -3\sqrt{x} + 6;\ g(x) = -\frac{3}{4}|x - 8| + 3}$

The flashing dot is at a height of 1.5 cm at the times t = 2.25 s, t = 6 s and t = 10 s.

10. A company's logo was drawn using the graphs of three square root functions as indicated in the figure on the right.

The rules of the functions f_1 and f_3 are respectively $f_1(x) = -\frac{4}{3}\sqrt{x} + 4$ and $f_3(x) = -\frac{1}{4}\sqrt{x} + 4$.

The x-coordinate of the intersection point of the functions f_2 and f_3 is 16. Knowing that point A is the vertex of the function f_2, what is the rule of the function f_2?



$f_3(16) = 3$; A(9, 0); f_2: $y = a\sqrt{x - 9}$

The rule of the function f_2 is: $y = \frac{3}{7}\sqrt{7(x - 9)}$

11. The value of one Kandev share fluctuated, over a one-month period, according to the rule of an absolute value function. At the opening of the market, this share was worth \$3.50. Twelve days later, it reaches its maximum value of \$8.

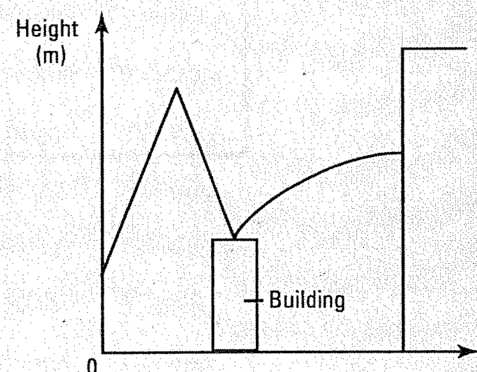
How many days go by between the moment the value of the share is worth \$5 for the first time and the moment it is worth \$2 on its descent?

$y = -\frac{3}{8}|x - 12| + 8$; $-\frac{3}{8}|x - 12| + 8 = 5$; $-\frac{3}{8}|x - 12| + 8 = 2$.

24 days.

12. The graph on the right illustrates a projectile's trajectory thrown from a height of 7 m. After 15 seconds, it reaches its maximum height of 40 m before descending onto the roof of an 18 m high building. The projectile bounces and, 4 seconds later, is at a height of 20 m.

The first trajectory follows the model of an absolute value function and the second one follows the model of a square root function whose vertex corresponds to the point where it hits the roof of the building.



The projectile hits the wall of another building at a height of 25 m. How many seconds after the projectile was thrown does it hit the wall of the second building?

$y = -2{,}2\,|x - 15| + 40$; $-2{,}2\,|x - 15| + 40 = 18$; $y = \sqrt{x - 25} + 18$.

The projectile hits the wall of the second building 74 s after it is thrown.

Chapter 6

Exponential and logarithmic functions

CHALLENGE 6

6.1 Basic exponential function

6.2 Exponential function

6.3 Basic logarithmic function

6.4 Logarithmic function

6.5 Logarithmic calculations

EVALUATION 6