

CHALLENGE 7

1. Trigonometric equations

a) $\tan t = \frac{b}{a}$ b) $\cotan t = \frac{a}{b}$ c) $\sec t = \frac{1}{a}$ d) $\csc t = \frac{1}{b}$

2. Determine the exact coordinates of the following trigonometric points.

a) $P\left(\frac{25\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ b) $P\left(-\frac{19\pi}{4}\right) = \left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$ c) $P\left(-\frac{16\pi}{3}\right) = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$

3. If $\tan t = \frac{12}{5}$ and $\pi \leq t \leq \frac{3\pi}{2}$, determine

a) $\sin t = \frac{-12}{13}$ b) $\cos t = \frac{-5}{13}$ c) $\sec t = \frac{-13}{5}$ d) $\csc t = \frac{-13}{12}$

4. Solve the equation $2 \sin \pi(x + 1) + 1 = 0$ in \mathbb{R} .

$$S = \left\{ \frac{-7}{6} + 2n \right\} \cup \left\{ \frac{1}{6} + 2n \right\}$$

5. Given the function $f(x) = -2 \cos \frac{\pi}{3}(x + 1) + 1$.

a) Determine

1. the amplitude of f . 2 2. the period of f . 6
 3. the domain of f . \mathbb{R} 4. the range of f . $[-1, 3]$

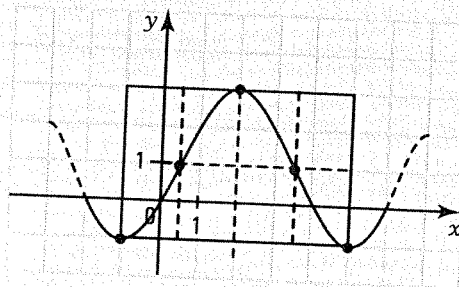
b) Graph the function f in the Cartesian plane.

c) Determine in \mathbb{R}

1. the zeros of f . $\{0 + 6n\} \cup \{4 + 6n\}$
 2. the sign of f . $f(x) \geq 0$ over $[6n, 4 + 6n]$; $f(x) \leq 0$ over $[-1 + 6n, 6n] \cup [4 + 6n, 5 + 6n]$
 3. the variation of f . $f \nearrow$ over $[-1 + 6n, 2 + 6n]$; $f \searrow$ over $[2 + 6n, 5 + 6n]$

d) Determine over $[11, 17]$

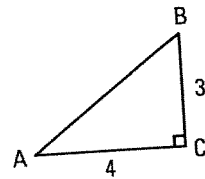
1. the zeros of f . 12 and 16
 2. the sign of f . $f(x) \geq 0$ over $[12, 16]$;
 $f(x) \leq 0$ over $[11, 12] \cup [16, 17]$
 3. the variation of f .
 $f \nearrow$ over $[11, 14]$; $f \searrow$ over $[14, 17]$



5.1 Trigonometric ratios in a right triangle

ACTIVITY 1 Trigonometric ratios

Consider the triangle ABC on the right.



- a) 1. What is the length of the hypotenuse? 5 units
 2. What can we say about the acute angles A and B? They are complementary.

- b) Determine the ratios
 1. $\sin A = \frac{3}{5}$ 2. $\cos A = \frac{4}{5}$ 3. $\tan A = \frac{3}{4}$

c) By inverting each of the preceding ratios, we define the ratios

$$\sec A = \frac{1}{\cos A}, \csc A = \frac{1}{\sin A} \text{ and } \cot A = \frac{1}{\tan A}. \text{ Determine}$$

1. $\sec A = \frac{5}{4}$ 2. $\csc A = \frac{5}{3}$ 3. $\cot A = \frac{4}{3}$

d) Verify the following trigonometric identities.

1. $\sin^2 A + \cos^2 A = 1$ $\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = 1$

2. $1 + \tan^2 A = \sec^2 A$ $1 + \left(\frac{3}{4}\right)^2 = 1 + \frac{9}{16} = \frac{25}{16} = \sec^2 A$

3. $1 + \cot^2 A = \csc^2 A$ $1 + \left(\frac{4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9} = \csc^2 A$

TRIGONOMETRIC RATIOS

- All right triangles verify the Pythagorean theorem:

$$a^2 + b^2 = c^2$$

- The acute angles of a right triangle are complementary.

$$m \angle A + m \angle B = 90^\circ$$

- We define the following ratios for the angle A.

$$\sin A = \frac{\text{measure of the opposite side}}{\text{measure of the hypotenuse}} = \frac{a}{c}$$

$$\cos A = \frac{\text{measure of the adjacent side}}{\text{measure of the hypotenuse}} = \frac{b}{c}$$

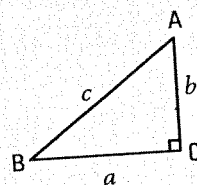
$$\tan A = \frac{\text{measure of the opposite side}}{\text{measure of the adjacent side}} = \frac{a}{b}$$

By inverting the preceding ratios, we get:

$$\sec A = \frac{1}{\cos A} = \frac{c}{b}$$

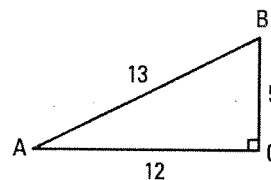
$$\csc A = \frac{1}{\sin A} = \frac{c}{a}$$

$$\cot A = \frac{1}{\tan A} = \frac{b}{a}$$



\overline{AB} is the hypotenuse.
 \overline{BC} is the opposite side to $\angle A$.
 \overline{AC} is the adjacent side to $\angle A$.

1. Consider the right triangle ABC on the right.



a) Determine the following ratios.

1. $\sin A = \frac{5}{13}$ 2. $\cos A = \frac{12}{13}$ 3. $\tan A = \frac{5}{12}$
 4. $\sec A = \frac{13}{12}$ 5. $\csc A = \frac{13}{5}$ 6. $\cot A = \frac{12}{5}$

b) Verify the trigonometric identities.

1. $\sin^2 A + \cos^2 A = 1$ $\left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = 1$

2. $1 + \tan^2 A = \sec^2 A$ $1 + \left(\frac{5}{12}\right)^2 = \left(\frac{13}{12}\right)^2$

3. $1 + \cot^2 A = \csc^2 A$ $1 + \left(\frac{12}{5}\right)^2 = \left(\frac{13}{5}\right)^2$

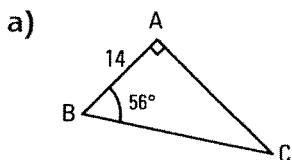
c) Verify that

1. $\tan A = \frac{\sin A}{\cos A} = \frac{5}{12} = \frac{\frac{5}{13}}{\frac{12}{13}}$ 2. $\cot A = \frac{\cos A}{\sin A} = \frac{12}{5} = \frac{\frac{12}{13}}{\frac{5}{13}}$

d) Verify that

1. $\sin A = \cos B$ 2. $\cos A = \sin B$ 3. $\tan A = \cot B$

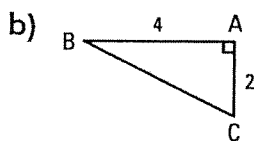
2. Solve the following triangles (round the measures of the sides and angles to the nearest tenth).



$m\overline{AC} = 20.8$

$m\overline{BC} = 25.0$

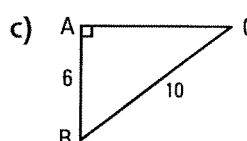
$m\angle C = 34^\circ$



$m\overline{BC} = 4.5$

$m\angle B = 26.6^\circ$

$m\angle C = 63.4^\circ$



$m\overline{AC} = 8$

$m\angle B = 53.1^\circ$

$m\angle C = 36.9^\circ$

ACTIVITY 2 Remarkable angles: 0° , 30° , 45° , 60° , 90°

a) The triangle ABC on the right is equilateral, with each side measuring 1 unit.

We have drawn the altitude AH.

1. Explain why $m\overline{BH} = 0.5$ u.

In an equilateral triangle, the altitude AH is also a median.

2. Explain why $m\angle ABC = m\angle BAC = m\angle ACB = 60^\circ$.

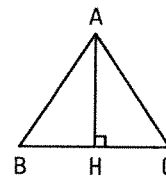
In an equilateral triangle, each angle measures 60° .

3. Explain why $m\angle BAH = 30^\circ$.

In an equilateral triangle, the altitude AH is also a perpendicular bisector.

4. Refer to the triangle ABH to show that $\sin 30^\circ = \frac{1}{2}$.

$\sin 30^\circ = \frac{0.5}{1} = \frac{1}{2}$



5. Explain why $m\overline{AH} = \frac{\sqrt{3}}{2}$.

$$m\overline{AH}^2 + m\overline{BH}^2 = m\overline{AB}^2 \text{ (Pythagoras)} \Rightarrow m\overline{AH}^2 = 1 - (0.5)^2 = \frac{3}{4} \Rightarrow m\overline{AH} = \frac{\sqrt{3}}{2}$$

6. Refer to the triangle ABH to show that $\sin 60^\circ = \frac{\sqrt{3}}{2}$.

$$\sin 60^\circ = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

7. Explain why $\cos 60^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

$$\cos 60^\circ = \sin 30^\circ = \frac{1}{2} \text{ (complementary angles); } \cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2} \text{ (complementary angles)}$$

8. Explain why $\tan 30^\circ = \frac{\sqrt{3}}{3}$ and $\tan 60^\circ = \sqrt{3}$

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}; \quad \tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

b) The given right triangle on the right is isosceles. The hypotenuse measures 1 unit.

1. What is the measure of the sides of the right angle?

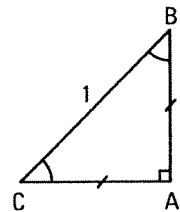
$$\text{Let } m\overline{AB} = m\overline{AC} = x$$

$$\text{We have: } x^2 + x^2 = 1 \text{ (Pythagoras); } 2x^2 = 1; x = \frac{\sqrt{2}}{2}$$

2. What is the measure of each acute angle? 45°

3. Show that $\sin 45^\circ = \frac{\sqrt{2}}{2}$ and that $\cos 45^\circ = \frac{\sqrt{2}}{2}$.

$$\sin 45^\circ = \sin B = \frac{\sqrt{2}}{2}; \quad \cos 45^\circ = \cos B = \frac{\sqrt{2}}{2}$$



$$\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

4. Explain why $\tan 45^\circ = 1$.

c) Using a calculator, verify that $\sin 0^\circ = 0$; $\sin 90^\circ = 1$; $\cos 0^\circ = 1$ and $\cos 90^\circ = 0$.

d) Explain why $\tan 90^\circ$ is not defined.

$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0}; \text{ Division by 0 is not defined.}$$

REMARKABLE ANGLES

Angles	0°	30°	45°	60°	90°
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	

Memorize the sine line.

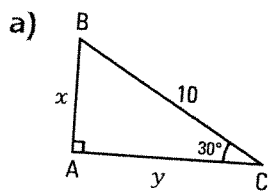
From the sine line, we deduce the cosine line, since $\cos x = \sin(90^\circ - x)$.

From the sine and cosine lines, we deduce the tangent line, since $\tan x = \frac{\sin x}{\cos x}$.

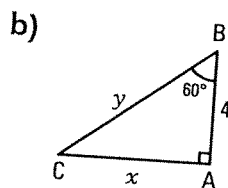
3. Complete the following table by giving the exact ratios.

Angles	0°	30°	45°	60°	90°
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	
Cosecant		2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent		$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

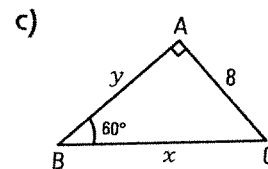
4. Determine the exact measures of x and y .



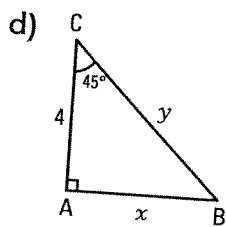
$$x = 5; y = 5\sqrt{3}$$



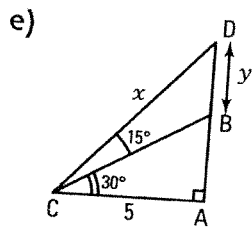
$$x = 4\sqrt{3}; y = 8$$



$$x = \frac{16\sqrt{3}}{3}; y = \frac{8\sqrt{3}}{3}$$



$$x = 4; y = 4\sqrt{2}$$

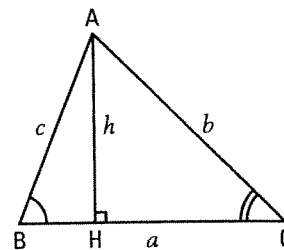


$$x = 5\sqrt{2}; y = 5 - \frac{5\sqrt{3}}{3}$$

7.2 Sine and cosine law

ACTIVITY 1 Sine law

a) In the right triangle ABC on the right, let a , b and c represent the measures of sides BC, AC and AB. The altitude AH is drawn and represented by h . Explain why



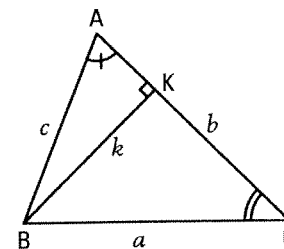
1. $h = c \sin B$. In the $\triangle ABH$, we have: $\sin B = \frac{h}{c}$.

2. $h = b \sin C$. In the $\triangle ACH$, we have: $\sin C = \frac{h}{b}$.

3. $c \sin B = b \sin C$. From equations 1 and 2, we deduce the 3rd equation.

4. $\frac{b}{\sin B} = \frac{c}{\sin C}$. Property of cross-products in a proportion.

b) In the same triangle ABC, the altitude BK is drawn and represented by k . Explain why



1. $k = c \sin A$. In the $\triangle ABK$, we have: $\sin A = \frac{k}{c}$.

2. $k = a \sin C$. In the $\triangle BCK$, we have: $\sin C = \frac{k}{a}$.

3. $c \sin A = a \sin C$. From equations 1 and 2, we deduce the 3rd equation.

4. $\frac{a}{\sin A} = \frac{c}{\sin C}$. Property of cross-products in a proportion.

c) Compare the three ratios $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$. What can be deduced?

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

SINE LAW

- The sides in a triangle are directly proportional to the sine of the opposite angles to these sides.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- The sine law can be used to find the measure of a missing side or angle.

1st case: Finding a side when we know two angles and a side.

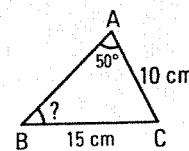
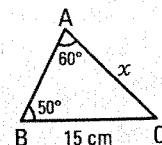
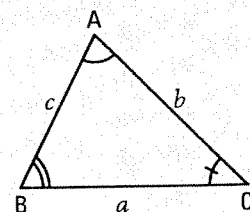
We calculate the measure x of AC.

$$\frac{x}{\sin 50^\circ} = \frac{15}{\sin 60^\circ} \Rightarrow x = \frac{15 \sin 50^\circ}{\sin 60^\circ} \approx 13.27 \text{ cm}$$

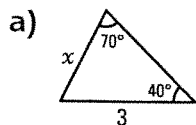
2nd case: Finding an angle when we know two sides and the opposite angle to one of these two sides.

We calculate the measure of angle B.

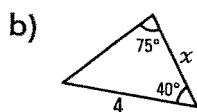
$$\frac{10}{\sin B} = \frac{15}{\sin 50^\circ} \Rightarrow \sin B = \frac{10 \sin 50^\circ}{15} = 0.5893 \Rightarrow m \angle B \approx 36^\circ$$



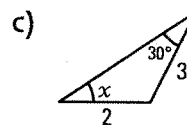
1. Use the sine law to calculate the value of x . (Round to the nearest tenth.)



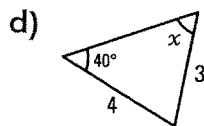
$x = 2.1$



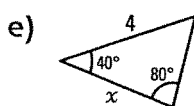
$x = 3.8$



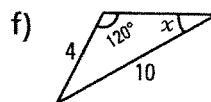
$x = 48.6^\circ$



$x = 59^\circ$



$x = 3.5$



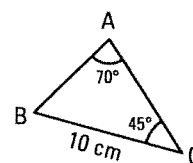
$x = 20.3^\circ$

2. Solve the following triangle. (Round each measure to the nearest tenth.)

$m \angle B = 65^\circ$

$m \overline{AB} = 7.5 \text{ cm}$

$m \overline{AC} = 9.6 \text{ cm}$

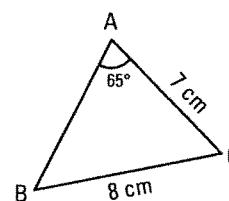


3. Solve the following triangle. (Round each measure to the nearest tenth.)

$m \angle B = 52.5^\circ$

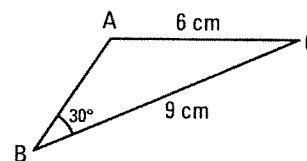
$m \angle C = 62.5^\circ$

$m \overline{AB} = 7.8 \text{ cm}$



4. Angle A in the triangle on the right is obtuse. Use the sine law to determine the measure of angle A.

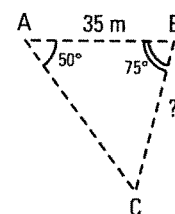
$\sin A = 0.75; m \angle A = 180^\circ - 48.6^\circ = 131.4^\circ$



5. Three boats are near each other. What distance separates boats B and C if the distance, to the nearest metre, separating boats A and B is 35m?

$\frac{\sin 55^\circ}{35} = \frac{\sin 50^\circ}{x} \Rightarrow x = \frac{35 \sin 50^\circ}{\sin 55^\circ} = 32.7 \text{ m}$

The distance separating B and C is 33 m.



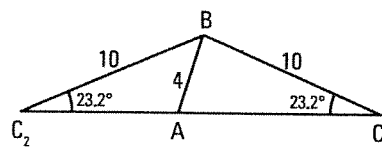
6. There exists two triangles where $m \angle C = 23.2^\circ$, $m \overline{AB} = 4$ and $m \overline{BC} = 10$. These two triangles ABC_1 and ABC_2 are represented on the right.

a) Solve both triangles.

$m \overline{AC}_1 = 9.9; m \overline{AC}_2 = 8.5$

$m \angle A_1 = 80^\circ, m \angle B_1 = 76.8^\circ$

$m \angle A_2 = 100^\circ, m \angle B_2 = 56.8^\circ$



b) Explain why the angles BAC_1 and BAC_2 have the same sine?

They are supplementary angles.

7. There are two possible ways to represent a triangle ABC with $m\angle C = 31.3^\circ$, $m\overline{AB} = 6$ and $m\overline{BC} = 10$, one where angle A is acute and one where angle A is obtuse. Indicate for each case the measure of each angle and each side of the triangle.

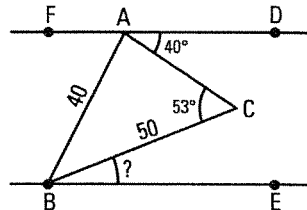
1st case: $m\angle A = 60^\circ$; $m\angle B = 89^\circ$; $m\overline{AC} = 11.5$

2nd case: $m\angle A = 120^\circ$; $m\angle B = 29^\circ$; $m\overline{AD} = 5.6$

8. Lines AD and BE on the right are parallel, $m\angle CAD = 40^\circ$, $m\angle ACB = 53^\circ$, $m\overline{AB} = 40$, $m\overline{BC} = 50$. What is, to the nearest degree, the measure of angle CBE?

$$\frac{\sin A}{50} = \frac{\sin 53^\circ}{40} \Rightarrow m\angle BAC = 86.7^\circ \Rightarrow m\angle FAB = 53.3^\circ$$

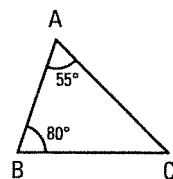
$$m\angle ABE = 53.3^\circ; m\angle ABC = 40.3^\circ; m\angle CBE = 13^\circ$$



9. The perimeter of triangle ABC on the right is equal to 153 cm. Find, to the nearest unit, each side of this triangle.

$$\frac{a}{\sin 55^\circ} = \frac{b}{\sin 80^\circ} = \frac{c}{\sin 45^\circ} = \frac{a+b+c}{\sin 55^\circ + \sin 80^\circ + \sin 45^\circ} = \frac{153}{2.511}$$

$$a = 50 \text{ cm}; b = 60 \text{ cm}; c = 43 \text{ cm}$$

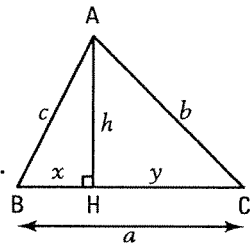


ACTIVITY 2 Cosine law

In the triangle ABC on the right, the altitude AH is drawn.

- a) Explain why

- $h = c \sin B$ and $x = c \cos B$. $\triangle ABH$ is a right triangle $\Rightarrow \sin B = \frac{h}{c}$ and $\cos B = \frac{x}{c}$.
- $h = b \sin C$ and $y = b \cos C$. $\triangle ACH$ is a right triangle $\Rightarrow \sin C = \frac{h}{b}$ and $\cos C = \frac{y}{b}$.



- b) Justify the steps of the following proof.

Steps	Justifications
$c^2 = x^2 + h^2$	Pythagorean Theorem applied to $\triangle ABH$
$c^2 = (a - y)^2 + h^2$	$a = x + y \Rightarrow x = a - y$
$c^2 = a^2 - 2ay + y^2 + h^2$	Expansion of the square
$c^2 = a^2 - 2ab \cos C + b^2 \cos^2 C + b^2 \sin^2 C$	Substitution: $y = b \cos C$ and $h = b \sin C$
$c^2 = a^2 - 2ab \cos C + b^2 (\cos^2 C + \sin^2 C)$	Common factor
$c^2 = a^2 - 2ab \cos C + b^2$	Fundamental identity ($\sin^2 C + \cos^2 C = 1$)
$c^2 = a^2 + b^2 - 2ab \cos C$	Commutative property of addition.

- c) Using a similar proof to b), prove that $b^2 = a^2 + c^2 - 2ac \cos B$.

$$b^2 = y^2 + h^2$$

$$b^2 = (a - x)^2 + h^2$$

$$b^2 = a^2 - 2ax + x^2 + h^2$$

$$b^2 = a^2 - 2ac \cos B + c^2 \cos^2 B + c^2 \sin^2 B$$

$$b^2 = a^2 - 2ac \cos B + c^2 (\cos^2 B + \sin^2 B)$$

$$b^2 = a^2 - 2ac \cos B + c^2$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

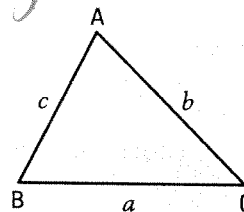
- d) Use the results obtained in b) and c) to complete the following

$$a^2 = b^2 + c^2 - 2bc \cos A$$

COSINE LAW

- The square of any side in a triangle is equal to the sum of the squares of the other two sides minus twice the product of these sides by the cosine of the angle between these two sides.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

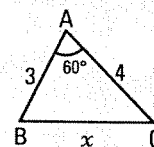


- The cosine law can be used to find the measure of a side or an angle.

1st case: Finding a side when the other two sides, and the angle contained between them, are given.

We calculate the measure x of side BC.

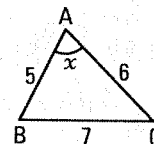
$$\begin{aligned} x^2 &= 3^2 + 4^2 - 2(3)(4) \cos 60^\circ = 9 + 16 - 24 \times \frac{1}{2} = 13 \\ x &= \sqrt{13} \approx 3.6 \end{aligned}$$



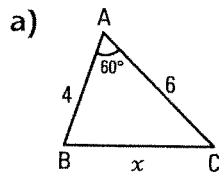
2nd case: Finding an angle when the three sides are given.

We calculate the measure of angle A.

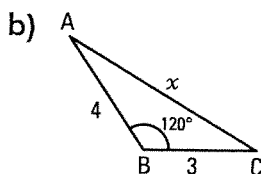
$$\begin{aligned} 7^2 &= 5^2 + 6^2 - 2(5)(6) \cos A \\ 49 &= 61 - 60 \cos A \Rightarrow \cos A = 0.2 \Rightarrow m \angle A = 78.5^\circ \end{aligned}$$



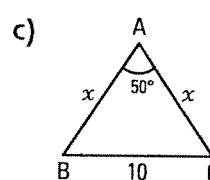
10. Use the cosine law to calculate the value of x . (Round to the nearest tenth.)



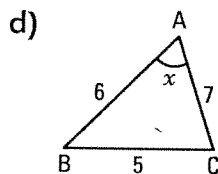
$$x = 5.3$$



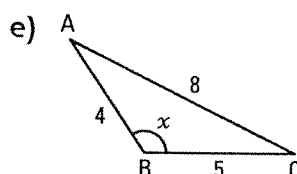
$$x = 6.1$$



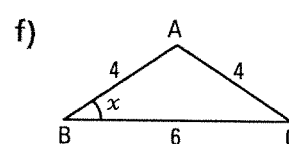
$$x = 11.8$$



$$x = 44.4^\circ$$



$$x = 125.1^\circ$$



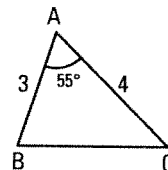
$$x = 41.4^\circ$$

11. Solve the following triangle.

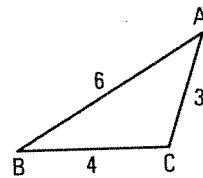
$$m\overline{BC} = 3.35$$

$$m \angle B = 78^\circ$$

$$m \angle C = 47.2^\circ$$



- 12.** Solve the triangle on the right. (Round each measure to the nearest tenth.)
 $m \angle A = 36.3^\circ$; $m \angle B = 26.4^\circ$; $m \angle C = 117.3^\circ$

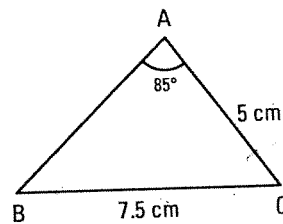


- 13.** In the triangle on the right, we have: $m\overline{AC} = 5$ cm, $m\overline{BC} = 7.5$ cm and $m \angle A = 85^\circ$.

What is, to the nearest tenth, the measure of side AB?

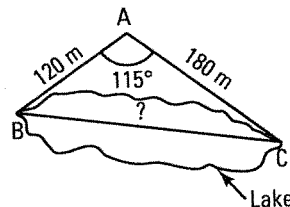
$m \angle B = 41.6^\circ$; $m \angle C = 53.4^\circ$; $m\overline{AB} = 6.04$

The measure of \overline{AB} , to the nearest tenth, is 6 cm.



- 14.** An engineer wants to determine the distance between two cottages on opposite sides of a lake. The engineer is located at point A and the cottages are at B and C. What is, to the nearest metre, the distance separating the two cottages?

255 m

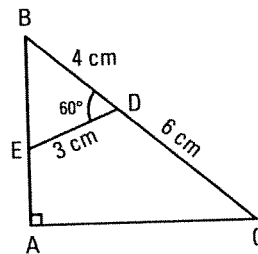


- 15.** In the triangle ABC on the right, segment DE is drawn.
 $m \angle BDE = 60^\circ$, $m\overline{BD} = 4$ cm, $m\overline{DE} = 3$ cm and $m\overline{DC} = 6$ cm.

What is, to the nearest hundredth, the measure of side AC?

$m\overline{BE} = \sqrt{13} \approx 3.61$ cm; $m \angle B \approx 46^\circ$;

$m\overline{AC} = 10 \sin 46^\circ = 7.19$ cm

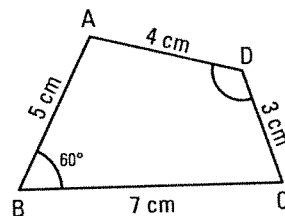


- 16.** In the quadrilateral ABCD on the right, we have: $m\overline{AB} = 5$ cm, $m\overline{BC} = 7$ cm, $m\overline{CD} = 3$ cm, $m\overline{AD} = 4$ cm and $m \angle ABC = 60^\circ$.

What is, to the nearest degree, the measure of angle ADC?

$m\overline{AC} = 6.24$ cm; $m \angle ADC = 125.7^\circ$

Angle D measures 126° to the nearest degree.



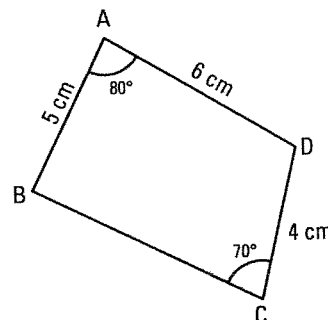
- 17.** In the quadrilateral on the right, we have: $m\overline{AB} = 5$ cm, $m\overline{AD} = 6$ cm, $m\overline{CD} = 4$ cm, $m \angle BAD = 80^\circ$ and $m \angle BCD = 70^\circ$.

What is, to the nearest tenth, the measure of segment BC?

$m\overline{BD} = 7.11$ cm; $m \angle DBC = 31.9^\circ$; $m \angle BDC = 78.1^\circ$

$m\overline{BC} = 7.40$ cm.

The measure of BC, to the nearest tenth, is 7.4 cm.



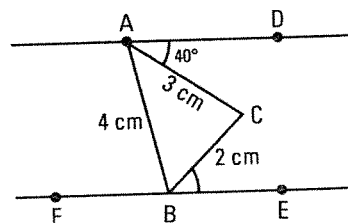
- 18.** In the figure on the right, lines AD and FE are parallel,
 $m\overline{AC} = 3$ cm, $m\overline{BC} = 2$ cm and $m\overline{AB} = 4$ cm.

What is, to the nearest degree, the measure of angle CBE?

$m \angle BAC \approx 29^\circ$, $m \angle ABC = 46.7^\circ$

$m \angle ABF = 69^\circ$, $m \angle CBE = 64.3^\circ$

Angle CBE measures 64° to the nearest degree.



7.3 Arc length

ACTIVITY 1 Units of angle measures

A turn is a natural unit for measuring angles. Certain fractions of turns define units of angle measures.

- a degree ($^\circ$) corresponds to the fraction $\frac{1}{360}$ of a turn. Therefore: 1 turn = 360° .

- a gradian (grad) corresponds to the fraction $\frac{1}{400}$ of a turn. Therefore: 1 turn = 400 grad.

- a radian (rad) corresponds to the fraction $\frac{1}{2\pi}$ of a turn (0.159... turn). Therefore: 1 turn = 2π rad.

a) Convert $\frac{1}{4}$ turn in

1. degrees 90° 2. gradians 100 grad 3. radians $\frac{\pi}{2} \text{ rd}$

b) Convert 180° in $\frac{1}{2}$ turn

1. turns $\frac{1}{2} \text{ tour}$ 2. gradians 200 gr 3. radians $\pi \text{ rd}$

RADIANS

- In a circle, a radian (rad) corresponds to the measure of the central angle that subtends an arc whose length is equal to the circle's radius.

We have: 1 turn = $360^\circ = 2\pi$ rad.

Remember that:

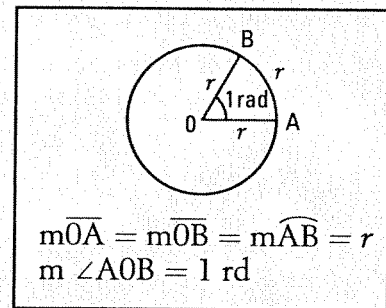
$$\pi \text{ rad} = 180^\circ$$

Therefore: $1 \text{ rad} = \frac{180^\circ}{\pi}$ and $1^\circ = \frac{\pi}{180^\circ} \text{ rad}$.

These relations enable you to convert radians into degrees and vice-versa.

Ex.: $3 \text{ rad} = 3 \left(\frac{180^\circ}{\pi} \right) = \frac{540^\circ}{\pi}$.

$$120^\circ = 120 \left(\frac{\pi}{180^\circ} \text{ rad} \right) = \frac{2\pi}{3} \text{ rad}.$$



- When the unit measure of an angle is not indicated, it is implied that the angle measure is in radians.

1. Express, in radians, the measures of the following angles.

a) 30° $\frac{\pi}{6} \text{ rad}$ b) 45° $\frac{\pi}{4} \text{ rad}$ c) 60° $\frac{\pi}{3} \text{ rad}$ d) 90° $\frac{\pi}{2} \text{ rad}$

e) 120° $\frac{2\pi}{3} \text{ rad}$ f) 135° $\frac{3\pi}{4} \text{ rad}$ g) 300° $\frac{5\pi}{3} \text{ rad}$ h) 390° $\frac{13\pi}{6} \text{ rad}$

2. Express, in degrees, the measures of the following angles.

a) 2 rad $\frac{360^\circ}{\pi}$ b) $\frac{\pi}{3} \text{ rad}$ 60° c) $\frac{7\pi}{6} \text{ rad}$ 210° d) $\frac{4\pi}{3} \text{ rad}$ 240°

e) 5 rad $\frac{900^\circ}{\pi}$ f) $\frac{2\pi}{5} \text{ rad}$ 72° g) $\frac{11\pi}{6} \text{ rad}$ 330° h) $\frac{5\pi}{2} \text{ rad}$ 450°

3. Convert according to the desired unit.

- a) $0.5 \text{ turn} = \frac{\pi}{2} \text{ rad}$ b) $60^\circ = \frac{1}{6} \text{ turn}$ c) $\frac{5\pi}{6} \text{ rad} = 150^\circ$
 d) $\frac{3\pi}{2} \text{ rad} = \frac{3}{4} \text{ turn}$ e) $150^\circ = \frac{5\pi}{6} \text{ rad}$ f) $\frac{2\pi}{3} \text{ rad} = \frac{1}{3} \text{ turn}$
 g) $2 \text{ turns} = 720^\circ$ h) $240^\circ = \frac{4\pi}{3} \text{ rad}$ i) $405^\circ = \frac{9\pi}{4} \text{ rad}$

4. Express, in radians and in terms of π , the measures of the angles that are multiples of 45° .

- a) $45^\circ = \frac{\pi}{4} \text{ rad}$ b) $90^\circ = \frac{\pi}{2} \text{ rad}$ c) $135^\circ = \frac{3\pi}{4} \text{ rad}$ d) $180^\circ = \pi \text{ rad}$
 e) $225^\circ = \frac{5\pi}{4} \text{ rad}$ f) $270^\circ = \frac{3\pi}{2} \text{ rad}$ g) $315^\circ = \frac{7\pi}{4} \text{ rad}$ h) $360^\circ = 2\pi \text{ rad}$

5. The following angles, in radians, are expressed in terms of $\frac{\pi}{6}$. Express them in degrees.

- a) $\frac{\pi}{6} \text{ rad} = 30^\circ$ b) $\frac{5\pi}{6} \text{ rad} = 150^\circ$ c) $\frac{7\pi}{6} \text{ rad} = 210^\circ$ d) $\frac{11\pi}{6} \text{ rad} = 330^\circ$

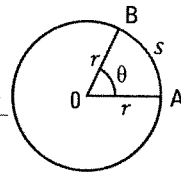
6. The following angles, in radians, are expressed in terms of $\frac{\pi}{3}$. Express them in degrees.

- a) $\frac{\pi}{3} \text{ rad} = 60^\circ$ b) $\frac{2\pi}{3} \text{ rad} = 120^\circ$ c) $\frac{4\pi}{3} \text{ rad} = 240^\circ$ d) $\frac{5\pi}{3} \text{ rad} = 300^\circ$

ACTIVITY 2 Arc length

Consider the circle on the right with radius r and central angle AOB.

- a) What is the measure of the arc AB if the central angle AOB measures
 1. 1 radian? r 2. 2 radians? $2r$ 3. 3 radians? $3r$
- b) If we let θ represent the measure, in radians, of the central angle AOB and s represent the measure of the arc AB, express s as a function of r and θ . $s = r\theta$
- c) If the circle has a radius of $r = 1$, what can be said of the measure s of the arc subtended by the central angle measuring θ radians?
 $s = \theta$, the length of the arc is equal to the measure of the central angle which subtends this arc.



ARC LENGTH

- Consider a circle with radius r and a central angle AOB which subtends the arc AB.

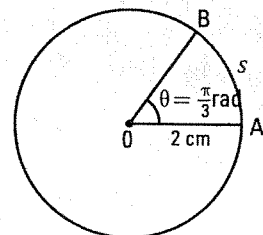
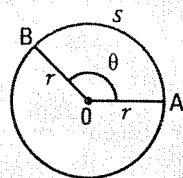
If θ represents the measure, in radians, of the central angle and s represents the length of the subtended arc, then

$$s = r\theta$$

- Note that r and s are expressed in the same units.

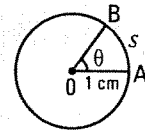
Ex.: On the right, we have: $r = 2 \text{ cm}$ and $\theta = \frac{\pi}{3} \text{ rad}$.

The arc AB measures $s = \frac{2\pi}{3} \text{ cm} \approx 2.09 \text{ cm}$.



- In a circle with a radius of 1 unit, the length of an arc is equal to the central angle which subtends this arc.

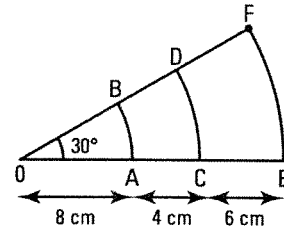
$$r = 1 \Rightarrow s = \theta$$



Ex.: $r = 1 \text{ cm}$ and $\theta = \frac{\pi}{3} \text{ rad} \Rightarrow s = \frac{\pi}{3} \text{ cm} \approx 1.05 \text{ cm}$.

7. By referring to the figure on the right, calculate the length of

a) $\widehat{AB} = \frac{4}{3}\pi \text{ cm}$ b) $\widehat{CD} = 2\pi \text{ cm}$ c) $\widehat{EF} = 3\pi \text{ cm}$



8. In a circle with radius r , a central angle AOB measuring θ subtends the arc AB of length s .

Complete the table of values on the right.

r	θ	s
2 cm	$\frac{2\pi}{3} \text{ rad}$	$\frac{4\pi}{3} \text{ cm}$
6 cm	$\frac{5\pi}{6} \text{ rad}$	$5\pi \text{ cm}$
12 cm	$\frac{5\pi}{6} \text{ rad}$	$10\pi \text{ cm}$

9. The radius of curvature of a railroad track is equal to 600 m. What is the central angle of a train's trajectory if it travels 1.8 km along this track? $\theta = 3 \text{ rad}$

10. The minute hand of a watch measures 0.8 cm. What is the distance traveled by the end of the minute hand in

a) 2 h 15 min? 0.47 cm b) 1 day? 120.6 cm c) 1 year? 440.2 m

11. The wheel of a bicycle turns at a speed of 150 turns per minute.

a) Express the speed in radians per second. $5\pi \text{ rad/s}$

- b) What is the distance traveled in 30 minutes if the radius of the wheel measures 40 cm?

$3600\pi \text{ m} \approx 11\,310 \text{ m}$

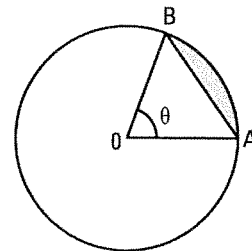
12. A circular segment is the surface on a disk between an arc and the chord which subtends this arc. Let r represent the radius of the circle on the right and θ represent the central angle that defines the circular segment.

- a) Show that the area A of the circular segment is: $A = \frac{r^2}{2}(\theta - \sin \theta)$.

$A = \text{area of the circular sector } \text{AOB} - \text{area of } \triangle \text{AOB}.$

$\text{Area of the circular sector } \text{AOB} = \frac{\theta}{2\pi}(\pi r^2) = \frac{\theta r^2}{2}; \text{ area } \triangle \text{AOB} = \frac{r^2}{2} \sin \theta.$

$A = \frac{r^2}{2}(\theta - \sin \theta).$



- b) Calculate the area A of a circular segment when $r = 10 \text{ cm}$ and $\theta = \frac{\pi}{6}$.

$A = \frac{10^2}{2} \left(\frac{\pi}{6} - \sin \frac{\pi}{6} \right) = 50 \left(\frac{\pi}{6} - \frac{1}{2} \right) = \frac{25\pi}{3} - 25 \approx 1.18 \text{ cm}^2.$

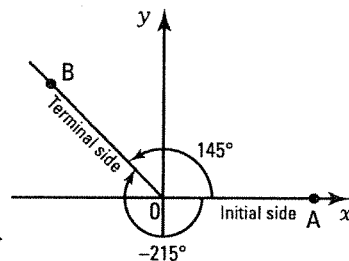
7.4 Trigonometric circle

ACTIVITY 1 Trigonometric angles

In the Cartesian plane on the right, the angle AOB is obtained by having the initial side OA undergo a rotation with centre O, angle of 145° in a positive direction (counter-clockwise). Such an angle is called a trigonometric angle. The side OB is called the terminal side.

Define a clockwise rotation which applies the initial side OA onto the terminal side OB.

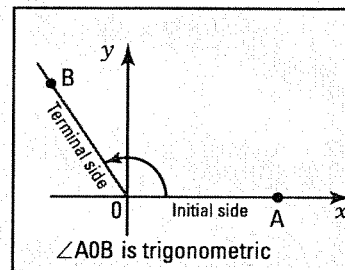
A rotation, centered at O, in a clockwise direction and of an angle of 215° .



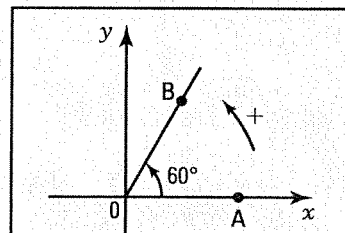
TRIGONOMETRIC ANGLE

In a trigonometric angle, we distinguish:

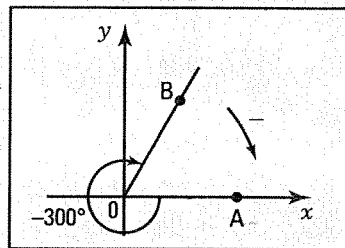
- the vertex, located at the centre O of the Cartesian plane.
- the side of the angle, called **initial side**, that is along the positive x-axis.
- the other side of the angle, called **terminal side**, obtained by a rotation centered at O of the initial side.



- If the rotation is performed in a **positive** direction (counter-clockwise), then the measure of the trigonometric angle is **positive**.



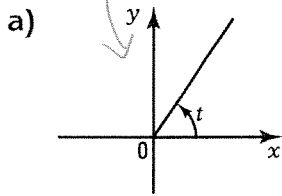
- If the rotation is performed in a **negative** direction (clockwise), then the measure of the trigonometric angle is **negative**.



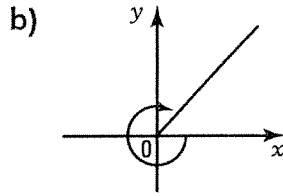
1. Consider the trigonometric angle AOB. What conditions must be met when considering

- a) the vertex O of the angle? The vertex O is the origin of the Cartesian plane.
- b) the point A if \overline{OA} is the initial side of the angle?
The point A must be located on the positive x-axis.
- c) the point B if \overline{OB} is the terminal side of the angle? None

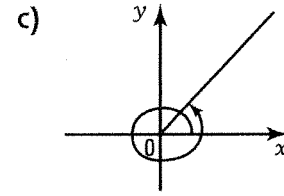
2. Determine the measure t , in degrees and radians, of each of the following trigonometric angles.



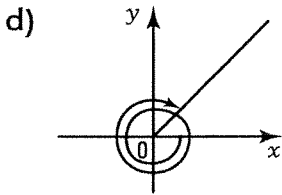
$45^\circ; \frac{\pi}{4} \text{ rad}$



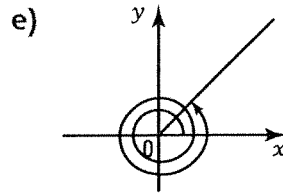
$-315^\circ; -\frac{7\pi}{4} \text{ rad}$



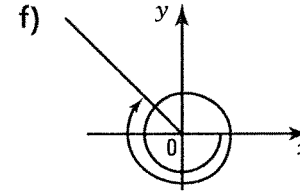
$405^\circ; \frac{9\pi}{4} \text{ rad}$



$-405^\circ; -\frac{9\pi}{4} \text{ rad}$



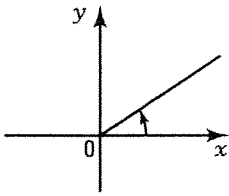
$765^\circ; \frac{17\pi}{4} \text{ rad}$



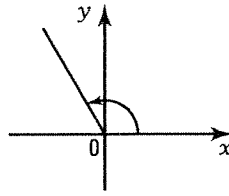
$-565^\circ; -\frac{13\pi}{4} \text{ rad}$

3. Represent each of the following trigonometric angles with the given measure.

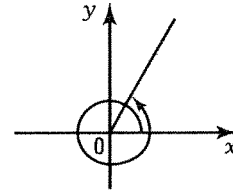
a) $\frac{\pi}{6} \text{ rad}$



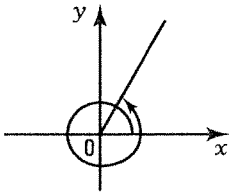
b) 120°



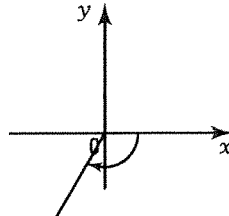
c) 420°



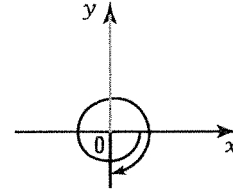
d) $\frac{7\pi}{3} \text{ rad}$



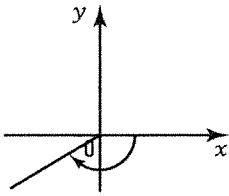
e) -120°



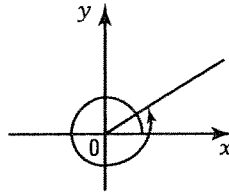
f) -450°



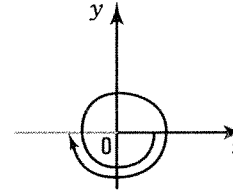
g) $-\frac{5\pi}{6} \text{ rad}$



h) $\frac{13\pi}{6} \text{ rad}$



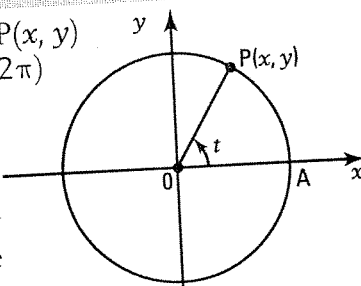
i) $-3\pi \text{ rad}$



ACTIVITY 3 Locating a trigonometric point

Consider the trigonometric circle on the right, a trigonometric point $P(x, y)$ and the measure t , in radians, of the trigonometric angle AOP . ($0 \leq t \leq 2\pi$)

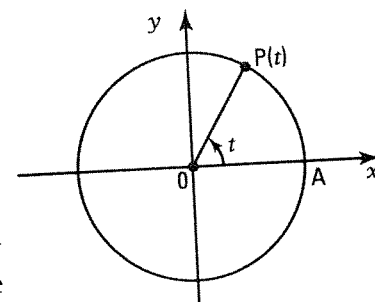
- a) What is the length of arc AP ? Justify your answer.
The unit circle has a radius of $r = 1$ unit. The length s of the arc AP is therefore $s = rt = 1 \times t = t$ units.



- b) Let $P(t)$ represent the trigonometric point associated with the trigonometric angle t . ($t \in \mathbb{R}$)

Determine the Cartesian coordinates (x, y) of the following trigonometric points $P(t)$.

1. $P(0)$ $(1, 0)$ 2. $P(\frac{\pi}{2})$ $(0, 1)$ 3. $P(\pi)$ $(-1, 0)$
 4. $P(\frac{3\pi}{2})$ $(0, -1)$ 5. $P(2\pi)$ $(1, 0)$ 6. $P(-\frac{\pi}{2})$ $(0, -1)$
 7. $P(-\pi)$ $(-1, 0)$ 8. $P(-\frac{3\pi}{2})$ $(0, 1)$ 9. $P(-2\pi)$ $(1, 0)$



- c) Is it true to say that for each real number t , there is a unique corresponding trigonometric point on the trigonometric circle? Yes
- d) Can we say that each trigonometric point $P(x, y)$ on the trigonometric circle corresponds to a unique trigonometric angle t ? No

LOCATING A TRIGONOMETRIC POINT

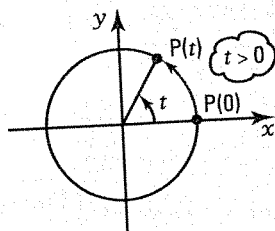
- Each real number t corresponds to a unique point on the unit circle written as $P(t)$.

$P(t)$ is the extremity of the arc whose origin is the point $P(0)$ and whose directed measure is equal to t .

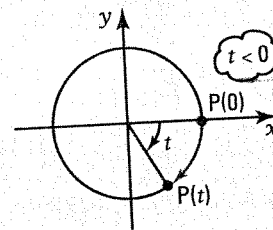
$P(x, y)$ is the Cartesian notation of this point.

Ex.: $P(0^\circ) = P(1, 0)$
 $P(90^\circ) = P(0, 1)$

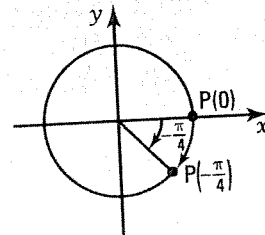
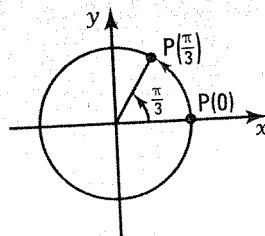
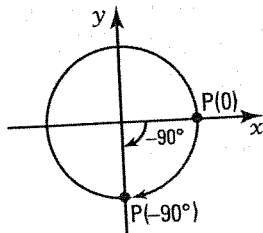
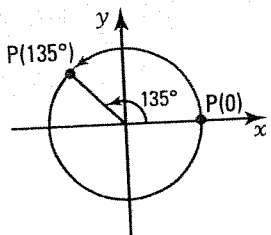
- If t is positive, we locate the point $P(t)$ by moving in a counter-clockwise direction.



- If t is negative, we locate the point $P(t)$ by moving in a clockwise direction.



Ex.:

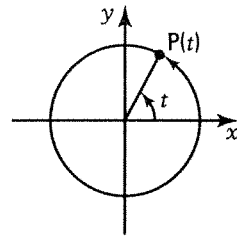


7. Indicate in which quadrant each of the following trigonometric points is located.

- a) $P(120^\circ)$ II b) $P(300^\circ)$ IV c) $P(400^\circ)$ I d) $P(-120^\circ)$ III
 e) $P\left(\frac{5\pi}{6}\right)$ II f) $P\left(\frac{4\pi}{3}\right)$ III g) $P\left(\frac{11\pi}{6}\right)$ IV h) $P\left(-\frac{3\pi}{4}\right)$ III

8. If $P(t)$ is a trigonometric point located in the 1st quadrant, deduce the quadrant that each of the following trigonometric points will be located in.

- a) $P(t + \pi)$ III b) $P\left(t + \frac{\pi}{2}\right)$ II c) $P\left(t + \frac{3\pi}{2}\right)$ IV
 d) $P(-t)$ IV e) $P(\pi - t)$ II f) $P\left(t - \frac{\pi}{2}\right)$ IV
 g) $P(t + 2\pi)$ I h) $P(t - 2\pi)$ I i) $P(t + 6\pi)$ I

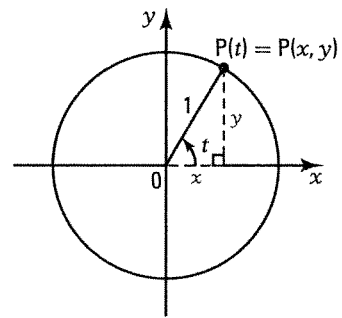


ACTIVITY 4 Cartesian coordinates of a trigonometric point

a) Consider a trigonometric point $P(t)$ located in the 1st quadrant;
 $(0 \leq t \leq \frac{\pi}{2})$.

Which trigonometric ratio enables you to calculate

1. the x -coordinate of the point $P(t)$? $x = \cos t$
 2. the y -coordinate of the point $P(t)$? $y = \sin t$

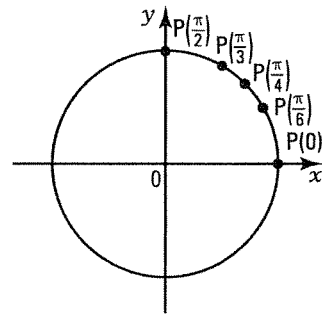


b) Determine, without using a calculator, the coordinates of the trigonometric points

1. $P(0) = (\cos 0, \sin 0) = (1, 0)$
 2. $P\left(\frac{\pi}{2}\right) = (\cos \frac{\pi}{2}, \sin \frac{\pi}{2}) = (0, 1)$

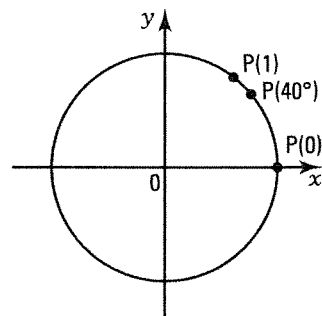
c) The points $P\left(\frac{\pi}{6}\right)$, $P\left(\frac{\pi}{4}\right)$ and $P\left(\frac{\pi}{3}\right)$ are trigonometric points in the 1st quadrant called remarkable trigonometric points. Determine the exact coordinates of these points without using a calculator.

1. $P\left(\frac{\pi}{6}\right) = (\cos \frac{\pi}{6}, \sin \frac{\pi}{6}) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
 2. $P\left(\frac{\pi}{4}\right) = (\cos \frac{\pi}{4}, \sin \frac{\pi}{4}) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
 3. $P\left(\frac{\pi}{3}\right) = (\cos \frac{\pi}{3}, \sin \frac{\pi}{3}) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



d) The points $P(1 \text{ rad})$ and $P(40^\circ)$ are trigonometric points located in the 1st quadrant. Using a calculator, determine the coordinates of these points to the nearest hundredth. Use the appropriate mode, rad or deg, which applies.

1. $P(1) = (\cos 1, \sin 1) = (0.54, 0.84)$
 2. $P(40^\circ) = (\cos 40, \sin 40) = (0.77, 0.64)$



- e) The results observed in a), valid for any trigonometric point in the 1st quadrant, are generalized and bring us to the definition of the Cartesian coordinates of any trigonometric point $P(t)$.

We have: $P(t) = (\cos t, \sin t)$ where $0 \leq t \leq 2\pi$.

Calculate, using a calculator, the Cartesian coordinates of

1. $P(100^\circ)$ $(-0.17, 0.98)$ 2. $P(200^\circ)$ $(-0.94, -0.34)$ 3. $P(300^\circ)$ $(0.5, -0.87)$

CARTESIAN COORDINATES OF A TRIGONOMETRIC POINT

- Given a trigonometric point $P(t)$. ($0 \leq t \leq 2\pi$)

- the x -coordinate of $P(t)$ is equal to $\cos t$.
- the y -coordinate of $P(t)$ is equal to $\sin t$.

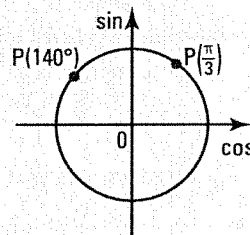
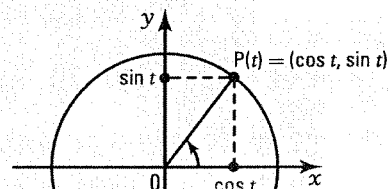
Note that:

$$P(t) = (\cos t, \sin t)$$

By convention, we call the x -axis the cosine axis, and the y -axis the sine axis.

Ex.: $P\left(\frac{\pi}{3}\right) = \left(\cos \frac{\pi}{3}, \sin \frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$P(140^\circ) = (\cos 140^\circ, \sin 140^\circ) = (-0.7660, 0.6428)$



9. For each of the following trigonometric points $P(t)$, indicate

- the quadrant in which the trigonometric point is located.
- the sign of $\cos t$ and the sign of $\sin t$.

a) $P(160^\circ)$

- II
- $(-, +)$

b) $P(350^\circ)$

- IV
- $(+, -)$

c) $P(-150^\circ)$

- III
- $(-, +)$

d) $P(750^\circ)$

- I
- $(+, +)$

e) $P\left(\frac{5\pi}{6}\right)$

- II
- $(-, +)$

f) $P\left(\frac{5\pi}{3}\right)$

- IV
- $(+, -)$

g) $P\left(-\frac{2\pi}{3}\right)$

- III
- $(-, -)$

h) $P\left(\frac{10\pi}{3}\right)$

- III
- $(-, -)$

10. Using a calculator, determine the coordinates of the following trigonometric points to the nearest thousandth.

a) $P(175^\circ)$ $(-0.996, 0.087)$

b) $P(625^\circ)$ $(-0.087, -0.996)$

c) $P\left(\frac{11\pi}{5}\right)$ $(0.809, 0.588)$

d) $P\left(-\frac{29\pi}{6}\right)$ $(-0.866, -0.5)$

11. Knowing that $P(t) = \left(\frac{3}{5}, \frac{4}{5}\right)$ is a trigonometric point, determine

a) $\cos t = \frac{3}{5}$

b) $\sin t = \frac{4}{5}$

c) $\tan t = \frac{4}{3}$

d) $\sec t = \frac{5}{3}$

e) $\csc t = \frac{5}{4}$

f) $\cot t = \frac{3}{4}$

12. Knowing that $P(t) = \left(\cos t, \frac{5}{13}\right)$ is a trigonometric point located in the 2nd quadrant, determine

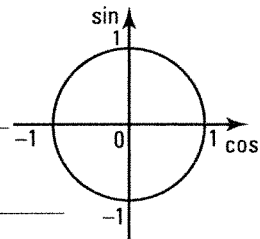
a) $\cos t = \frac{-12}{13}$ b) $\sec t = \frac{-13}{12}$ c) $\csc t = \frac{5}{13}$
 d) $\tan t = \frac{-5}{12}$ e) $\cot t = \frac{-12}{5}$

13. For each of the following trigonometric points, give the two possible values for the missing coordinate.

a) $P\left(\frac{1}{2}, \dots\right) \pm \frac{\sqrt{3}}{2}$ b) $P\left(\dots, \frac{\sqrt{3}}{2}\right) \pm \frac{1}{2}$ c) $P(\dots, 0.6) \pm 0.8$
 d) $P\left(\frac{-5}{13}, \dots\right) \pm \frac{12}{13}$ e) $P\left(\frac{2}{3}, \dots\right) \pm \frac{\sqrt{5}}{3}$ f) $P\left(\dots, \frac{\sqrt{2}}{2}\right) \pm \frac{\sqrt{2}}{2}$

14. A trigonometric point $P(t)$ has an x -coordinate of $\cos t = 0.8$.

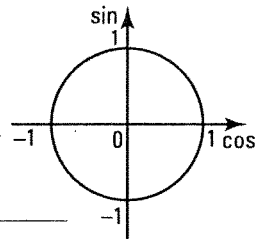
- a) If the point $P(t)$ is located in the 1st quadrant, $x^2 + y^2 = 1$
- determine the y -coordinate $\sin t$. $\sin t = 0.6$
 - deduce, in degrees, the value of t knowing that $0 \leq t \leq 90^\circ$.
 $t = 36.9^\circ$
 - deduce, in degrees, the value of t knowing that $360^\circ \leq t \leq 450^\circ$.
 $t = 396.9^\circ$



- b) If the point $P(t)$ is located in the 4th quadrant,
- determine the y -coordinate $\sin t$. $\sin t = -0.6$
 - deduce, in degrees, the value of t knowing that $270^\circ \leq t \leq 360^\circ$. $t = 323.1^\circ$
 - deduce, in degrees, the value of t knowing that $630^\circ \leq t \leq 720^\circ$. $t = 683.1^\circ$

15. A trigonometric point $P(t)$ has an x -coordinate of $\cos t = -0.6$.

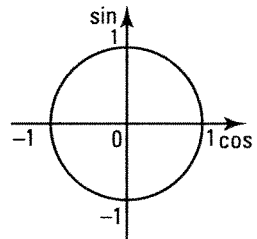
- a) If the point $P(t)$ is located in the 2nd quadrant,
- determine the y -coordinate $\sin t$. $\sin t = 0.8$
 - deduce, in degrees, the value of t knowing that $90^\circ \leq t \leq 180^\circ$.
 ~~$t = 53.1^\circ$~~ 126.87°
 - deduce, in degrees, the value of t knowing that $450^\circ \leq t \leq 540^\circ$.
 ~~$t = 413.1^\circ$~~ 186.87°



- b) If the point $P(t)$ is located in the 3rd quadrant,
- determine the y -coordinate $\sin t$. $\sin t = -0.8$
 - deduce, in degrees, the value of t knowing that $180^\circ \leq t \leq 270^\circ$. $t = 233.1^\circ$
 - deduce, in degrees, the value of t knowing that $540^\circ \leq t \leq 630^\circ$. $t = 593.1^\circ$

16. A trigonometric point $P(t)$ has a y -coordinate of $\sin t = \frac{5}{13}$.

- a) If the point $P(t)$ is located in the 1st quadrant,
- determine the x -coordinate $\cos t$. $\cos t = \frac{12}{13}$
 - deduce, in degrees, the value of t knowing that $0^\circ \leq t \leq 90^\circ$. $t = 22.6^\circ$



- b) If the point $P(t)$ is located in the 2nd quadrant,
- determine the x -coordinate $\cos t$. $\cos t = \frac{-12}{13}$
 - deduce, in degrees, the value of t knowing that $90^\circ \leq t \leq 180^\circ$. $t = 157.4^\circ$

17. A trigonometric point $P(t)$ has a y -coordinate of $\sin t = \frac{-4}{5}$.

a) If the point $P(t)$ is located in the 3rd quadrant,

1. determine the x -coordinate $\cos t$. $\cos t = \frac{-3}{5}$

2. deduce, in degrees, the value of t knowing that $180^\circ \leq t \leq 270^\circ$. $t = 233.1^\circ$

b) If the point $P(t)$ is located in the 4th quadrant,

1. determine the x -coordinate $\cos t$. $\cos t = \frac{3}{5}$

2. deduce, in degrees, the value of t knowing that $270^\circ \leq t \leq 360^\circ$. $t = 306.9^\circ$

ACTIVITY 5 Properties of trigonometric points

a) Consider the trigonometric points represented on the unit circle on the right.

1. Name the geometric transformation that applies the point $P(t)$ onto the trigonometric point

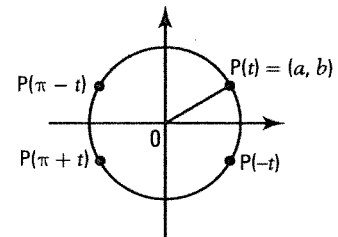
1) $P(\pi - t)$ Reflection about the y -axis.

2) $P(-t)$ Reflection about the x -axis.

3) $P(\pi + t)$ Reflection about the y -axis, followed by a reflection about the x -axis.

2. Deduce the Cartesian coordinates of the points

1) $P(\pi - t)$ $(-a, b)$ 2) $P(-t)$ $(a, -b)$ 3) $P(\pi + t)$ $(-a, -b)$



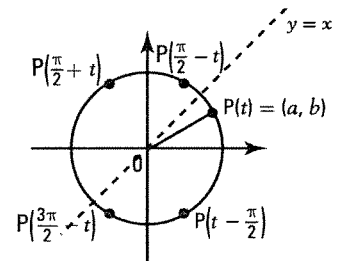
b) Consider the trigonometric points on the right.

1. Find the image of the point $P(t)$ by the reflection about the line $y = x$, and deduce the coordinates of this image.

$P\left(\frac{\pi}{2} - t\right) = (b, a)$

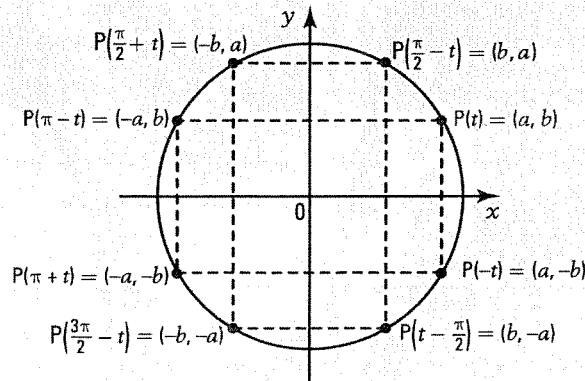
2. Deduce the coordinates of the trigonometric points

1) $P\left(\frac{\pi}{2} + t\right)$ $(-b, a)$ 2) $P\left(\frac{3\pi}{2} - t\right)$ $(-b, -a)$ 3) $P\left(t - \frac{\pi}{2}\right)$ $(b, -a)$



PROPERTIES OF TRIGONOMETRIC POINTS

Given a trigonometric point $P(t) = (a, b)$, geometric transformations enable you to deduce the Cartesian coordinates of the following trigonometric points.



18. Consider the trigonometric point $P(t) = \left(\frac{5}{13}, \frac{12}{13}\right)$.

a) Deduce the coordinates of the following trigonometric points.

1. $P(-t)$ $\underline{\left(\frac{5}{13}, -\frac{12}{13}\right)}$ 2. $P(\pi - t)$ $\underline{\left(-\frac{5}{13}, \frac{12}{13}\right)}$ 3. $P(\pi + t)$ $\underline{\left(-\frac{5}{13}, -\frac{12}{13}\right)}$

b) Deduce

1. $\cos t$ $\underline{\frac{5}{13}}$ 2. $\sin t$ $\underline{\frac{12}{13}}$ 3. $\cos(-t)$ $\underline{\frac{5}{13}}$
 4. $\sin(-t)$ $\underline{-\frac{12}{13}}$ 5. $\cos(\pi - t)$ $\underline{-\frac{5}{13}}$ 6. $\sin(\pi + t)$ $\underline{-\frac{12}{13}}$

19. In each of the following cases, choose the correct answer from the following 4 possible answers: $\cos t$, $\sin t$, $-\cos t$ and $-\sin t$.

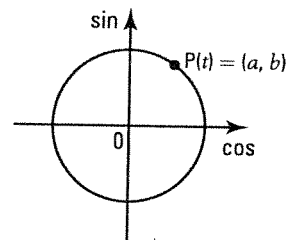
a) $\cos(-t)$ $\underline{\cos t}$ b) $\sin(-t)$ $\underline{-\sin t}$ c) $\cos(\pi + t)$ $\underline{-\cos t}$ d) $\sin(\pi + t)$ $\underline{-\sin t}$
 e) $\cos(\pi + t)$ $\underline{-\cos t}$ f) $\sin(\pi - t)$ $\underline{\sin t}$ g) $\cos\left(\frac{\pi}{2} - t\right)$ $\underline{\sin t}$ h) $\sin\left(\frac{\pi}{2} - t\right)$ $\underline{\cos t}$
 i) $\cos\left(\frac{\pi}{2} + t\right)$ $\underline{-\sin t}$ j) $\sin\left(\frac{\pi}{2} + t\right)$ $\underline{\cos t}$ k) $\cos\left(t - \frac{\pi}{2}\right)$ $\underline{\sin t}$ l) $\sin\left(t - \frac{\pi}{2}\right)$ $\underline{-\cos t}$

ACTIVITY 6 Periodicity of trigonometric points

Given the trigonometric point $P(t) = (a, b)$.

a) Determine the coordinates of the following trigonometric points.

1. $P(t + 2\pi)$ $\underline{(a, b)}$ 2. $P(t + 4\pi)$ $\underline{(a, b)}$ 3. $P(t + 6\pi)$ $\underline{(a, b)}$
 4. $P(t - 2\pi)$ $\underline{(a, b)}$ 5. $P(t - 4\pi)$ $\underline{(a, b)}$ 6. $P(t - 6\pi)$ $\underline{(a, b)}$



b) 1. What can we say about the 6 trigonometric points defined in a)?

They are coincident.

2. What is the smallest positive real number a such that $P(t) = P(t + a)$? $\underline{2\pi}$

3. Is it true to say that $P(t) = P(t + 2\pi n)$, $n \in \mathbb{Z}$? $\underline{\text{True}}$

c) If $n \in \mathbb{Z}$, is it true to say that

1. $\cos t = \cos(t + 2\pi n)$? $\underline{\text{True}}$ 2. $\sin t = \sin(t + 2\pi n)$? $\underline{\text{True}}$

PERIODICITY OF TRIGONOMETRIC POINTS

- Given a trigonometric point $P(t)$. The smallest positive real number p such that $P(t + p) = P(t)$ is called the period.

This period is equal to 2π , that is

$$P(t + 2\pi) = P(t), \forall t \in \mathbb{R}$$

- Note that $P(t) = P(t + 2\pi n)$, $n \in \mathbb{Z}$.

Thus, ... $P(t - 4\pi) = P(t - 2\pi) = P(t) = P(t + 2\pi) = P(t + 4\pi)$...

- Consequently, $\forall t \in \mathbb{R}, \forall n \in \mathbb{Z}$, we have:

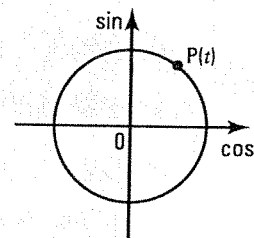
$$\cos(t + 2\pi n) = \cos t$$

and

$$\sin(t + 2\pi n) = \sin t$$

Ex.: If $P(t) = (0.6; 0.8)$ then $P(t + 10\pi) = (0.6; 0.8)$,

$\cos(t + 10\pi) = 0.6$ and $\sin(t + 10\pi) = 0.8$.



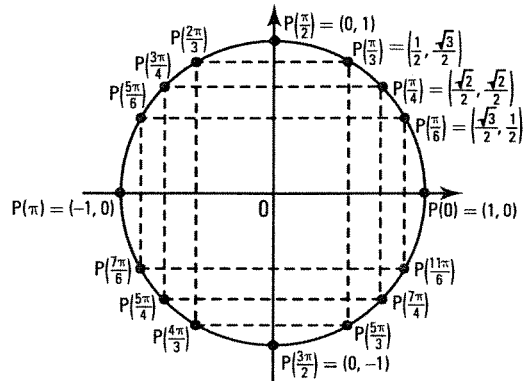
Remon 2π - 2π for each

20. Write each of the following trigonometric points in the form $P(t)$ where $0 \leq t < 2\pi$.

- a) $P\left(\frac{13\pi}{6}\right)$ $P\left(\frac{\pi}{6}\right)$ b) $P(2\pi)$ $P(0)$ c) $P\left(\frac{17\pi}{4}\right)$ $P\left(\frac{\pi}{4}\right)$
- d) $P\left(\frac{-5\pi}{3}\right)$ $P\left(\frac{\pi}{3}\right)$ e) $P\left(\frac{-10\pi}{3}\right)$ $P\left(\frac{2\pi}{3}\right)$ f) $P\left(\frac{121\pi}{6}\right)$ $P\left(\frac{\pi}{6}\right)$

ACTIVITY 7 Cartesian coordinates of the remarkable trigonometric points

The three remarkable trigonometric points of the 1st quadrant $P\left(\frac{\pi}{6}\right)$, $P\left(\frac{\pi}{4}\right)$ and $P\left(\frac{\pi}{3}\right)$ are represented on the right.



a) Locate on the unit circle the remarkable points of the 2nd quadrant and give the Cartesian coordinates of these points.

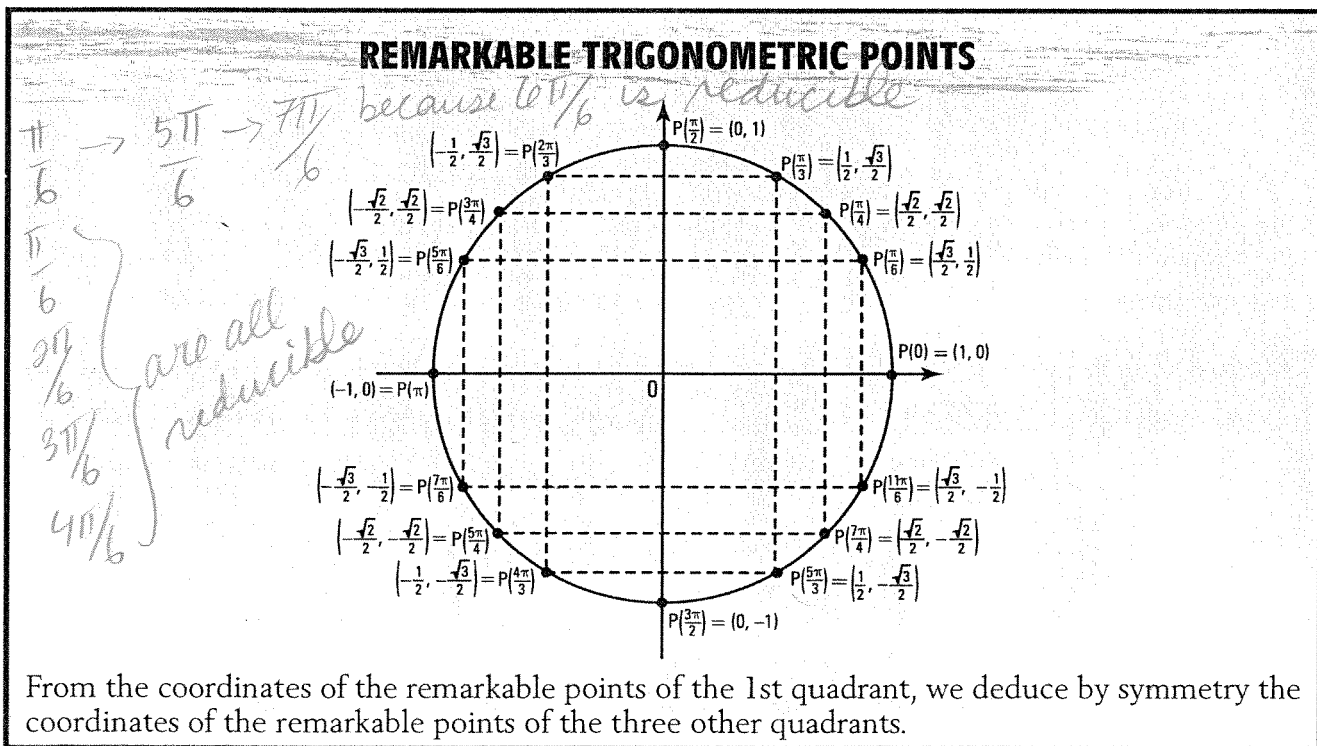
- $P\left(\frac{2\pi}{3}\right)$ $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
- $P\left(\frac{3\pi}{4}\right)$ $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
- $P\left(\frac{5\pi}{6}\right)$ $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

b) Locate on the trigonometric circle the remarkable points of the 3rd quadrant and give the Cartesian coordinates of these points.

- $P\left(\frac{7\pi}{6}\right)$ $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
- $P\left(\frac{5\pi}{4}\right)$ $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
- $P\left(\frac{4\pi}{3}\right)$ $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

c) Locate on the trigonometric circle the remarkable points of the 4th quadrant and give the Cartesian coordinates of these points.

- $P\left(\frac{5\pi}{3}\right)$ $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
- $P\left(\frac{7\pi}{4}\right)$ $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
- $P\left(\frac{11\pi}{6}\right)$ $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$



21. Determine the Cartesian coordinates of the following trigonometric points.

a) $P\left(-\frac{\pi}{6}\right) \underline{\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)}$ b) $P\left(-\frac{3\pi}{4}\right) \underline{\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)}$ c) $P\left(-\frac{4\pi}{3}\right) \underline{\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$
 d) $P\left(\frac{7\pi}{3}\right) \underline{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$ e) $P\left(\frac{31\pi}{6}\right) \underline{\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)}$ f) $P\left(-\frac{13\pi}{4}\right) \underline{\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)}$

22. From the Cartesian coordinates of the remarkable trigonometric points, determine the exact value of

a) $\sin\left(\frac{2\pi}{3}\right) \underline{\frac{\sqrt{3}}{2}}$ b) $\cos\left(\frac{7\pi}{6}\right) \underline{-\frac{\sqrt{3}}{2}}$ c) $\sin\left(\frac{3\pi}{4}\right) \underline{\frac{\sqrt{2}}{2}}$
 d) $\sin\left(\frac{3\pi}{2}\right) \underline{-1}$ e) $\cos\left(\frac{11\pi}{6}\right) \underline{\frac{\sqrt{3}}{2}}$ f) $\cos\left(\frac{4\pi}{3}\right) \underline{-\frac{1}{2}}$

23. Determine the exact value of

a) $\sin\left(-\frac{\pi}{6}\right) \underline{-\frac{1}{2}}$ b) $\cos\left(-\frac{\pi}{3}\right) \underline{\frac{1}{2}}$ c) $\sin\left(\frac{13\pi}{6}\right) \underline{\frac{1}{2}}$
 d) $\cos\left(\frac{19\pi}{4}\right) \underline{-\frac{\sqrt{2}}{2}}$ e) $\sin\left(-\frac{7\pi}{3}\right) \underline{-\frac{\sqrt{3}}{2}}$ f) $\cos\left(-\frac{17\pi}{4}\right) \underline{\frac{\sqrt{2}}{2}}$

24. Determine the exact value of

a) $\tan\left(\frac{2\pi}{3}\right) \underline{-\sqrt{3}}$ b) $\cot\left(\frac{5\pi}{6}\right) \underline{-\sqrt{3}}$
 c) $\sec\left(\frac{7\pi}{6}\right) \underline{-\frac{2\sqrt{3}}{3}}$ d) $\csc\left(\frac{11\pi}{6}\right) \underline{-2}$

25. Knowing that $0 \leq t \leq 2\pi$, determine the two values of t such that

a) $\cos t = \frac{1}{2} \underline{\frac{\pi}{3} \text{ or } \frac{5\pi}{3}}$ b) $\sin t = \frac{\sqrt{3}}{2} \underline{\frac{\pi}{3} \text{ or } \frac{2\pi}{3}}$
 c) $\cos t = \frac{-\sqrt{3}}{2} \underline{\frac{5\pi}{6} \text{ or } \frac{7\pi}{6}}$ d) $\sin t = -\frac{1}{2} \underline{\frac{7\pi}{6} \text{ or } \frac{11\pi}{6}}$

26. Find t if

a) $\sin t = \frac{1}{2}$ and $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2} \underline{t = \frac{5\pi}{6}}$ b) $\cos t = \frac{-1}{2}$ and $\pi \leq t \leq \frac{3\pi}{2} \underline{t = \frac{4\pi}{3}}$
 c) $\sin t = \frac{-\sqrt{3}}{2}$ and $\frac{3\pi}{2} \leq t \leq 2\pi \underline{t = \frac{5\pi}{3}}$ d) $\cos t = \frac{1}{2}$ and $0 \leq t \leq \frac{\pi}{2} \underline{t = \frac{\pi}{3}}$

27. Knowing that $0 \leq t \leq 360^\circ$, find the two values of t (to the nearest tenth) such that

a) $\cos t = 0.8 \underline{36.9^\circ \text{ or } 323.1^\circ}$ b) $\cos t = -0.6 \underline{126.9^\circ \text{ or } 233.1^\circ}$
 c) $\sin t = 0.2 \underline{11.5^\circ \text{ or } 168.5^\circ}$ d) $\sin t = -0.4 \underline{203.6^\circ \text{ or } 336.4^\circ}$

28. Knowing that $0 \leq t \leq 2\pi$, find the two values of t (to the nearest hundredth) such that

a) $\sin t = 0.7 \underline{0.78 \text{ rad or } 2.37 \text{ rad}}$ b) $\sin t = -0.6 \underline{5.64 \text{ rad or } 3.79 \text{ rad}}$
 c) $\cos t = 0.2 \underline{1.37 \text{ rad or } 4.91 \text{ rad}}$ d) $\cos t = -0.8 \underline{2.50 \text{ rad or } 3.79 \text{ rad}}$

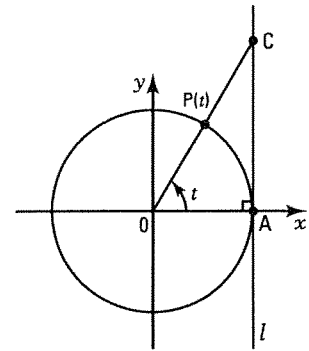
ACTIVITY 8 Trigonometric circle and the tangent axis

- a) On the trigonometric circle on the right, line l is drawn perpendicular to the x -axis at the point $A(1, 0)$.

Given a trigonometric point $P(t)$ of the first quadrant, we designate C as the intersection of the extension of the radius OP and the line l .

Explain why $\tan t = \overline{mAC}$.

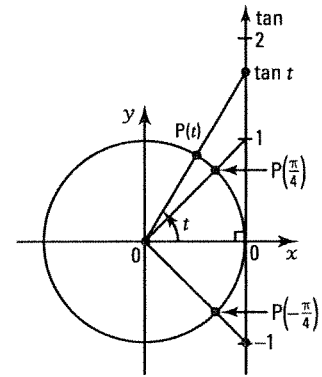
$$\text{In the right triangle } OAC, \text{ we have: } \tan t = \frac{\overline{mAC}}{\overline{mOA}} = \frac{\overline{mAC}}{1}.$$



- b) On the trigonometric circle on the right, we have directed and scaled the line l . The chosen scale corresponds to the radius of the unit circle. The resulting line l , scaled and directed, is called the **tangent axis**.

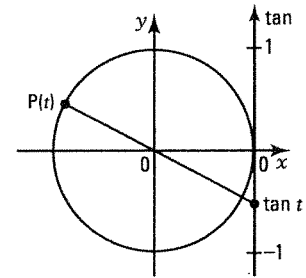
Given a trigonometric point $P(t)$, we locate $\tan t$ on the tangent axis by extending the radius OP (see a)).

1. Locate the point $P\left(\frac{\pi}{4}\right)$ and verify by construction that $\tan \frac{\pi}{4} = 1$.
2. Locate the point $P\left(-\frac{\pi}{4}\right)$ and verify by construction that $\tan\left(-\frac{\pi}{4}\right) = -1$.



- c) Given a trigonometric point $P(t)$ in the 2nd quadrant, explain how to locate $\tan t$ on the tangent axis and deduce the sign of $\tan t$.

We extend OP . The intersection of the extension of OP and the tangent axis corresponds to $\tan t$. We have: $\tan t < 0$.



TRIGONOMETRIC CIRCLE AND THE TANGENT AXIS

- The trigonometric circle and tangent axis are represented on the right. Given a trigonometric point $P(t)$. To locate $\tan t$ on the tangent axis, we extend the radius OP . The intersection of the extension of the radius OP and the tangent axis enables you to locate $\tan t$.

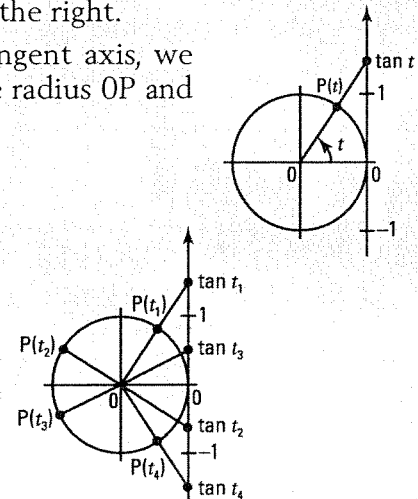
- Note that

$$0 \leq t < \frac{\pi}{2} \Rightarrow \tan t \geq 0.$$

$$\frac{\pi}{2} < t \leq \pi \Rightarrow \tan t \leq 0.$$

$$\pi \leq t < \frac{3\pi}{2} \Rightarrow \tan t \geq 0.$$

$$\frac{3\pi}{2} < t \leq 2\pi \Rightarrow \tan t \leq 0.$$



21. Determine the Cartesian coordinates of the point P(t) on the unit circle.

a) $P\left(-\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

d) $P\left(\frac{7\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

22. From the Cartesian coordinates of the point P(t) on the unit circle, determine the value of

a) $\sin\left(\frac{2\pi}{3}\right)$

d) $\sin\left(\frac{3\pi}{2}\right)$

23. Determine the value of

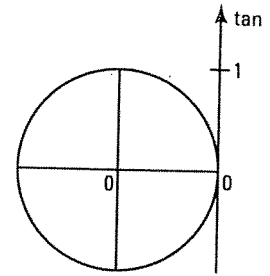
a)



ed if

ring cases.

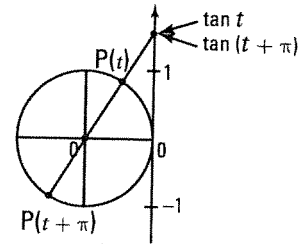
$t = \frac{3\pi}{4}$ or $t = \frac{7\pi}{4}$



$t + \pi$.

compare $\tan t$

$\tan(t + \pi) = \tan t$ Yes



Find all solutions for t in each of the following.

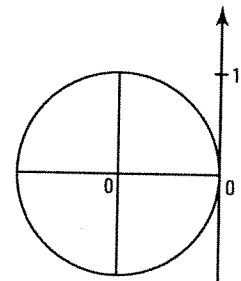
c) $\tan t = \frac{\sqrt{3}}{3}$ $\frac{\pi}{6}$ or $\frac{7\pi}{6}$

e) $\tan t = 0$ 0 or π

b) $\tan t = -\sqrt{3}$ $\frac{2\pi}{3}$ or $\frac{5\pi}{3}$

d) $\tan t = \frac{-\sqrt{3}}{3}$ $\frac{5\pi}{6}$ or $\frac{11\pi}{6}$

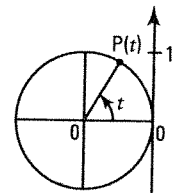
f) $\tan t = 0,7$ 0.61 or 3.75



32. a) The trigonometric circle and a trigonometric point P(t) in the 1st quadrant are represented on the right.

As t approaches $\frac{\pi}{2}$, what can we say about tan t?

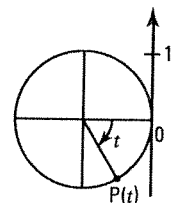
tan t increases indefinitely while remaining positive.



b) The trigonometric circle and a trigonometric point P(t) in the 4th quadrant are represented on the right.

As t approaches $\frac{-\pi}{2}$, what can we say about tan t?

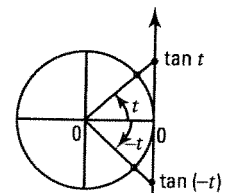
tan t decreases indefinitely while remaining negative.



33. a) Place the trigonometric points P(t) and P(-t) on the trigonometric circle.

b) Compare $\tan(-t)$ and $\tan t$. $\tan(-t) = -\tan t$.

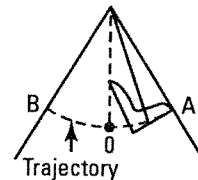
c) Is it true to say that for any real t, we have: $\tan(-t) = -\tan t$. Yes



7.5 Periodic functions

ACTIVITY 1 Movement of a swing

Initially ($t = 0$), the seat of a swing is in position A. We notice that it takes 8 seconds for the swing to go back and forth 5 times.



- a) Does the seat periodically follow the same trajectory? Yes
If yes, each return is called a cycle.
- b) The required time p to return to point A for the first time, ie the duration of one cycle, is called the period of the movement. Determine the period p of the movement.
 $p = 1.6$ s

- c) Complete the table below which gives the position $P(t)$ (A, 0 or B) on the movement as a function of elapsed time t since the start.

t	0	0.4	0.8	1.2	1.6	2	2.4	2.8	3.2	4.8	6.4	8
$P(t)$	A	0	B	0	A	0	B	0	A	A	A	A

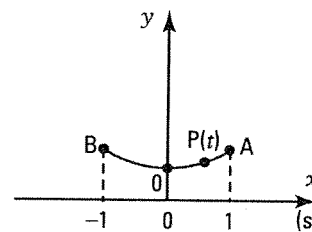
- d) Verify that for any value of t ($0 \leq t \leq 8$), we have: $P(t) = P(t + p)$ where p is the period of the movement.

Various answers

- e) The frequency of the movement is the number of cycles per unit of time. Given that we observe here 5 cycles in 8 seconds, calculate the frequency F , ie the number of cycles per second.

$$F = \frac{5}{8} = 0.625 \text{ cycle/sec}$$

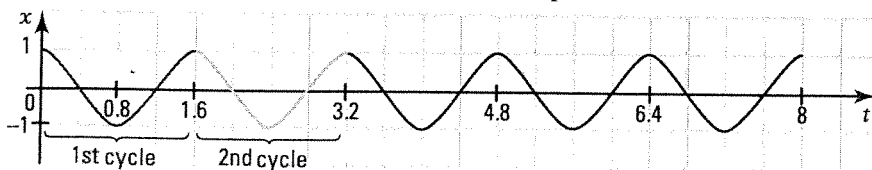
- f) The trajectory of the swing's seat is represented by the arc AB in the Cartesian plane to the right. The points A, 0 and B have x -coordinates 1, 0 and -1 respectively.



- In what interval is the x -coordinate of a point $P(t)$ on the trajectory? $-1 \leq x \leq 1$
- Complete the table of values below which gives the x -coordinate of a point $P(t)$ on the trajectory as a function of elapsed time t since the start.

t	0	0.4	0.8	1.2	1.6	2	2.4	2.8	3.2	4.8	6.4	8
x	1	0	-1	0	1	0	-1	0	1	1	1	1

3. We have represented two cycles of the function f which gives the x -coordinate of a point $P(t)$ on the trajectory as a function of elapsed time t since the start. Draw 3 other cycles.



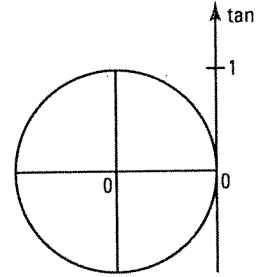
29. Consider the trigonometric point $P(t)$.

Determine in which quadrant(s) $P(t)$ is located if

1. $\tan t > 0$. In the 1st or 3rd quadrant.
2. $\tan t < 0$. In the 2nd or 4th quadrant.

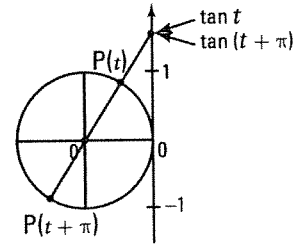
b) If $0 \leq t \leq 2\pi$, determine t in each of the following cases.

1. $\tan t = 1$ $t = \frac{\pi}{4}$ or $t = \frac{5\pi}{4}$
2. $\tan t = -1$ $t = \frac{3\pi}{4}$ or $t = \frac{7\pi}{4}$



30. Consider the trigonometric circle on the right.

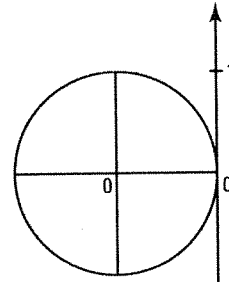
- a) Randomly place a trigonometric point $P(t)$.
- b) Deduce the location of the trigonometric point $P(t + \pi)$.
- c) Locate on the tangent axis $\tan t$ and $\tan(t + \pi)$ and compare $\tan t$ and $\tan(t + \pi)$. $\tan t = \tan(t + \pi)$
- d) Is it true to say that for any real t , we have: $\tan(t + \pi) = \tan t$? Yes



31. Consider the trigonometric point $P(t)$.

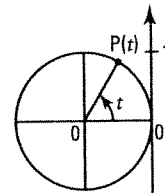
If $0 \leq t \leq 2\pi$, determine the two possible solutions for t in each of the following cases.

- a) $\tan t = \sqrt{3}$ $\frac{\pi}{3}$ or $\frac{4\pi}{3}$
- b) $\tan t = -\sqrt{3}$ $\frac{2\pi}{3}$ or $\frac{5\pi}{3}$
- c) $\tan t = \frac{\sqrt{3}}{3}$ $\frac{\pi}{6}$ or $\frac{7\pi}{6}$
- d) $\tan t = \frac{-\sqrt{3}}{3}$ $\frac{5\pi}{6}$ or $\frac{11\pi}{6}$
- e) $\tan t = 0$ 0 or π
- f) $\tan t = 0,7$ $0,61$ or $3,75$



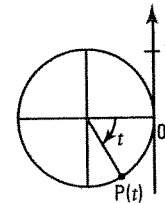
32. a) The trigonometric circle and a trigonometric point $P(t)$ in the 1st quadrant are represented on the right.

As t approaches $\frac{\pi}{2}$, what can we say about $\tan t$?
 $\tan t$ increases indefinitely while remaining positive.



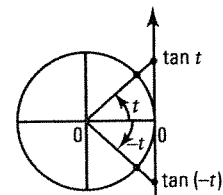
b) The trigonometric circle and a trigonometric point $P(t)$ in the 4th quadrant are represented on the right.

As t approaches $\frac{-\pi}{2}$, what can we say about $\tan t$?
 $\tan t$ decreases indefinitely while remaining negative.



33. a) Place the trigonometric points $P(t)$ and $P(-t)$ on the trigonometric circle.

- b) Compare $\tan(-t)$ and $\tan t$. $\tan(-t) = -\tan t$.
- c) Is it true to say that for any real t , we have: $\tan(-t) = -\tan t$. Yes



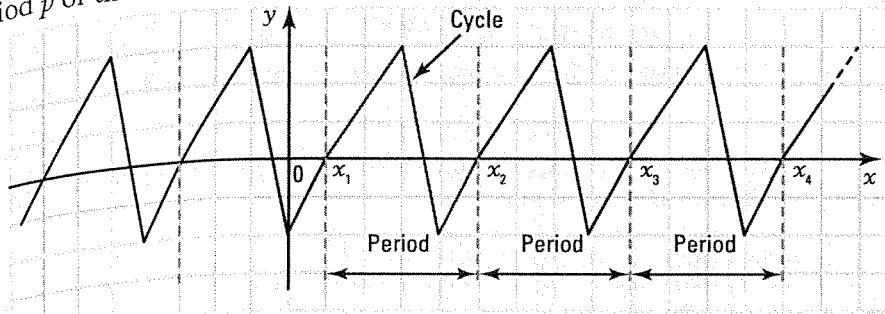
PERIODIC FUNCTIONS

- A function f is called periodic if there exists a real number a such that for any value x of the domain of f , we have:

$$f(x + a) = f(x)$$

The smallest positive value of a for which $f(x + a) = f(x)$ is called the period (denoted p) of the function.

- One cycle of a periodic function is the smallest portion of the graph that, by repetition, forms the curve representing the function. The distance between the extremities of the cycle is equal to the period p of the function.



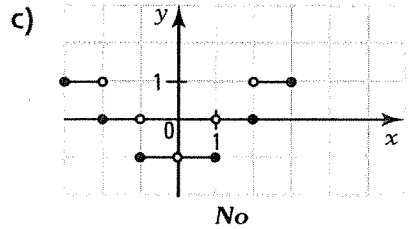
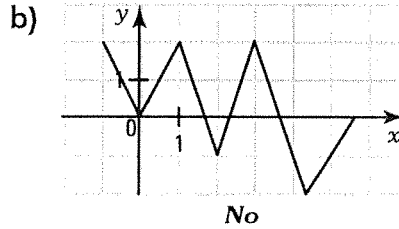
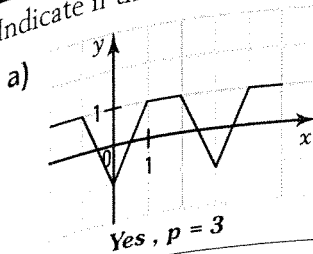
$$p = x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = \dots$$

- The frequency, denoted F , is the reciprocal of the period.

$$F = \frac{1}{p}$$

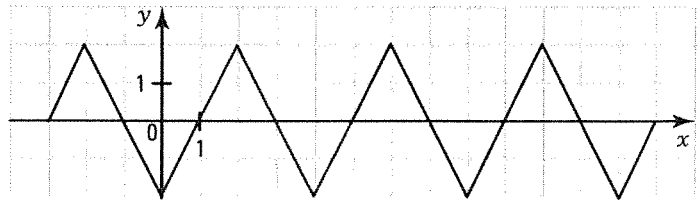
In a situation where the variable x represents time, the period represents the duration of one cycle and the frequency represents the number of cycles per unit of time.

1. Indicate if the following functions are periodic. If yes, indicate the period p of the function.



2. One cycle of a periodic function f is represented on the right.

- a) Determine the period p and the frequency F of this function.
- $p = 4, F = \frac{1}{4}$



- b) Complete the graph of this function for $x \in [-3, 13]$.

- c) Determine
1. $\text{ran } f$. $[-2, 2]$ 2. $\min f$. -2 3. $\max f$. 2

- d) Determine
1. $f(15)$. 0 2. $f(25)$. 0 3. $f(42)$. 2 4. $f(32)$. -2

- e) What are the zeros of f over the interval $[0, 10]$? $1, 3, 5, 7 \text{ and } 9$

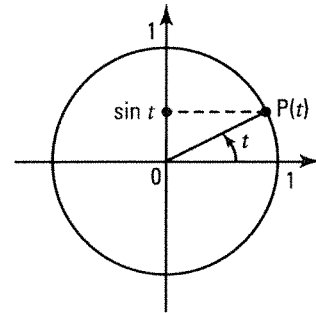
7.6 Sine function

ACTIVITY 1 Basic sine function

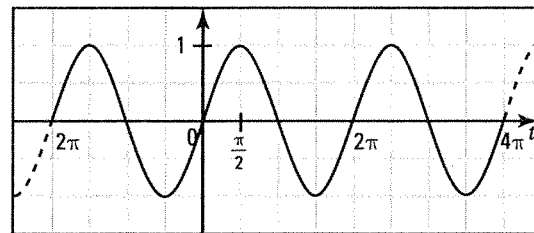
The basic sine function, denoted \sin , has the rule $y = \sin t$.

a) Complete the table of values below when t varies from -2π to 4π .

t	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$\sin t$	0	1	0	-1	0	1	0	-1	0	1	0	-1	0



b) We have represented the function $y = \sin t$ on the right. Is this function periodic? If yes, what is the period p of the function? Yes, $p = 2\pi$



c) For this function, determine

- the domain. \mathbb{R}
- the range. $[-1, 1]$
- the maximum. 1
- the minimum. -1

d) The amplitude A of a function f is equal to half the difference between the maximum and the minimum of the function, i.e. $A = \frac{\max f - \min f}{2}$. What is the amplitude A of the sine function? 1

e) When $t \in [0, 2\pi]$, determine, for the sine function,

- the zeros. $0, \pi$ and 2π
- the sign. $\sin t \geq 0$ si $0 \leq t \leq \pi$ and $\sin t \leq 0$ if $\pi \leq t \leq 2\pi$
- the variation. $\sin \nearrow$ if $t \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$; $\sin \searrow$ if $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

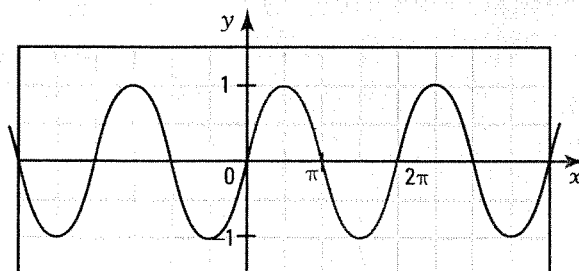
f) Verify the property $\forall t \in \mathbb{R}: \sin(-t) = -\sin t$, using the graph or trigonometric circle.

BASIC SINE FUNCTION

- The sine function, denoted \sin , is defined by

$$\begin{aligned} \sin: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = \sin x \end{aligned}$$

- The sine function is a periodic function with period 2π . $\sin(x + 2\pi) = \sin x$



One cycle of the basic sine function is represented in green over $[0, 2\pi]$.

• The amplitude of a sine function f is given by $A = \frac{\max f - \min f}{2}$.

The amplitude of the basic sine function is: $A = 1$.

• We have:

– domain = \mathbb{R} , range = $[-1, 1]$.

– zeros over $[0, 2\pi]$: $0, \pi$ and 2π .

– sign over $[0, 2\pi]$: $\sin x \geq 0$ if $x \in [0, \pi]$ and $\sin x \leq 0$ if $x \in [\pi, 2\pi]$.

– variation over $[0, 2\pi]$: $\sin \nearrow$ if $x \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$; $\sin \searrow$ if $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

– extrema: $\max = 1$; $\min = -1$.

• For any real x , we have: $\sin(-x) = -\sin x$. The basic sine function is therefore considered to be an odd function.

1. Consider the function $f(x) = \sin x$.

a) Find the zeros of f when

1. $x \in [-2\pi, 4\pi]$. $-2\pi, -\pi, 0, \pi, 2\pi, 3\pi$ et 4π
 $\{ \pi n \}, n \in \mathbb{Z}$

2. $x \in \mathbb{R}$.

b) Solve the inequality $\sin x \geq 0$ when

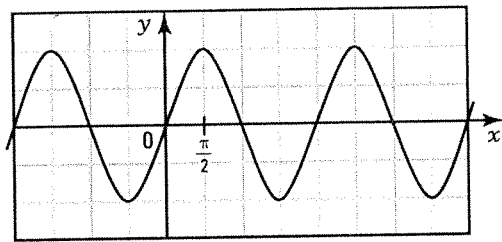
1. $x \in [-2\pi, 4\pi]$. $S = [-2\pi, -\pi] \cup [0, \pi] \cup [2\pi, 3\pi]$
 $S = [2\pi n, \pi + 2\pi n], n \in \mathbb{Z}$

2. $x \in \mathbb{R}$.

c) Find the values of x for which the function f is increasing when

1. $x \in [-2\pi, 4\pi]$. $[-2\pi, -\frac{3\pi}{2}] \cup [-\frac{\pi}{2}, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, \frac{5\pi}{2}] \cup [\frac{7\pi}{2}, 4\pi]$
 $[-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n], n \in \mathbb{Z}$

2. $x \in \mathbb{R}$.



ACTIVITY 2 Equation $\sin \theta = k$

The function $y = \sin x$ is represented on the right when $x \in [-2\pi, 4\pi]$.

a) 1. By referring to the trigonometric circle, solve the equation $\sin \theta = \frac{1}{2}$ when $\theta \in [0, 2\pi]$.

$S = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\}$

2. Explain how to find, from the solutions in 1, the solutions to the equation $\sin \theta = \frac{1}{2}$ when

1) $\theta \in [2\pi, 4\pi]$.

Add the period 2π to each solution.

2) $\theta \in [-2\pi, 0]$.

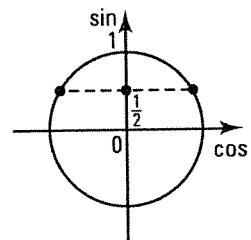
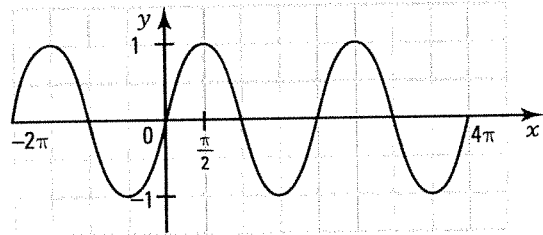
Subtract the period 2π to each solution.

3. Verify that the solution set S to the equation $\sin \theta = \frac{1}{2}$ over \mathbb{R} is described by

$S = \left\{ \dots, -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \dots \right\}$ (enumeration).

or by

$S = \left\{ \frac{\pi}{6} + 2\pi n \right\} \cup \left\{ \frac{5\pi}{6} + 2\pi n \right\}$ where $n \in \mathbb{Z}$ (set-builder notation).



- b) By referring to the trigonometric circle, and using a calculator, solve the equation $\sin \theta = 0.4$ when
- $\theta \in [0, 2\pi]$. $S = \{0.41; 2.73\}$
 - $\theta \in [4\pi, 6\pi]$. $S = \{12.98; 15.29\}$
 - $\theta \in \mathbb{R}$. $S = \{0.41 + 2\pi n\} \cup \{2.73 + 2\pi n\}, n \in \mathbb{Z}$

c) Solve the inequality $\sin \theta \geq \frac{1}{2}$ when

- $\theta \in [0, 2\pi]$. $S = \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$
- $\theta \in [0, 4\pi]$. $S = \left[\frac{\pi}{6}, \frac{5\pi}{6}\right] \cup \left[\frac{13\pi}{6}, \frac{17\pi}{6}\right]$
- $\theta \in \mathbb{R}$. $S = \left[\frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n\right], n \in \mathbb{Z}$

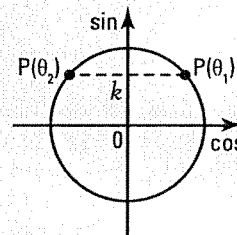
EQUATION $\sin \theta = k, \quad -1 \leq k \leq 1$

- When $\theta \in [0, 2\pi]$, the equation $\sin \theta = k$ yields 2 solutions θ_1 and θ_2 .

$$\theta_1 = \sin^{-1} k \text{ and } \theta_2 = \pi - \theta_1$$

- When $\theta \in \mathbb{R}$, the equation $\sin \theta = k$ yields an infinite number of solutions. The solution set S is defined in notation by

$$S = \{\theta_1 + 2\pi n\} \cup \{\theta_2 + 2\pi n\} \text{ where } n \in \mathbb{Z}$$



Ex.: The equation $\sin \theta = \frac{\sqrt{2}}{2}$ has the solution set:

- $S = \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ when $\theta \in [0, 2\pi]$
- $S = \left\{\frac{\pi}{4} + 2\pi n\right\} \cup \left\{\frac{3\pi}{4} + 2\pi n\right\}, n \in \mathbb{Z}$ when $\theta \in \mathbb{R}$.

2. Solve the following equations over

- | | | | | |
|----|-------------------------------|---|--|--|
| | 1. $[0, 2\pi]$ | 2. $[2\pi, 4\pi]$ | 3. \mathbb{R} | |
| a) | $\sin x = \frac{\sqrt{3}}{2}$ | 1. $S = \left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ | 2. $S = \left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ | 3. $S = \left\{\frac{\pi}{3} + 2\pi n\right\} \cup \left\{\frac{2\pi}{3} + 2\pi n\right\}, n \in \mathbb{Z}$ |
| b) | $\sin x = 1$ | 1. $S = \left\{\frac{\pi}{2}\right\}$ | 2. $S = \left\{\frac{5\pi}{2}\right\}$ | 3. $S = \left\{\frac{\pi}{2} + 2\pi n\right\}, n \in \mathbb{Z}$ |
| c) | $\sin x = -\frac{1}{2}$ | 1. $S = \left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ | 2. $S = \left\{\frac{19\pi}{6}, \frac{23\pi}{6}\right\}$ | 3. $S = \left\{\frac{7\pi}{6} + 2\pi n\right\} \cup \left\{\frac{11\pi}{6} + 2\pi n\right\}, n \in \mathbb{Z}$ |
| d) | $\sin x = 0.6$ | 1. $S = \{0.64; 2.5\}$ | 2. $S = \{6.92; 8.78\}$ | 3. $S = \{0.64 + 2\pi n\} \cup \{2.5 + 2\pi n\}, n \in \mathbb{Z}$ |

3. Solve the following inequalities over

- | | | | |
|----|----------------------------------|---|---|
| | 1. $[0, 2\pi]$ | 2. \mathbb{R} | |
| a) | $\sin x \geq \frac{\sqrt{2}}{2}$ | 1. $S = \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ | 2. $S = \left[\frac{\pi}{4} + 2\pi n, \frac{5\pi}{4} + 2\pi n\right], n \in \mathbb{Z}$ |
| b) | $\sin x \leq \frac{1}{2}$ | 1. $S = \left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$ | 2. $S = \left[2\pi n, \frac{\pi}{6} + 2\pi n\right] \cup \left[\frac{5\pi}{6} + 2\pi n, 2\pi + 2\pi n\right], n \in \mathbb{Z}$ |

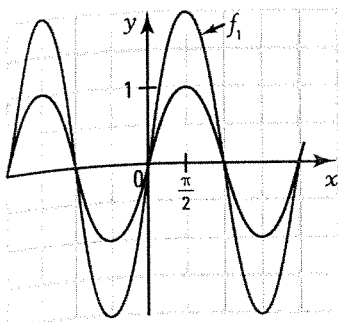
c) $\sin x \leq -\frac{1}{2}$ 1. $S = \left[\frac{7\pi}{6}, \frac{11\pi}{6} \right]$ 2. $S = \left[\frac{7\pi}{6} + 2\pi n, \frac{11\pi}{6} + 2\pi n \right]$

d) $\sin x \leq 1$ 1. $S = [0, 2\pi]$ 2. $S = \mathbb{R}$

ACTIVITY 3 Function $y = a \sin x$

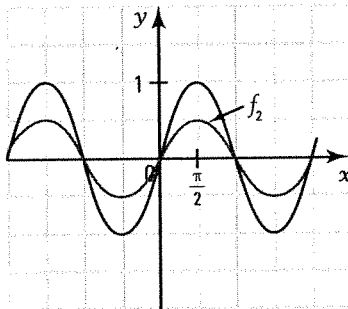
- a) For each of the following functions,
- determine the value of parameter a .
 - indicate how to obtain the graph of the function from the graph of $y = \sin x$.
 - determine the amplitude.

$$f_1(x) = 2 \sin x$$



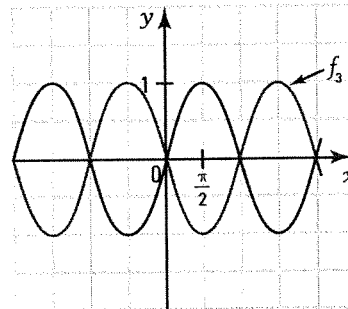
- $a = 2$
- Vertical stretch
- Amplitude = 2

$$f_2(x) = \frac{1}{2} \sin x$$



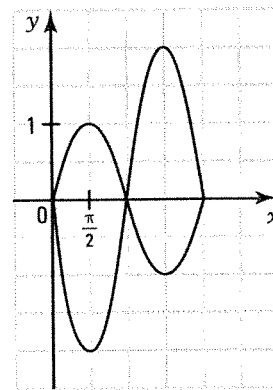
- $a = \frac{1}{2}$
- Vertical reduction
- Amplitude = $\frac{1}{2}$

$$f_3(x) = -\sin x$$



- $a = -1$
- Reflection about the x-axis
- Amplitude = 1

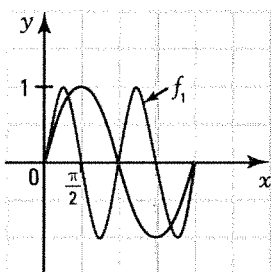
- b) 1. From the graph of the function $y = \sin x$ ($0 \leq x \leq 2\pi$), deduce the graph of $y = -2 \sin x$. Explain the procedure.
- We perform a vertical stretch of factor 2.
 - We perform a reflection about the x-axis of the resulting figure.
2. What is the amplitude of the function $y = -2 \sin x$? 2
- c) What is the relationship between the parameter a of the function $y = -2 \sin x$ and the amplitude of the function? Amplitude = $|a|$



ACTIVITY 4 Function $y = \sin bx$

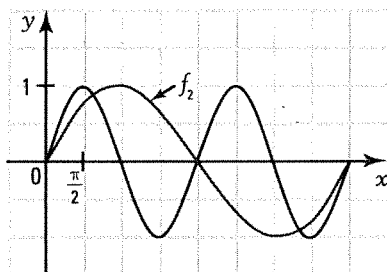
- a) For each of the following functions,
- determine the value of parameter b .
 - indicate how to obtain the graph of the function from the graph of $y = \sin x$.
 - determine the period.

$$f_1(x) = \sin 2x$$



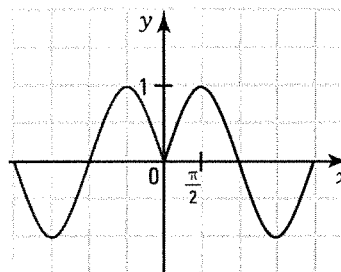
- $b = 2$
- Horizontal reduction
- Period = π

$$f_2(x) = \sin \frac{1}{2}x$$



- $b = \frac{1}{2}$
- Horizontal stretch
- Period = 4π

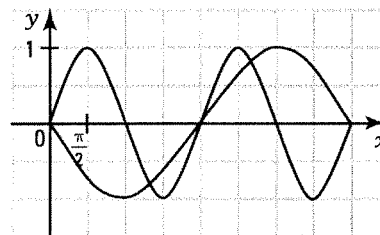
$$f_3(x) = \sin(-x)$$



- $b = -1$
- Reflection about the y-axis
- Period = 2π

- b) 1. From the graph of the function $y = \sin x$ ($0 \leq x \leq 4\pi$), deduce the graph of $y = \sin\left(-\frac{1}{2}x\right)$. Explain the procedure.

- We perform a horizontal stretch of factor 2.
- We perform a reflection about the x-axis of the resulting figure.



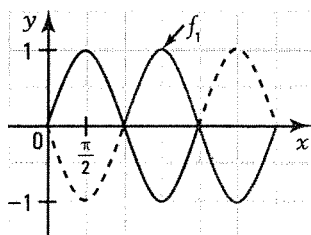
2. What is the period of the function $y = \sin\left(-\frac{1}{2}x\right)$? 4π

- c) What is the relationship between the parameter b of the function $y = \sin bx$ and the period of the function? $\text{Period} = \frac{2\pi}{|b|}$

ACTIVITY 5 Function $y = \sin(x - h)$

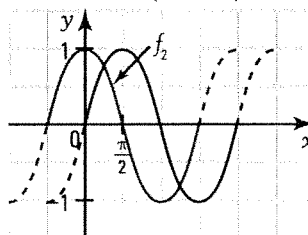
- a) For each of the following functions,
- determine the value of parameter h .
 - indicate how to obtain the graph of the function from the graph of $y = \sin x$.

$$f_1(x) = \sin(x - \pi)$$



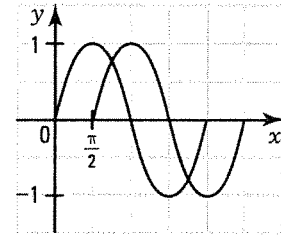
- $h = \pi$
- Horizontal translation of π units to the right.

$$f_2(x) = \sin\left(x + \frac{\pi}{2}\right)$$



- $h = -\frac{\pi}{2}$
- Horizontal translation of $\frac{\pi}{2}$ units to the left.

- b) From the graph of the function $y = \sin x$ ($0 \leq x \leq 2\pi$), deduce the graph of $y = \sin\left(x - \frac{\pi}{2}\right)$. Explain the procedure.
 We perform a horizontal translation of $\frac{\pi}{2}$ units to the right.

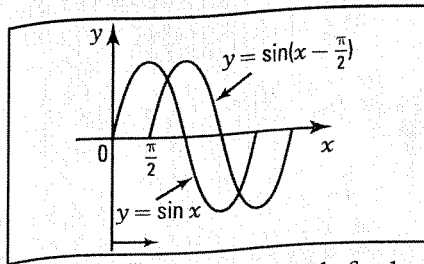


PHASE SHIFT

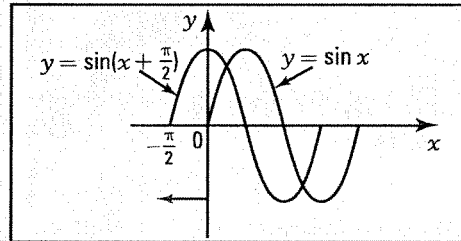
The graph of the function $y = \sin(x - h)$ is deduced from the graph of the function $y = \sin x$ by a horizontal translation of

- $|h|$ units to the right if $h > 0$.
- $|h|$ units to the left if $h < 0$.

Such a shift of the basic function is called the phase shift.



- The curve $y = \sin x$ is phase shifted $|h|$ units to the right.

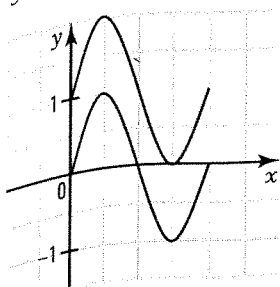


- The curve $y = \sin x$ is phase shifted $|h|$ units to the left.

ACTIVITY 6 Function $y = \sin x + k$

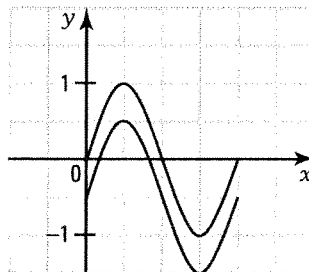
From the graph of $y = \sin x$ ($0 \leq x \leq 2\pi$), deduce, explaining the procedure, the graph of

a) $y = \sin x + 1$.



Perform a vertical translation of 1 unit upward.

b) $y = \sin x - \frac{1}{2}$.



Perform a vertical translation of 0.5 units downward.

SINUSOIDAL FUNCTION $y = a \sin b(x - h) + k$

- The graph of the function $y = a \sin b(x - h) + k$ is deduced from the graph of the basic function $y = \sin x$ by the transformation $(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$.

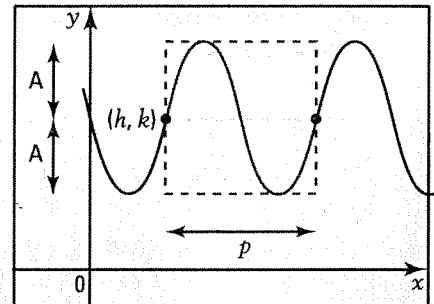
- We have:

period: $p = \frac{2\pi}{|b|}$; amplitude: $A = |a|$;

domain = \mathbb{R} ;

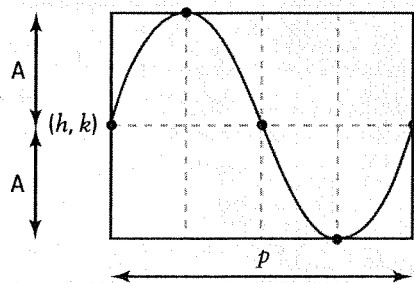
range = $[k - A, k + A]$.

- To draw one cycle of the graph of $y = a \sin b(x - h) + k$,
 - we identify the point (h, k) , starting point of one cycle.
 - we draw a rectangle with a height equal to $2A$, twice the amplitude, and a length equal to the period p .
 - we draw one cycle inside the rectangle.

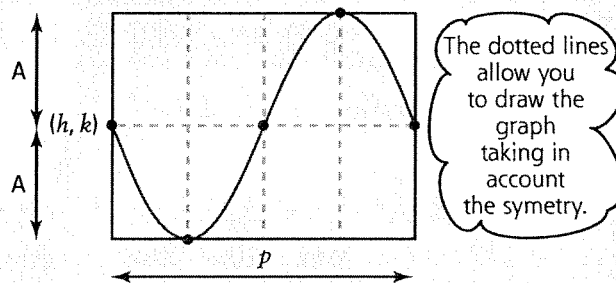


There are two situations depending on the signs of a and b .

$ab > 0$
Increasing cycle from the start.



$ab < 0$
Decreasing cycle from the start.

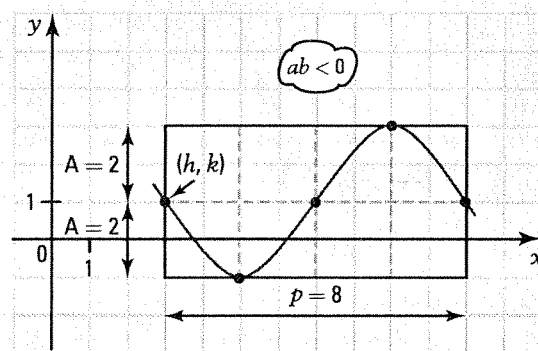


Ex.: Draw one cycle of the function

$$y = -2 \sin \frac{\pi}{4}(x - 3) + 1.$$

We have: $a = -2$, $b = \frac{\pi}{4}$, $h = 3$, $k = 1$

- starting point of the cycle: $(h, k) = (3, 1)$.
- amplitude: $A = |a| = 2$.
- period: $p = \frac{2\pi}{|b|} = 8$.

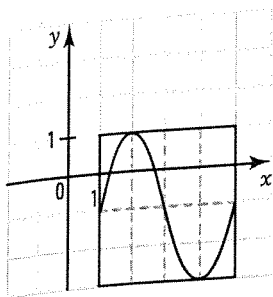


- 4.** Consider the functions $f(x) = a \sin b(x - h) + k$ and $g(x) = -a \sin -b(x - h) + k$. Explain why the functions f and g are equal.

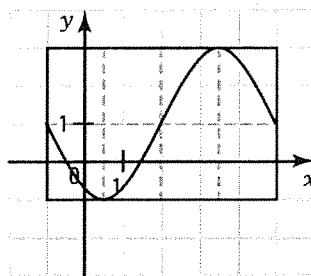
$$-a \sin -b(x - h) + k = -a[-\sin b(x - h)] + k = a \sin b(x - h) + k \text{ since } \sin(-t) = -\sin t.$$

5. Draw one cycle of each of the following functions.

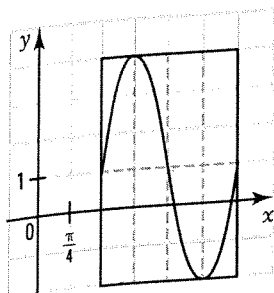
a) $y = 2 \sin \frac{\pi}{2}(x - 1) - 1$



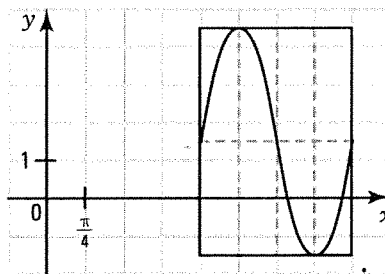
b) $y = -2 \sin \frac{\pi}{3}(x + 1) + 1$



c) $y = 3 \sin 2\left(x - \frac{\pi}{2}\right) + 1$



d) $y = -3 \sin -2(x - \pi) + \frac{3}{2}$



ACTIVITY 7 Finding the zeros of the function $y = a \sin b(x - h) + k$

A portion of the graph of the function

$f(x) = -2 \sin \frac{\pi}{3}(x - 1) + 1$

- a) What is the period p of the function? $p = 6$
 b) 1. How many zeros does the function f have?
An infinite number.

2. How many zeros does the function f have when $x \in [1, 7]$?
Two zeros.

- c) Justify the steps for finding the zeros of f when $x \in [1, 7]$.

1. $-2 \sin \frac{\pi}{3}(x - 1) + 1 = 0$

2. $\sin \frac{\pi}{3}(x - 1) = \frac{1}{2}$

3. $\frac{\pi}{3}(x - 1) = \frac{\pi}{6}$ or $\frac{\pi}{3}(x - 1) = \frac{5\pi}{6}$

4. $x = 1.5$ or $x = 3.5$

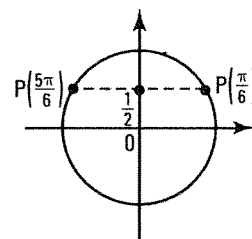
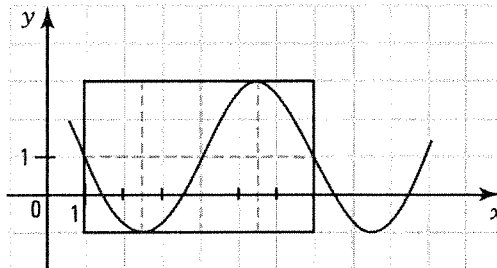
We set $f(x) = 0$.

We isolate $\sin b(x - h)$.

We solve the equation $\sin \theta = \frac{1}{2}$

knowing that $\theta = \frac{\pi}{3}(x - 1)$.

We deduce the solutions for x .



- d) From the zeros of f obtained over $[1, 7]$, explain how to deduce the zeros of f located on the next cycle, i.e. when $x \in [7, 13]$.

You need to add the period to each zero. The zeros over $[7, 13]$ are 7.5 and 9.5.

- e) Verify that the set of zeros of f is $\{\dots; -4.5; -2.5; 1.5; 3.5; 7.5; 9.5; \dots\}$
 or $\{1.5 + 6n\} \cup \{3.5 + 6n\}, n \in \mathbb{Z}$.

FINDING THE ZEROS OF THE FUNCTION $f(x) = a \sin b(x - h) + k$

To determine the zeros of $f(x) = a \sin b(x - h) + k$,

1. we establish the period of the function.

$$p = \frac{2\pi}{|b|}$$

2. we solve the equation $f(x) = 0$.

- Isolate $\sin b(x - h)$.
- Find the angles $b(x - h)$ over the interval $[0, 2\pi]$ that verify the equation.
- Isolate x .
- Determine the solution set, taking the period into consideration.

Ex.: $f(x) = 2 \sin \frac{\pi}{2}(x - 1) - 1$

$$p = 4$$

$$2 \sin \frac{\pi}{2}(x - 1) - 1 = 0$$

$$\sin \frac{\pi}{2}(x - 1) = \frac{1}{2}$$

$$\frac{\pi}{2}(x - 1) = \frac{\pi}{6} \text{ or } \frac{\pi}{2}(x - 1) = \frac{5\pi}{6}$$

$$x = \frac{4}{3} \text{ or } x = \frac{8}{3}$$

$$S = \left\{ \frac{4}{3} + 4n \right\} \cup \left\{ \frac{8}{3} + 4n \right\}$$

6. The following functions have the rule $f(x) = a \sin b(x - h) + k$.

Find the zeros of each function over

1. the interval $[h, h + p]$ where p is the period of the function.
2. the set of all real numbers.

a) $f(x) = 2 \sin \frac{\pi}{6}(x - 2) + 1$

$$\sin \frac{\pi}{6}(x - 2) = \frac{-1}{2}$$

$$\frac{\pi}{6}(x - 2) = \frac{7\pi}{6} \text{ or } \frac{\pi}{6}(x - 2) = \frac{11\pi}{6}$$

$$x = 9 \text{ or } x = 13$$

1. $S = \{9, 13\}$ over $[2, 14]$

2. $S = \{9 + 12n\} \cup \{13 + 12n\}$

b) $f(x) = -2 \sin \frac{\pi}{3}(x + 1) + \frac{1}{2}$

$$\sin \frac{\pi}{3}(x + 1) = 0.25$$

$$\frac{\pi}{3}(x + 1) = 0.25 \text{ or } \frac{\pi}{3}(x + 1) = 2.89$$

$$x = -0.76 \text{ or } x = 1.76$$

1. $S = \{-0.76; 1.76\}$ over $[-1, 5]$

2. $S = \{-0.76 + 6n\} \cup \{1.76 + 6n\}$

c) $f(x) = \sin 2(x - \pi) + 1$

$$\sin 2(x - \pi) = -1$$

$$2(x - \pi) = \frac{3\pi}{2}$$

$$x = \frac{7\pi}{4}$$

1. $S = \left\{ \frac{7\pi}{4} \right\}$ over $[\pi, 2\pi]$

2. $S = \left\{ \frac{7\pi}{4} + \pi n \right\}$

d) $f(x) = 6 \sin \left(x + \frac{\pi}{2} \right) - 3$

$$\sin \left(x + \frac{\pi}{2} \right) = \frac{1}{2}$$

$$x + \frac{\pi}{2} = \frac{\pi}{6} \text{ or } x + \frac{\pi}{2} = \frac{5\pi}{6}$$

$$x = -\frac{\pi}{3} \text{ or } x = \frac{\pi}{3}$$

1. $S = \left\{ -\frac{\pi}{3}, \frac{\pi}{3} \right\}$ over $\left[-\frac{\pi}{2}, \frac{3\pi}{2} \right]$

2. $S = \left\{ -\frac{\pi}{3} + 2\pi n \right\} \cup \left\{ \frac{\pi}{3} + 2\pi n \right\}$

e) $f(x) = -3 \sin \frac{\pi}{4}x + 6$

$$\sin \frac{\pi}{4}x = 2$$

1. $S = \emptyset$

2. $S = \emptyset$

f) $f(x) = -2 \sin \frac{\pi}{8}(x + 2) + \sqrt{2}$

$$\sin \frac{\pi}{8}(x + 2) = \frac{\sqrt{2}}{2}$$

$$\frac{\pi}{8}(x + 2) = \frac{\pi}{4} \text{ or } \frac{\pi}{8}(x + 2) = \frac{3\pi}{4}$$

1. $S = \{0, 4\}$ over $[-2, 14]$

2. $S = \{0 + 16n\} \cup \{4 + 16n\}$

7. Determine the zeros of the function $f(x) = -2 \sin \frac{\pi}{12}(x+5) - 1$ over the interval $[90, 150]$.

$$\sin \frac{\pi}{12}(x+5) = -\frac{1}{2}$$

$$\frac{\pi}{12}(x+5) = \frac{7\pi}{6} \text{ or } \frac{\pi}{12}(x+5) = \frac{11\pi}{6}$$

$$x = 9 \text{ or } x = 17$$

The zeros are: 105, 113, 129, 137.

ACTIVITY 8 Study of the function $f(x) = a \sin b(x-h) + k$

One cycle of the function $f(x) = 2 \sin \frac{\pi}{6}(x-1) + 1$ is represented below.

a) Determine
1. the period. 12 2. the amplitude. 2

b) Determine
1. dom f . \mathbb{R} 2. ran f . $[-1, 3]$

3. the zeros of f over $[1, 13]$. 8 and 12

4. the zeros of f over \mathbb{R} . $[8 + 12n] \cup [12 + 12n]$

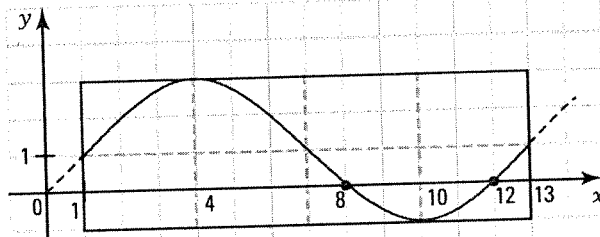
5. the sign of f over $[1, 13]$.
 $f(x) \geq 0$ over $[1, 8] \cup [12, 13]$ et $f(x) \leq 0$ over $[8, 12]$

6. the sign of f over \mathbb{R} .
 $f(x) \geq 0$ over $[1 + 12n, 8 + 12n] \cup [12 + 12n, 13 + 12n]$

7. the variation of f over $[1, 13]$.
 $f \nearrow$ over $[1, 4] \cup [10, 13]$ et $f \searrow$ over $[4, 10]$

8. the variation of f over \mathbb{R} .
 $f \nearrow$ over $[1 + 12n, 4 + 12n] \cup [10 + 12n, 13 + 12n]$

9. the maximum and minimum of f . $\max f = 3; \min f = -1$



STUDY OF THE FUNCTION $f(x) = a \sin b(x-h) + k$

Given $f(x) = -2 \sin 2\left(x - \frac{\pi}{6}\right) + 1$. We have:

• dom $f = \mathbb{R}$; ran $f = [-1, 3]$

• période $p = \pi$; amplitude $A = 2$

• zeros of f : $\left\{\frac{\pi}{4} + \pi n\right\} \cup \left\{\frac{7\pi}{12} + \pi n\right\}$

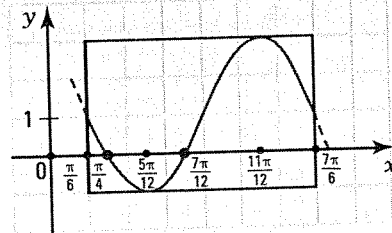
• sign of f .
 $f(x) \geq 0$ over $\left[\frac{\pi}{6} + \pi n, \frac{\pi}{4} + \pi n\right] \cup \left[\frac{7\pi}{12} + \pi n, \frac{7\pi}{6} + \pi n\right]$

$f(x) \leq 0$ over $\left[\frac{\pi}{4} + \pi n, \frac{7\pi}{12} + \pi n\right]$

• variation of f .
 $f \searrow$ over $\left[\frac{\pi}{6} + \pi n, \frac{5\pi}{12} + \pi n\right] \cup \left[\frac{11\pi}{12} + \pi n, \frac{7\pi}{6} + \pi n\right]$

$f \nearrow$ over $\left[\frac{5\pi}{12} + \pi n, \frac{11\pi}{12} + \pi n\right]$

• $\min f = -1; \max f = 3$.



8. For each of the following functions, determine

1. the period, 2. the amplitude, 3. the range of the function.

a) $f(x) = -2 \sin \frac{\pi}{8}(x-5) + 3$

1. $p = 16$
2. $A = 2$
3. $\text{ran } f = [1, 5]$

b) $f(x) = 3 \sin 12\left(x + \frac{\pi}{2}\right) + 5$

1. $p = \frac{\pi}{6}$
2. $A = 3$
3. $\text{ran } f = [2, 8]$

c) $f(x) = 5 \sin \frac{4\pi}{3}(x+1) - 4$

1. $p = \frac{3}{2}$
2. $A = 5$
3. $\text{ran } f = [-9, 1]$

d) $f(x) = 10 \sin \frac{6}{5}\left(x - \frac{\pi}{4}\right) + 4$

1. $p = \frac{5\pi}{3}$
2. $A = 10$
3. $\text{ran } f = [-6, 14]$

9. Determine the initial value of the following functions.

a) $f(x) = 4 \sin \frac{\pi}{6}(x+1) - 3$ -1

b) $f(x) = -2 \sin \frac{\pi}{3}(x-2) + 2$ $\sqrt{3} + 2$

c) $f(x) = 2 \sin 2\left(x - \frac{\pi}{4}\right) + 4$ 2

d) $f(x) = 3 \sin \pi(x+5) - 1$ -1

10. For each of the following functions, determine, over \mathbb{R} , the interval over which the function is positive.

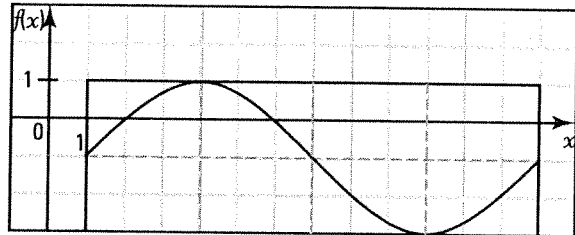
a) $f(x) = 2 \sin \frac{\pi}{6}(x-1) - 1$

Zeros: $\sin \frac{\pi}{6}(x-1) = \frac{1}{2}$

$$\frac{\pi}{6}(x-1) = \frac{\pi}{6} \text{ or } \frac{\pi}{6}(x-1) = \frac{5\pi}{6}$$

$$x = 2 \qquad x = 6$$

$f(x) \geq 0$ over $[2 + 12n, 6 + 12n]$



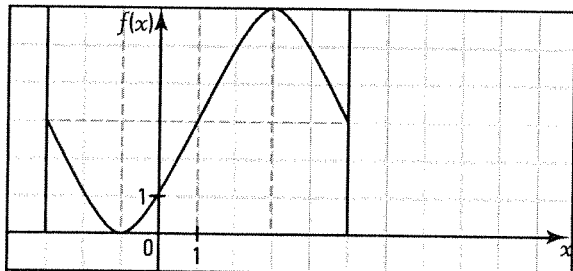
b) $f(x) = -3 \sin \frac{\pi}{4}(x+3) + 3$

Zeros: $\sin \frac{\pi}{4}(x+3) = 1$

$$\frac{\pi}{4}(x+3) = \frac{\pi}{2}$$

$$x = -1$$

$f(x) \geq 0$ over \mathbb{R}



c) $f(x) = 2 \sin \frac{1}{3}\left(x - \frac{\pi}{2}\right) + 1$

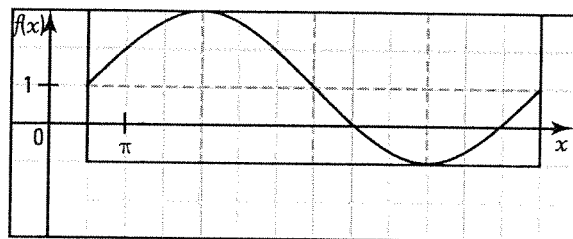
Zéros: $\sin \frac{1}{3}\left(x - \frac{\pi}{2}\right) = -\frac{1}{2}$

$$\frac{1}{3}\left(x - \frac{\pi}{2}\right) = \frac{7\pi}{6} \text{ or } \frac{1}{3}\left(x - \frac{\pi}{2}\right) = \frac{11\pi}{6}$$

$$x = 4\pi$$

$$x = 6\pi$$

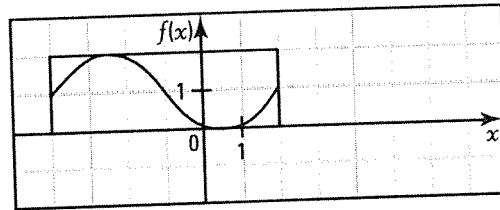
$f(x) \geq 0$ over $\left[\frac{\pi}{2} + 6\pi n, 4\pi + 6\pi n\right] \cup \left[6\pi + 6\pi n, \frac{13\pi}{2} + 6\pi n\right]$



11. For each of the following functions, determine the interval over which the function is decreasing.

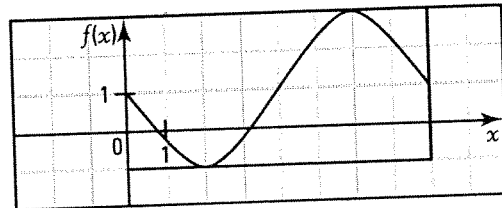
a) $f(x) = \sin \frac{\pi}{3}(x + 4) + 1$ over $[-4, 2]$.

$f \searrow$ over $[-2.5, 0.5]$



b) $f(x) = -2 \sin \frac{\pi}{4}x + 1$ over $[0, 8]$

$f \searrow$ over $[0, 2] \cup [6, 8]$



12. In an experiment, we define a function I which gives the electrical current across a cable, expressed in amperes, as a function of time x , expressed in seconds. The function I is defined by the rule: $I(x) = 4 \sin \frac{\pi}{6}(x + 8) + 4$.

A light bulb lights up when the intensity of the current is equal to 6 amperes.

Determine at what times the light bulb lights up during the first 30 seconds of the experiment.

$4 \sin \frac{\pi}{6}(x + 8) + 4 = 6$

The light bulb lights up at the moments

$\sin \frac{\pi}{6}(x + 8) = \frac{1}{2}$

5 s, 9 s, 17 s, 21 s and 29 s.

$\frac{\pi}{6}(x + 8) = \frac{\pi}{6}$ or $\frac{\pi}{6}(x + 8) = \frac{5\pi}{6}$

$x = -7$ $x = -3$

13. In a company, the number of parts assembled varies according to a sinusoidal function defined by the rule: $f(x) = 200 \sin \left(\frac{\pi}{24}x \right) + 15$ where x represents the number of days elapsed since the beginning of the year.

Over the course of the first 50 days, for how many days the number of parts assembled was greater than or equal to 115?

$200 \sin \left(\frac{\pi}{24} \right) x + 15 = 115$

$\sin \frac{\pi}{24}x = \frac{1}{2}$

$\frac{\pi}{24}x = \frac{\pi}{6}$ or $\frac{\pi}{24}x = \frac{5\pi}{6}$

During 17 days, from the 4th day to the 20th day, inclusively.

$x = 4$ $x = 20$

ACTIVITY 9 Finding the rule $y = a \sin b(x - h) + k$

Consider the function f represented on the right.

a) Determine

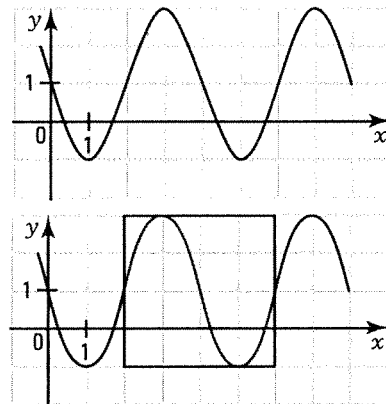
1. the period of f . 4 2. the amplitude of f . 2

b) Determine the rule of the function f when we choose a cycle starting at the point $(h, k) = (2, 1)$.

Explain your procedure: Since the cycle is increasing from the start, we deduce that $ab > 0$.

$$p = \frac{2\pi}{|b|} = 4 \Rightarrow |b| = \frac{\pi}{2}; A = |a| = 2$$

$$\text{Thus, } f(x) = 2 \sin \frac{\pi}{2}(x - 2) + 1 \text{ or } f(x) = -2 \sin \frac{-\pi}{2}(x - 2) + 1$$

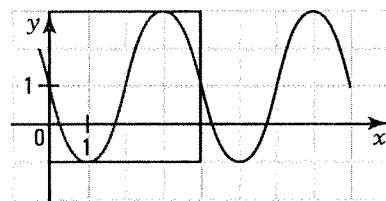


c) Determine the rule of the function f when we choose a cycle starting at the point $(h, k) = (0, 1)$.

Explain your procedure: Since the cycle is decreasing from the start, we deduce that $ab < 0$.

$$|b| = \frac{\pi}{2}; |a| = 2$$

$$\text{Thus, } f(x) = -2 \sin \frac{\pi}{2}x + 1 \text{ or } f(x) = 2 \sin \frac{-\pi}{2}x + 1$$



d) Is it true to say that the writing of the rule varies depending on the chosen cycle but that all of the resulting rules represent the same function? Yes.

FINDING THE RULE $y = a \sin b(x - h) + k$

From the graph of the function on the right, we deduce that: $|a| = 1.5$ and $|b| = 2$.

The writing of the function's rule depends on your choice of the starting point (h, k) of one cycle.

Consider two cases:

1st case: $(h, k) = \left(\frac{\pi}{4}, 1\right)$

The cycle is increasing from the start $\Rightarrow ab > 0$.

Thus, $a = 1.5$ and $b = 2$ or $a = -1.5$ and $b = -2$.

The rule is written in two ways:

$$y = 1.5 \sin 2\left(x - \frac{\pi}{4}\right) + 1 \text{ or } y = -1.5 \sin -2\left(x - \frac{\pi}{4}\right) + 1.$$

2nd case: $(h, k) = \left(\frac{3\pi}{4}, 1\right)$

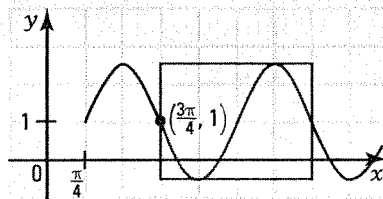
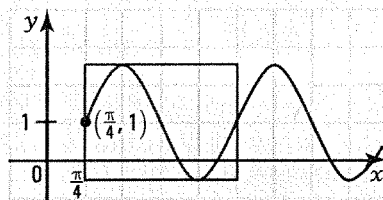
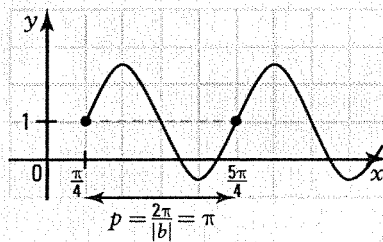
The cycle is decreasing from the start $\Rightarrow ab < 0$.

Thus, $a = 1.5$ and $b = -2$ or $a = -1.5$ and $b = 2$.

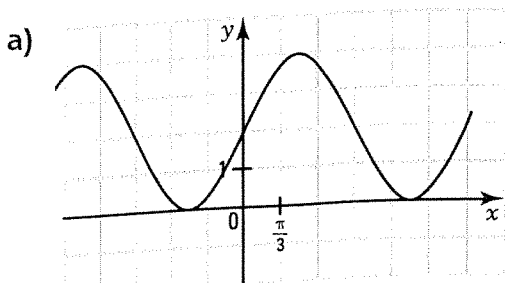
The rule is written in two ways:

$$y = 1.5 \sin -2\left(x - \frac{3\pi}{4}\right) + 1 \text{ or } y = -1.5 \sin 2\left(x - \frac{3\pi}{4}\right) + 1.$$

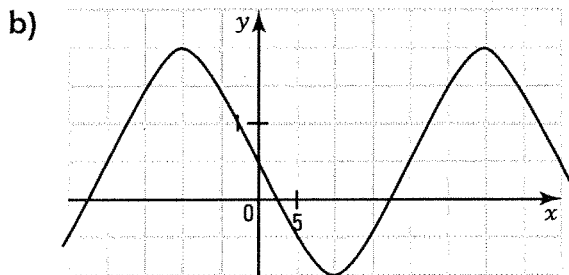
Note that the different ways of writing the rule represent the same function.



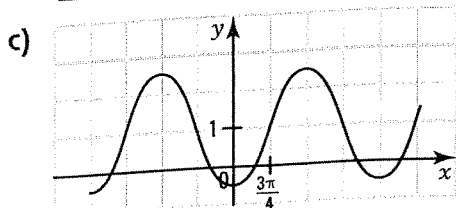
- 14.** Find a rule of the form $y = a \sin b(x - h) + k$ for each of the following functions.



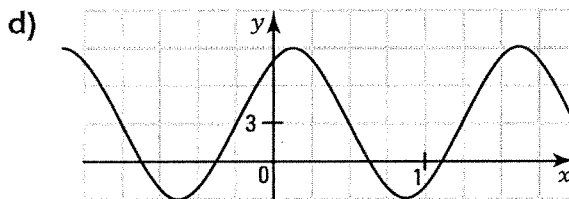
For example, $y = 2 \sin x + 2$



For example, $y = -1.5 \sin \frac{\pi}{20} x + 0.5$



For example, $y = -1.5 \sin \frac{2}{3} \left(x + \frac{3\pi}{4} \right) + 1$



For example, $y = 6 \sin \frac{4\pi}{3} \left(x + \frac{1}{4} \right) + 3$

- 15.** The waves of an artificial lake are observed in a laboratory setting as indicated in the figure on the right. The graph represents the movement of one wave where t is time, expressed in seconds, and $h(t)$ is the height of the wave, in metres.



What is the height of the wave after 3 seconds?

$(h, k) = (0; 0.5)$ $h(t) = \sin \frac{\pi}{4} t + 0.5$; $h(3) \approx 1.21$ m

- 16.** Raphael is playing with a yo-yo for 30 seconds. The height of the yo-yo, in metres, varies as a function of time t , in seconds, according to the rule of a sinusoidal function. Initially, the yo-yo is at its maximum height of 2 m. After 4 seconds, the yo-yo reaches its minimum height of 1 m for the first time.

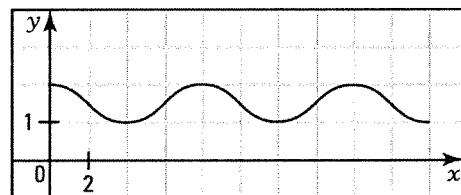
After how many seconds does Raphael's yo-yo reach a height of 1.8 m for the first time?

Rule of the function:

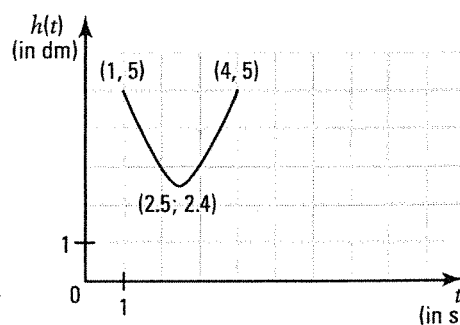
for example, $y = -0.5 \sin \frac{\pi}{4} (x - 2) + 1.5$.

$-0.5 \sin \frac{\pi}{4} (x - 2) + 1.5 = 1.8$

$x = 1.18$. After 1.18 seconds, the yo-yo reaches a height of 1.8 m for the first time.



- 17.** The flashing light of an electronic game follows the trajectory of a sinusoidal function as indicated in the graph on the right. If t represents the time (in seconds) and $h(t)$ represents the height (in dm) of the light, determine at what height the light is at the moment $t = 6$ s?



$(h, k) = (1.5)$

$y = -2.6 \sin \frac{\pi}{3} (x - 1) + 5$

$A = 2.6$ $a = -2.6$

$b = \frac{2\pi}{p} = \frac{2\pi}{6} = \frac{\pi}{3}$

At the moment $t = 6$ s, the light will be at a height of 7.25 dm.

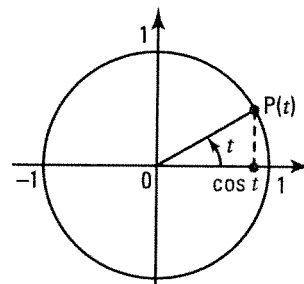
7.7 Cosine function

ACTIVITY 1 Basic cosine function

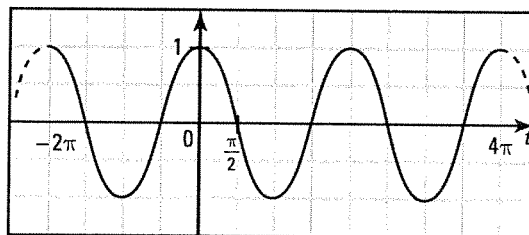
The basic cosine function, denoted \cos , has the rule $y = \cos t$.

a) Complete the table of values below when t varies from -2π to 4π .

t	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$\cos t$	1	0	-1	0	1	0	-1	0	1	0	-1	0	1



b) We have represented the function $y = \cos t$ on the right. Is this function periodic? If yes, what is the period p of the function? **Yes, $p = 2\pi$**



c) For this function, determine

- the domain. \mathbb{R}
- the range. $[-1, 1]$
- the maximum. **1**
- the minimum. **-1**

d) What is the amplitude A of the cosine function? **1**

e) When $t \in [0, 2\pi]$, determine, for the basic cosine function,

- the zeros. $\frac{\pi}{2}$ and $\frac{3\pi}{2}$
- the sign. $\cos t \geq 0$ if $t \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$; $\cos t \leq 0$ if $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$
- the variation. $\cos \searrow$ if $t \in [0, \pi]$; $\cos \nearrow$ if $t \in [\pi, 2\pi]$

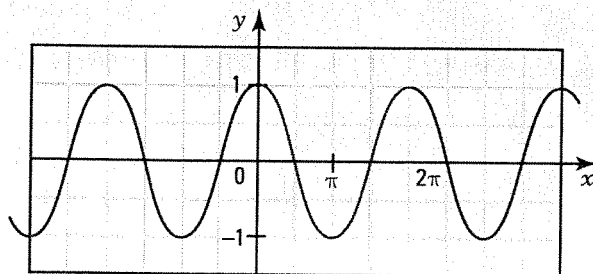
f) Verify, using the graph or the trigonometric circle, the property: $\forall t \in \mathbb{R}: \cos(-t) = \cos(t)$.

BASIC COSINE FUNCTION

- The cosine function, denoted \cos , is defined by

$$\begin{aligned} \cos: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = \cos x \end{aligned}$$

- The cosine function is a periodic function with period 2π . $\cos(x + 2\pi) = \cos x$



One cycle of the basic cosine function is represented in green over $[0, 2\pi]$.

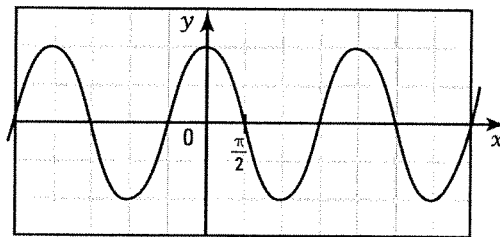
- The amplitude of the basic cosine function is: $A = 1$.

- We have:
 - domain = \mathbb{R} , range = $[-1, 1]$.
 - zeros over $[0, 2\pi]$: $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.
 - sign over $[0, 2\pi]$: $\cos x \geq 0$ if $x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ and $\cos x \leq 0$ if $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.
 - variation over $[0, 2\pi]$: $\cos \searrow$ if $x \in [0, \pi]$; $\cos \nearrow$ if $x \in [\pi, 2\pi]$.
 - extrema: max = 1; min = -1.
- For any real x , we have: $\cos(-x) = \cos x$. The basic cosine function is therefore considered to be an **even function**.

1. Consider the function $f(x) = \cos x$.

a) Find the zeros of f when

- $x \in [-2\pi, 4\pi]$. $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ and $\frac{7\pi}{2}$
- $x \in \mathbb{R}$. $\left\{\frac{\pi}{2} + \pi n\right\}, n \in \mathbb{Z}$



b) Solve the inequality $\cos x \geq 0$ when

- $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$. $S = \left[-\frac{5\pi}{2}, -\frac{3\pi}{2}\right] \cup \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$
- $x \in \mathbb{R}$. $\left[-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n\right]$

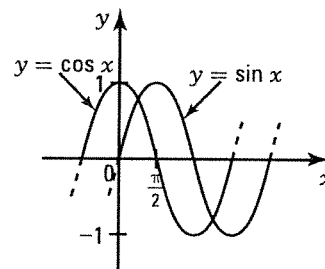
c) Find the values of x for which the function f is increasing when

- $x \in [-2\pi, 4\pi]$. $[-\pi, 0] \cup [\pi, 2\pi] \cup [3\pi, 4\pi]$
- $x \in \mathbb{R}$. $[\pi + 2\pi n, 2\pi + 2\pi n], n \in \mathbb{Z}$

2. The functions $y = \sin x$ and $y = \cos x$ are represented on the right.

- Verify that $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.
- Complete.

The graph of the function $y = \cos x$ is deduced from the graph of $y = \sin x$ by by a phase shift to the left of $\frac{\pi}{2}$ units.

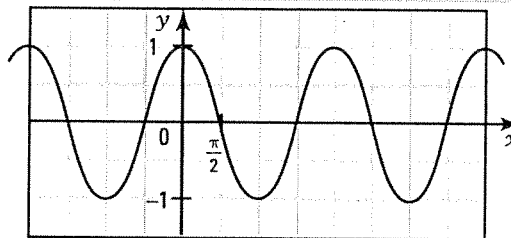


ACTIVITY 2 Equation $\cos \theta = k$

The function $y = \cos x$ is represented on the right when $x \in [-2\pi, 4\pi]$.

- By referring to the trigonometric circle, solve the equation $\cos \theta = \frac{1}{2}$ when $\theta \in [0, 2\pi]$.

$$S = \left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$$



2. Explain how to find, from the solutions in 1, the solutions to the equation $\cos \theta = \frac{1}{2}$ when

1) $\theta \in [2\pi, 4\pi]$. Add the period 2π to each solution.

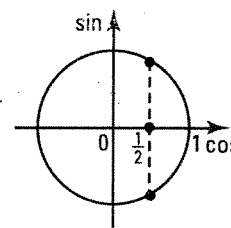
2) $\theta \in [-2\pi, 0]$. Subtract the period 2π to each solution.

3. Verify that the solution set S to the equation $\cos \theta = \frac{1}{2}$ over \mathbb{R} is described by

$$S = \left\{ \dots, -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \dots \right\} \text{ (enumeration).}$$

or by

$$S = \left\{ \frac{\pi}{3} + 2\pi n \right\} \cup \left\{ \frac{5\pi}{3} + 2\pi n \right\} \text{ where } n \in \mathbb{Z} \text{ (set-builder notation).}$$



b) By referring to the trigonometric circle and using a calculator, solve the equation $\cos \theta = 0.4$ when

$$S = \{1.16; 5.12\}$$

1. $\theta \in [0, 2\pi]$. $S = \{13.73; 17.69\}$

2. $\theta \in [4\pi, 6\pi]$. $S = \{1.16 + 2\pi n\} \cup \{5.12 + 2\pi n\}, n \in \mathbb{Z}$

3. $\theta \in \mathbb{R}$. _____

c) Solve the inequality $\cos \theta \geq \frac{1}{2}$ when

1. $\theta \in [0, 2\pi]$. $S = \left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$

2. $\theta \in [0, 4\pi]$. $S = \left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right] \cup \left[2\pi, \frac{7\pi}{3}\right] \cup \left[\frac{11\pi}{3}, 4\pi\right]$

3. $\theta \in \mathbb{R}$. $S = \left[0 + 2\pi n, \frac{\pi}{3} + 2\pi n\right] \cup \left[\frac{5\pi}{3} + 2\pi n, 2\pi + 2\pi n\right], n \in \mathbb{Z}$

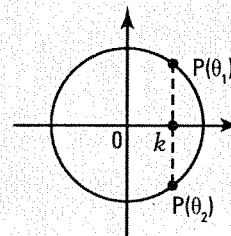
EQUATION $\cos \theta = k, -1 \leq k \leq 1$

• When $\theta \in [0, 2\pi[$, the equation $\cos \theta = k$ yields 2 solutions θ_1 and θ_2 .

$$\theta_1 = \cos^{-1} k \text{ and } \theta_2 = 2\pi - \theta_1$$

• When $\theta \in \mathbb{R}$, the equation $\cos \theta = k$ yields an infinite number of solutions. The solution set S is defined by

$$S = \{\theta_1 + 2\pi n\} \cup \{\theta_2 + 2\pi n\} \text{ where } n \in \mathbb{Z}$$



Ex.: The equation $\cos \theta = \frac{\sqrt{3}}{2}$ has the solution set:

$$- S = \left\{ \frac{\pi}{6}, \frac{11\pi}{6} \right\} \text{ when } \theta \in [0, 2\pi[$$

$$- S = \left\{ \frac{\pi}{6} + 2\pi n \right\} \cup \left\{ \frac{11\pi}{6} + 2\pi n \right\}, n \in \mathbb{Z} \text{ when } \theta \in \mathbb{R}.$$

3. Solve the following equations over

1. $[0, 2\pi]$

2. $[2\pi, 4\pi]$

3. \mathbb{R}

a) $\cos x = \frac{\sqrt{3}}{2}$ 1. $\left\{ \frac{\pi}{6}, \frac{11\pi}{6} \right\}$ 2. $\left\{ \frac{13\pi}{6}, \frac{23\pi}{6} \right\}$ 3. $\left\{ \frac{\pi}{6} + 2\pi n \right\} \cup \left\{ \frac{11\pi}{6} + 2\pi n \right\}, n \in \mathbb{Z}$

b) $\cos x = 1$ 1. $\{0, 2\pi\}$ 2. $\{2\pi, 4\pi\}$ 3. $\{2\pi n\}, n \in \mathbb{Z}$

c) $\cos x = -\frac{\sqrt{2}}{2}$ 1. $\left\{\frac{3\pi}{4}, \frac{5\pi}{4}\right\}$ 2. $\left\{\frac{11\pi}{4}, \frac{13\pi}{4}\right\}$ 3. $\left\{\frac{3\pi}{4} + 2\pi n\right\} \cup \left\{\frac{5\pi}{4} + 2\pi n\right\}, n \in \mathbb{Z}$

d) $\cos x = 0.6$ 1. $[0.93; 5.35]$ 2. $[7.21; 11.63]$ 3. $[0.93 + 2\pi n] \cup [5.35 + 2\pi n], n \in \mathbb{Z}$

4. Solve the following inequalities over

1. $[0, 2\pi]$ 2. \mathbb{R}

a) $\cos x \geq \frac{\sqrt{3}}{2}$ 1. $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{11\pi}{6}, 2\pi\right]$ 2. $\left[2\pi n, \frac{\pi}{6} + 2\pi n\right] \cup \left[\frac{11\pi}{6} + 2\pi n, 2\pi + 2\pi n\right]$

b) $\cos x \leq \frac{1}{2}$ 1. $\left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$ 2. $\left[\frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n\right]$

c) $\cos x \leq -\frac{1}{2}$ 1. $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$ 2. $\left[\frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n\right]$

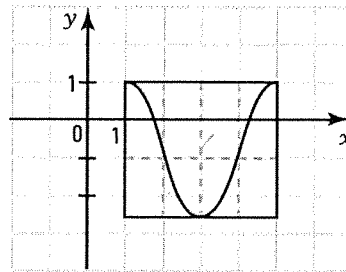
d) $\cos x \leq 1$ 1. $[0, 2\pi]$ 2. \mathbb{R}

ACTIVITY 3 Sinusoidal function $y = a \cos b(x - h) + k$

Consider the function $y = 2 \cos \frac{\pi}{2}(x - 1) - 1$.

a) Identify the parameters a, b, h and k .

$a = 2, b = \frac{\pi}{2}, h = 1, k = -1$



b) For this function, determine

1. the period. 4 2. the amplitude. 2

c) Draw one cycle of this function using the point (h, k) as the starting point.

SINUSOIDAL FUNCTION $y = a \cos b(x - h) + k$

- The graph of the function $y = a \cos b(x - h) + k$ is deduced from the graph of the basic function $y = \cos x$ by the transformation $(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$.

- We have:

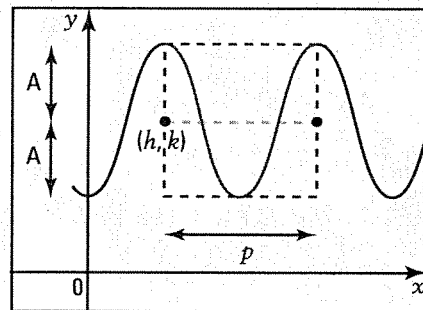
period: $p = \frac{2\pi}{|b|}$; amplitude: $A = |a|$;

domain = \mathbb{R} ;

range = $[k - A, k + A]$

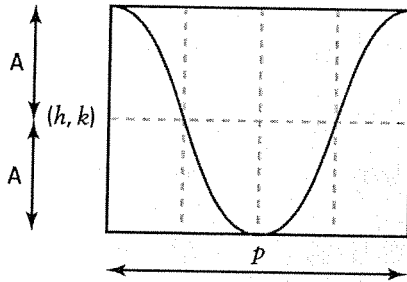
- To draw one cycle of the graph of $y = a \cos b(x - h) + k$,

- we identify the point (h, k) .
- we draw a rectangle with a height equal to $2A$, twice the amplitude, and a length equal to the period p .
- we draw one cycle inside the rectangle.

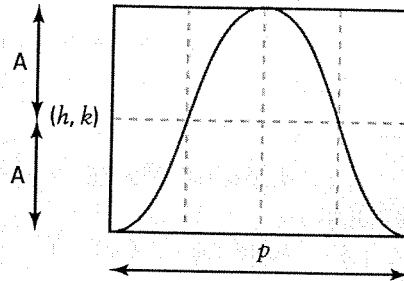


There are two situations depending on the sign of a .

$a > 0$
Decreasing cycle from the start.
Starting point: $(h, k + A)$.



$a < 0$
Increasing cycle from the start.
Starting point: $(h, k - A)$.

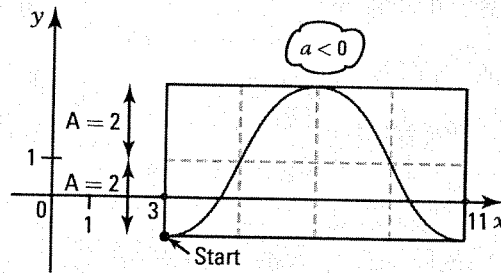


Ex.: Draw one cycle of the function

$$y = -2 \cos \frac{-\pi}{4}(x - 3) + 1.$$

We have: $a = -2$, $b = \frac{-\pi}{4}$, $h = 3$, $k = 1$

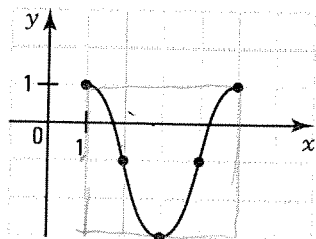
- starting point of the cycle: $(h, k - A) = (3, -1)$.
- amplitude: $A = |a| = 2$.
- period: $p = \frac{2\pi}{|b|} = 8$.



5. Consider the functions $f(x) = a \cos b(x - h) + k$ and $g(x) = a \cos -b(x - h) + k$. Explain why the functions f and g are equal.
 $a \cos -b(x - h) + k = a \cos b(x - h) + k$ since $\cos(-t) = \cos t$.

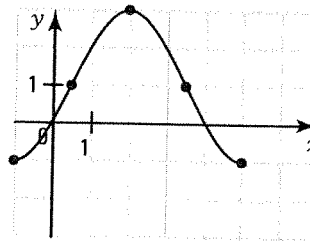
6. Draw one cycle of each of the following functions.

a) $y = 2 \cos \frac{\pi}{2}(x - 1) - 1$

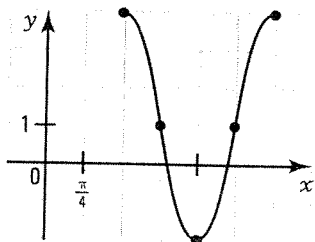


$(1, 1)$
 $|a| = 2$
period = $\frac{2\pi}{|\pi/2|}$
 $= 2\pi \cdot \frac{2}{\pi}$
 $= 4$

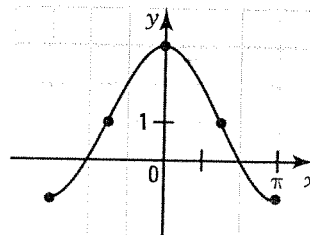
b) $y = -2 \cos \frac{\pi}{3}(x + 1) + 1$



c) $y = 3 \cos 2\left(x - \frac{\pi}{2}\right) + 1$

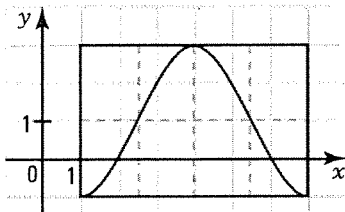


d) $y = -2 \cos -(x + \pi) + 1$



ACTIVITY 4 Finding the zeros of the function $y = a \cos b(x - h) + k$

A portion of the graph of the function $f(x) = -2 \cos \frac{\pi}{3}(x - 1) + 1$ is represented on the right.



- a) What is the period p of the function? $p = 6$
 b) 1. How many zeros does the function f have? An infinite number.
 2. How many zeros does the function f have when $x \in [1, 7]$?

Two zeros.

- c) Justify the steps for finding the zeros of f when $x \in [1, 7]$.

1. $-2 \cos \frac{\pi}{3}(x - 1) + 1 = 0$

We set $f(x) = 0$.

2. $\cos \frac{\pi}{3}(x - 1) = \frac{1}{2}$

We isolate $\cos b(x - h)$.

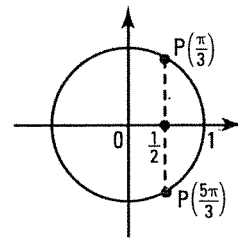
3. $\frac{\pi}{3}(x - 1) = \frac{\pi}{3}$ or $(x - 1) = \frac{5\pi}{3}$

We solve the equation $\cos \theta = \frac{1}{2}$

knowing that $\theta = \frac{\pi}{3}(x - 1)$.

4. $x = 2$ or $x = 6$

We deduce the solutions for x .



- d) From the zeros of f obtained over $[1, 7]$, explain how to deduce the zeros of f located on the next cycle, i.e. when $x \in [7, 13]$.

You need to add the period to each zero. The zeros over $[7, 13]$ are 8 and 12.

- e) Verify that the set of zeros of f is $\{\dots, -4, 0, 2, 6, 8, 12, \dots\}$
 or $\{2 + 6n\} \cup \{6 + 6n\}, n \in \mathbb{Z}$.

FINDING THE ZEROS OF THE FUNCTION $f(x) = a \cos b(x - h) + k$

To determine the zeros of $f(x) = a \cos b(x - h) + k$,

Ex.: $f(x) = 2 \cos \frac{\pi}{2}(x - 1) - 1$

1. we establish the period p of the function.

$p = 4$

$p = \frac{2\pi}{|b|}$

$2 \cos \frac{\pi}{2}(x - 1) - 1 = 0$

2. we solve the equation $f(x) = 0$.

$\cos \frac{\pi}{2}(x - 1) = \frac{1}{2}$

- Isolate $\cos b(x - h)$.
- Find the angles $b(x - h)$ over the interval $[0, 2\pi]$ that verify the equation.

$\frac{\pi}{2}(x - 1) = \frac{\pi}{3}$ or $\frac{\pi}{2}(x - 1) = \frac{5\pi}{3}$

$x = \frac{5}{3}$ or $x = \frac{13}{3}$

- Isolate x .
- Determine the solution set, taking the period into consideration.

$S = \left\{ \frac{5}{3} + 4n \right\} \cup \left\{ \frac{13}{3} + 4n \right\}$

7. The following functions have the rule $f(x) = a \cos b(x - h) + k$. Find the zeros of each function over

- the interval $[h, h + p]$ where p is the period of the function.
- the set of all real numbers.

a) $f(x) = 2 \cos \frac{\pi}{6}(x - 2) + 1$

$$\cos \frac{\pi}{6}(x - 2) = -\frac{1}{2}$$

$$\frac{\pi}{6}(x - 2) = \frac{2\pi}{3} \text{ or } \frac{\pi}{6}(x - 2) = \frac{4\pi}{3}$$

$$x = 6 \text{ or } x = 10$$

$$1. S = \{6, 10\}$$

$$2. S = \{6 + 12n\} \cup \{10 + 12n\}$$

b) $f(x) = -2 \cos \frac{\pi}{3}(x + 1) + \frac{1}{2}$

$$\cos \frac{\pi}{3}(x + 1) = \frac{1}{4}$$

$$\frac{\pi}{3}(x + 1) = 1.32 \text{ or } \frac{\pi}{3}(x + 1) = 4.97$$

$$x = 0.26 \text{ or } x = 3.75$$

$$1. S = \{0.26; 3.75\}$$

$$2. S = \{0.26 + 6n\} \cup \{3.75 + 6n\}$$

c) $f(x) = \cos 2(x - \pi) + 1$

$$\cos 2(x - \pi) = -1$$

$$2(x - \pi) = \pi$$

$$x = \frac{3\pi}{2}$$

$$1. S = \left\{ \frac{3\pi}{2} \right\}$$

$$2. S = \left\{ \frac{3\pi}{2} + \pi n \right\}$$

d) $f(x) = 2 \cos \left(x + \frac{\pi}{2} \right) - \sqrt{3}$

$$\cos \left(x + \frac{\pi}{2} \right) = \frac{\sqrt{3}}{2}$$

$$x + \frac{\pi}{2} = \frac{\pi}{6} \text{ or } x + \frac{\pi}{2} = \frac{11\pi}{6}$$

$$x = -\frac{\pi}{3} \text{ or } x = \frac{4\pi}{3}$$

$$1. S = \left\{ -\frac{\pi}{3}, \frac{4\pi}{3} \right\}$$

$$2. S = \left\{ -\frac{\pi}{3} + 2\pi n \right\} \cup \left\{ \frac{4\pi}{3} + 2\pi n \right\}$$

e) $f(x) = -3 \cos \frac{\pi}{4}x + 6$

$$\cos \frac{\pi}{4}x = 2$$

$$1. S = \emptyset$$

$$2. S = \emptyset$$

f) $f(x) = -2 \cos \frac{\pi}{8}(x + 2) - \sqrt{2}$

$$\cos \frac{\pi}{8}(x + 2) = -\frac{\sqrt{2}}{2}$$

$$\frac{\pi}{8}(x + 2) = \frac{3\pi}{4} \text{ or } \frac{\pi}{8}(x + 2) = \frac{5\pi}{4}$$

$$x = 4 \text{ or } x = 8$$

$$1. S = \{4, 8\}$$

$$2. S = \{4 + 16n\} \cup \{8 + 16n\}$$

8. Determine the zeros of the function $f(x) = -2 \cos \frac{\pi}{12}(x + 5) - 1$ over the interval $[66, 126]$.

$$\cos \frac{\pi}{12}(x + 5) = -\frac{1}{2}$$

$$\frac{\pi}{12}(x + 5) = \frac{2\pi}{3} \text{ or } \frac{\pi}{12}(x + 5) = \frac{4\pi}{3}$$

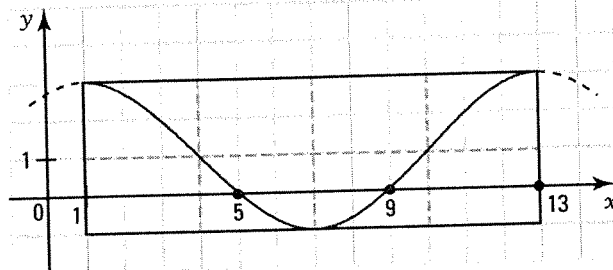
$$x = 3 \text{ or } x = 11$$

The zeros over the interval $[66, 126]$ are 75, 83, 99, 107, 123.

3 + period $\frac{2\pi}{\pi/12} = 24$
 $3 + 24 + 24 + 24$

ACTIVITY 5 Study of the function $f(x) = a \cos b(x - h) + k$

One cycle of the function $f(x) = 2 \cos \frac{\pi}{6}(x - 1) + 1$ is represented below.



a) Determine

1. the period. 12 2. the amplitude. 2

b) Determine

1. dom f . \mathbb{R} 2. ran f . $[-1, 3]$

3. the zeros of f over $[1, 13]$. 5 and 9

4. the zeros of f over \mathbb{R} . $\{5 + 12n\} \cup \{9 + 12n\}$

5. the sign of f over $[1, 13]$. $f(x) \geq 0$ over $[1, 5] \cup [9, 13]$ et $f(x) \leq 0$ over $[5, 9]$

6. the sign of f over \mathbb{R} . $f(x) \geq 0$ over $[1 + 12n, 5 + 12n] \cup [9 + 12n, 13 + 12n]$

7. the variation of f over $[1, 13]$. $f \searrow$ over $[1, 7]$ et $f \nearrow$ over $[7, 13]$

8. the variation of f over \mathbb{R} . $f \searrow$ over $[1 + 12n, 7 + 12n]$ et $f \nearrow$ over $[7 + 12n, 13 + 12n]$

9. the maximum and minimum of f . $\max f = 3; \min f = -1$

STUDY OF THE FUNCTION $f(x) = a \cos b(x - h) + k$

Given $f(x) = -2 \cos 2\left(x - \frac{\pi}{6}\right) + 1$. We have:

- dom $f = \mathbb{R}$; ran $f = [-1, 3]$
- period $p = \pi$; amplitude $A = 2$

- zeros of f : $\left\{\frac{\pi}{3} + \pi n\right\} \cup \left\{\pi + \pi n\right\}$

- sign of f .

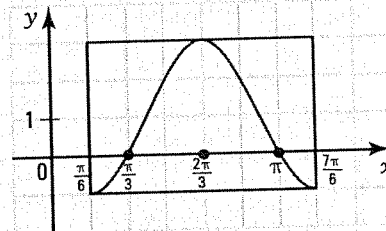
$$f(x) \leq 0 \text{ over } \left[\frac{\pi}{6} + \pi n, \frac{\pi}{3} + \pi n\right] \cup \left[\pi + \pi n, \frac{7\pi}{6} + \pi n\right]$$

$$f(x) \geq 0 \text{ over } \left[\frac{\pi}{3} + \pi n, \pi + \pi n\right]$$

- variation of f .

$$f \nearrow \text{ over } \left[\frac{\pi}{6} + \pi n, \frac{2\pi}{3} + \pi n\right]; f \searrow \text{ over } \left[\frac{2\pi}{3} + \pi n, \frac{7\pi}{6} + \pi n\right]$$

- min $f = -1$; max $f = 3$.



9. For each of the following functions, determine

1. the period,

a) $f(x) = 2 \cos \frac{3\pi}{4}(x - 1) + 5$

1. $p = \frac{8}{3}$

2. $A = 2$

3. $\text{ima } f = [3, 7]$

2. the amplitude,

3. the range of the function.

b) $f(x) = 5 \cos 4(x + 2) - 12$

1. $p = \frac{\pi}{2}$

2. $A = 5$

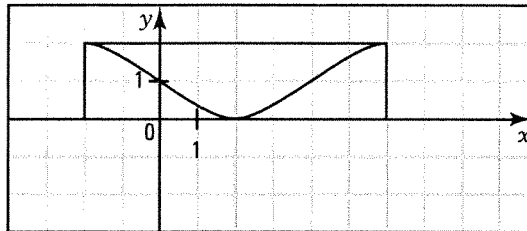
3. $\text{ima } f = [-17, -7]$

10. Determine the initial value of the following functions.

a) $f(x) = 2 \cos \frac{\pi}{3}(x+1) + 1$ 2 b) $f(x) = 5 \cos 2\left(x - \frac{\pi}{2}\right) + 1$ -4

11. Determine, over \mathbb{R} , the interval over which the function $f(x) = \cos \frac{\pi}{4}(x+2) + 1$ is increasing.

$f \nearrow$ over $[2 + 8n, 6 + 8n]$



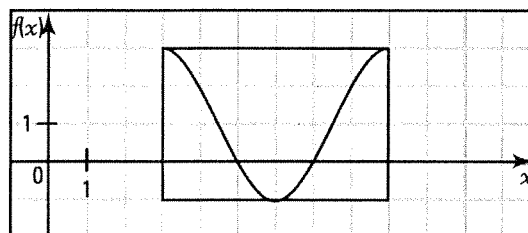
12. Determine, over \mathbb{R} , the interval over which the function $f(x) = 2 \cos \frac{\pi}{3}(x-3) + 1$ is negative.

Zeros: $\cos \frac{\pi}{3}(x-3) = -\frac{1}{2}$

$\frac{\pi}{3}(x-3) = \frac{2\pi}{3}$ or $\frac{\pi}{3}(x-3) = \frac{4\pi}{3}$

$x = 5$ or $x = 7$

$f(x) \leq 0$ over $[5 + 6n, 7 + 6n]$



13. On a boat, a sailor observes the movement of the waves while on a sailing expedition. The height of a wave (in m) can be expressed as a function of the time (in s) since the start of the observation by the rule:

$$h(t) = 2 \cos \frac{\pi}{6}(t-4) + 1$$

The sailor observes a wave for 30 seconds. Determine at which moments the wave will be at a height of 2 metres.

$2 \cos \frac{\pi}{6}(t-4) + 1 = 2$

$\cos \frac{\pi}{6}(t-4) = \frac{1}{2}$

$\frac{\pi}{6}(t-4) = \frac{\pi}{3}$ or $\frac{\pi}{6}(t-4) = \frac{5\pi}{3}$ *The wave will be at a height of 2 m at the moments $t = 2$ s, $t = 6$ s, $t = 14$ s, $t = 18$ s and $t = 26$ s.*

$t = 6$ or $t = 14$

ACTIVITY 6 Finding the rule $y = a \cos b(x-h) + k$

Consider the function f represented on the right.

a) Determine

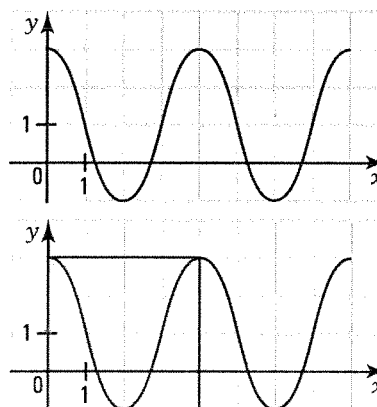
1. the period of f . 4 2. the amplitude of f . 2

b) Determine the rule of the function f when we choose a cycle starting at the point $(h, k+A) = (0, 3)$.

Explain your procedure: Since the cycle is decreasing from the start, we deduce that $a > 0$. Thus $a = 2$ et $k = 1$

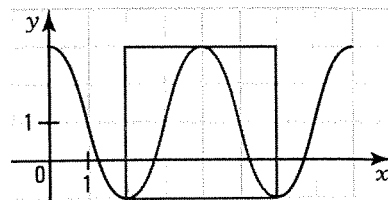
$p = 4 \Rightarrow |b| = \frac{\pi}{2}$.

Thus, $f(x) = 2 \cos \frac{\pi}{2}x + 1$ or $f(x) = 2 \cos -\frac{\pi}{2}x + 1$



- c) Determine the rule of the function f when we choose a cycle starting at the point $(h, k - A) = (2, -1)$.

$$y = -2 \cos \frac{\pi}{2}(x - 2) + 1$$

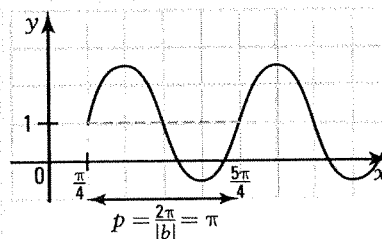


- d) Write the rule of the function in the form $y = a \sin b(x - h) + k$.

For example, $y = -2 \sin \frac{\pi}{2}(x - 1) + 1$

FINDING THE RULE $y = a \cos b(x - h) + k$

From the graph of the function on the right, we deduce that: $|a| = 1.5$, $|b| = 2$ and $k = 1$.
The writing of the function's rule depends on your choice of the starting point of one cycle.



Consider two cases:

1st case: starting point: $(h, k + A) = (\frac{\pi}{2}, 2.5)$

The cycle is decreasing from the start $\Rightarrow a < 0$.

Thus, $a = -1.5$ and $b = 2$ or $a = -1.5$ and $b = -2$.

The rule is written in two ways:

$$y = 1.5 \cos 2\left(x - \frac{\pi}{2}\right) + 1 \text{ or } y = 1.5 \cos -2\left(x - \frac{\pi}{2}\right) + 1.$$

2nd case: starting point: $(h, k - A) = (\pi, -0.5)$.

The cycle is increasing from the start $\Rightarrow a > 0$.

Thus, $a = 1.5$ and $b = 2$ or $a = 1.5$ and $b = -2$.

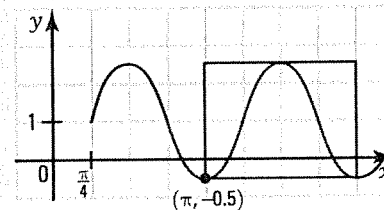
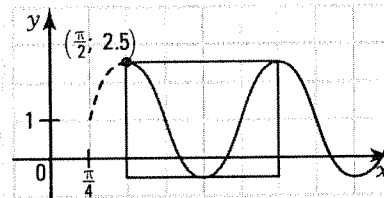
The rule is written in two ways:

$$y = -1.5 \cos 2(x - \pi) + 1 \text{ or } y = -1.5 \cos -2(x - \pi) + 1.$$

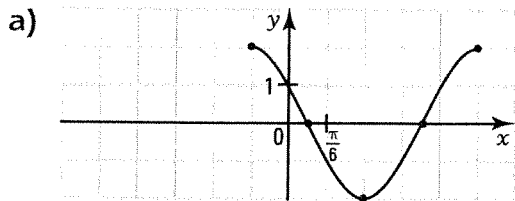
Note that this function can be described by a rule written in the form $y = a \sin b(x - h) + k$.

For example, $y = 1.5 \sin 2\left(x - \frac{\pi}{4}\right) + 1$.

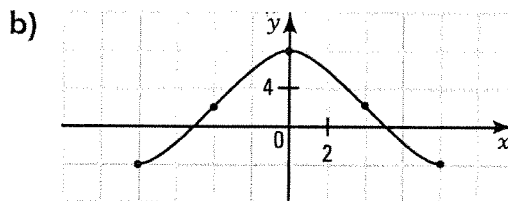
Thus, any sinusoidal function can be described by a rule written in the form $y = a \sin b(x - h) + k$ or $y = a \cos b(x - h) + k$.



14. Find a rule of the form $y = a \cos b(x - h) + k$ for each of the following functions.



For example, $y = 2 \cos 2\left(x + \frac{\pi}{6}\right)$



For example, $y = -6 \cos \frac{\pi}{8}(x + 8) + 2$

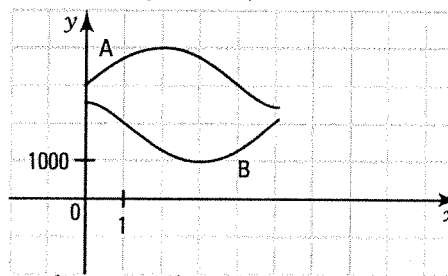
15. The populations of two neighboring villages A and B vary according to the model of a sinusoidal function which gives the population P of the village as a function of time t , in years, since the year 2000.

In the year 2000, the two villages had 3000 and 2500 inhabitants respectively.

The graph on the right shows the progression of the population of each village.

Village A reaches its maximum population of 4125 after 2 years and village B reaches its minimal population of 1000 after 3 years.

What will be the difference in population between these two villages in the year 2005?



Rule corresponding to village A.

Rule corresponding to village B.

$(h, k) = (0, 3000)$

$(h, k + A) = (0, 2500)$

$A = 1125$

$A = 750$

$p = 8 \Rightarrow b = \frac{\pi}{4}$

$p = 6 \Rightarrow b = \frac{\pi}{3}$

$y = 1125 \sin \frac{\pi}{4}x + 3000$

$y = 750 \cos \frac{\pi}{3}x + 1750$

In 2005, $y = 2205$ inhabitants

In 2005, $y = 2125$ inhabitants

The difference in their populations will be 80 inhabitants.

7.8 Tangent function

ACTIVITY 1 Basic tangent function

The basic tangent function, denoted \tan , has the rule $y = \tan t$.

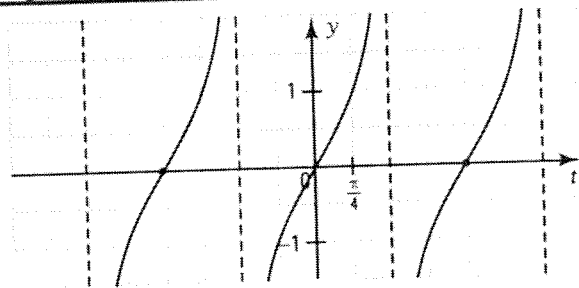
a) Complete the table of values below when t varies from $-\frac{3\pi}{2}$ to $\frac{3\pi}{2}$.

t	$-\frac{3\pi}{2}$	$-\frac{5\pi}{4}$	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
$\tan t$		-1	0	1		-1	0	1		-1	0	1	

b) Explain why $\tan t$ does not exist when $t = \frac{\pi}{2}$.
 $\tan \frac{\pi}{2} = \frac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2}} = \frac{1}{0}$. Division by zero is not defined.

c) We have represented the function $y = \tan t$ on the right. Is this function periodic? If yes, what is the period p of the function? $p = \pi$

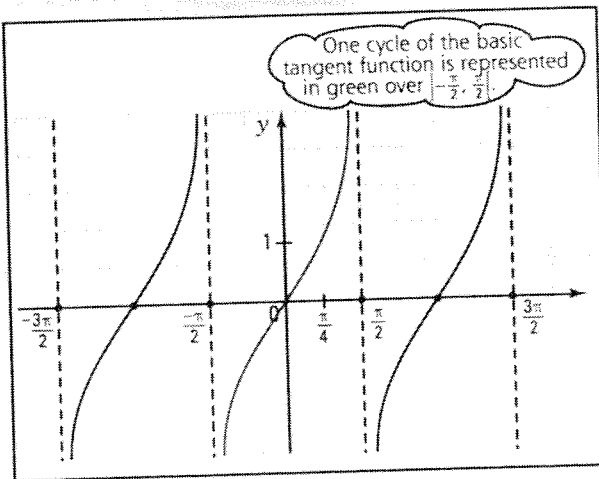
d) The graph of the tangent function has vertical asymptotes. What are the equations of the tangent function's asymptotes when $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$?
 $t = -\frac{\pi}{2}$ and $t = \frac{\pi}{2}$



- e) When $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, determine, for the basic tangent function,
- the domain. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
 - the range. \mathbb{R}
 - the zero. 0
 - the sign. $\tan t \leq 0$ if $-\frac{\pi}{2} < t \leq 0$ and $\tan t \geq 0$ if $0 \leq t < \frac{\pi}{2}$.
 - the variation. $\tan t$ is increasing if $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- f) Does the basic tangent function have any extrema? **No**

BASIC TANGENT FUNCTION

- The tangent function, denoted \tan , is defined by
 $\tan: \mathbb{R} \rightarrow \mathbb{R}$
 $x \mapsto y = \tan x$
- The basic tangent function is a periodic function with period π .
 $\tan(x + \pi) = \tan x$
- The tangent function has an infinite number of vertical asymptotes with equations:
 $x = \frac{\pi}{2} + \pi n$ ($n \in \mathbb{Z}$).
 Note that the distance separating two consecutive asymptotes is the period π .



- We have:
 - domain = $\mathbb{R} \setminus \left\{ \frac{\pi}{2} + \pi n \right\}$; range = \mathbb{R} .
 - zero over $\left| -\frac{\pi}{2}, \frac{\pi}{2} \right|$: 0.
 - sign over $\left| -\frac{\pi}{2}, \frac{\pi}{2} \right|$: $\tan x \leq 0$ if $-\frac{\pi}{2} < x \leq 0$ and $\tan x \geq 0$ if $0 \leq x < \frac{\pi}{2}$.
 - variation: $\tan x$ over $\left| -\frac{\pi}{2}, \frac{\pi}{2} \right|$.
 - extrema: none
- For any real x , we have: $\tan(-x) = -\tan x$. The basic tangent function is therefore considered to be an odd function.

1. Consider the function $y = \tan x$.

a) Find the equations of the asymptotes when

1. $x \in \left| -\frac{\pi}{2}, \frac{\pi}{2} \right|$. $x = \frac{-\pi}{2}$ and $x = \frac{\pi}{2}$

2. $x \in \mathbb{R}$. $x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$

b) Find the zeros of f when

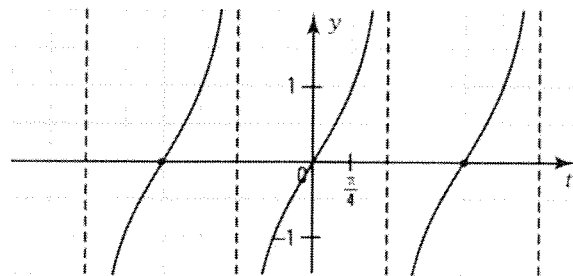
1. $x \in \left| -\frac{3\pi}{2}, \frac{3\pi}{2} \right|$. $-\pi, 0, \pi$

2. $x \in \mathbb{R}$. $\{n\pi\}, n \in \mathbb{Z}$

c) Solve the inequality $\tan x \geq 0$ when

1. $x \in \left| -\frac{3\pi}{2}, \frac{3\pi}{2} \right|$. $\left[-\pi, \frac{-\pi}{2} \right] \cup \left[0, \frac{\pi}{2} \right] \cup \left[\pi, \frac{3\pi}{2} \right]$

2. $x \in \mathbb{R}$. $\left[\pi n, \frac{\pi}{2} + \pi n \right]$



ACTIVITY 2 Equation $\tan \theta = k$

The function $y = \tan x$ is represented on the right when

$x \in \left| -\frac{3\pi}{2}, \frac{3\pi}{2} \right|$.

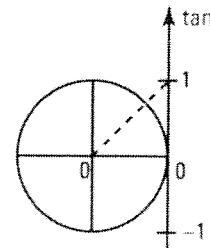
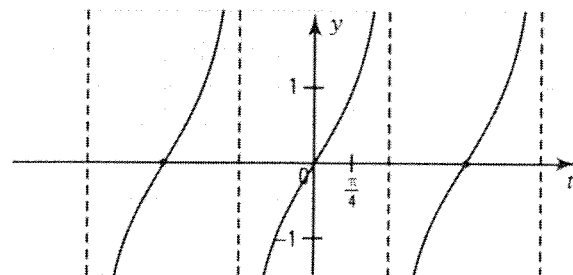
a) 1. By referring to the trigonometric circle, solve the equation $\tan \theta = 1$ when

$\theta \in \left| -\frac{\pi}{2}, \frac{\pi}{2} \right|$. $S = \left\{ \frac{\pi}{4} \right\}$

2. Explain how to find, from the solutions in 1, the solution to $\tan \theta = 1$ when

1) $\theta \in \left| \frac{\pi}{2}, \frac{3\pi}{2} \right|$. *Add the period π to the solution.*

2) $\theta \in \left| -\frac{3\pi}{2}, -\frac{\pi}{2} \right|$. *Subtract the period π from the solution.*



3. Verify that the solution set S to the equation $\tan \theta = 1$ over \mathbb{R} is described by

$$S = \left\{ \dots, -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \dots \right\} \text{ (enumeration).}$$

$$\text{or } S = \left\{ \frac{\pi}{4} + \pi n \right\} \text{ where } n \in \mathbb{Z} \text{ (set-builder notation).}$$

b) Using a calculator, solve the equation $\tan \theta = 2$ when

1. $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$. 1.11 2. $\theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2} \right]$. 7.39 3. $\theta \in \mathbb{R}$. 1.11 + πn

c) Solve the inequality $\tan \theta \geq 2$ when

1. $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$. 1.11; $\frac{\pi}{2}$

2. $\theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2} \right]$. 7.39; $\frac{5\pi}{2}$

3. $\theta \in \mathbb{R}$. 1.11 + πn ; $\frac{\pi}{2} + \pi n$

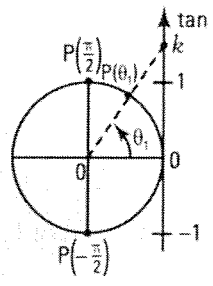
EQUATION $\tan \theta = k, k \in \mathbb{R}$

• When $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$, the equation $\tan \theta = k$ yields one unique solution

$$\theta_1 = \tan^{-1} k$$

• When $\theta \in \mathbb{R}$, the equation $\tan \theta = k$ yields an infinite number of solutions. The solution set S is defined in set-builder notation by

$$S = \{ \theta_1 + \pi n \} \text{ where } n \in \mathbb{Z}$$



Ex.: The equation $\tan \theta = \sqrt{3}$ has the solution set:

- $S = \left\{ \frac{\pi}{3} \right\}$ when $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$.

- $S = \left\{ \frac{\pi}{3} + \pi n \right\}$ when $\theta \in \mathbb{R}$.

2. Solve the following equations over

1. $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ 2. $\left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$ 3. \mathbb{R}

a) $\tan x = \frac{\sqrt{3}}{3}$ 1. $S = \left\{ \frac{\pi}{6} \right\}$ 2. $S = \left\{ \frac{7\pi}{6} \right\}$ 3. $S = \left\{ \frac{\pi}{6} + \pi n \right\}$

b) $\tan x = -1$ 1. $S = \left\{ -\frac{\pi}{4} \right\}$ 2. $S = \left\{ \frac{3\pi}{4} \right\}$ 3. $S = \left\{ -\frac{\pi}{4} + \pi n \right\}$

c) $\tan x = -\sqrt{3}$ 1. $S = \left\{ -\frac{\pi}{3} \right\}$ 2. $S = \left\{ \frac{2\pi}{3} \right\}$ 3. $S = \left\{ -\frac{\pi}{3} + \pi n \right\}$

d) $\tan x = 5$ 1. $S = \{1.37\}$ 2. $S = \{4.51\}$ 3. $S = \{1.37 + \pi n\}$

period = $\frac{\pi}{|b|}$

4. For each of the following functions, determine
 1. the period. 2. the equations of 2 consecutive asymptotes. 3. the domain.

a) $f(x) = 2 \tan \frac{\pi}{4}(x-1) + 1$

b) $f(x) = -3 \tan \frac{2\pi}{3}(x+2) - 5$

1. $p = 4$

$\frac{\pi}{\pi/4} = 4$

1. $p = \frac{3}{2}$

add period

2. $x = -1$ and $x = 3$

$\frac{\pi}{4}(x-1) = \frac{\pi}{2}$

2. $x = \frac{-11}{4}$ and $x = \frac{-5}{4}$

3. $\mathbb{R} \setminus \{3 + 4n\}$

$x-1 = 2$

3. $\mathbb{R} \setminus \left\{ \frac{-5}{4} + \frac{3}{2}n \right\}$

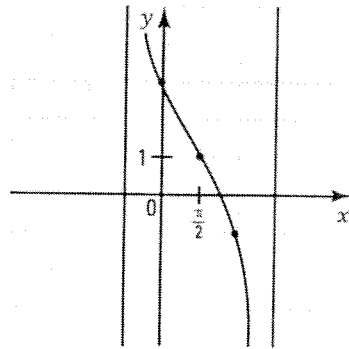
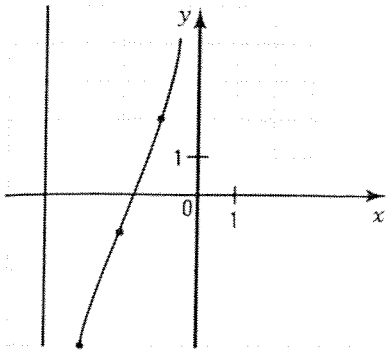
$x = 3$

$x = \frac{3}{2} - 1 = -\frac{1}{2}$

5. Draw one cycle of each of the following functions.

a) $f(x) = 3 \tan \frac{\pi}{4}(x+2) - 1$

b) $f(x) = -2 \tan \frac{1}{2}\left(x - \frac{\pi}{2}\right) + 1$



6. Determine the zeros of the following functions over \mathbb{R} .

a) $f(x) = -2 \tan \frac{\pi}{6}(x-1) + 2$

b) $f(x) = 3 \tan \frac{\pi}{3}(x+2) + \sqrt{3}$

$\tan \frac{\pi}{6}(x-1) = 1$

$\tan \frac{\pi}{3}(x+2) = -\frac{\sqrt{3}}{3}$

$\frac{\pi}{6}(x-1) = \frac{\pi}{4}$

$\frac{\pi}{3}(x+2) = -\frac{\pi}{6}$

$x = \frac{5}{2}$

$x = -\frac{5}{2}$

zeros: $\frac{5}{2} + 6n$

zeros: $-\frac{5}{2} + 3n$

7. Determine the initial value of the following functions.

a) $f(x) = 3 \tan \frac{\pi}{2}\left(x + \frac{1}{2}\right) - 1$

b) $f(x) = 3 \tan \frac{1}{2}\left(x - \frac{\pi}{2}\right) + 2$

2

-1

- Determine the interval over which the function $f(x) = 4 \tan \frac{\pi}{4}(x-1) + 4$ is positive.

zero: $\tan \frac{\pi}{4}(x-1) = -1$

$f(x) \geq 0$ over the interval $[0 + 4n, 3 + 4n[$.

$\frac{\pi}{4}(x-1) = -\frac{\pi}{4}$

$x = 0$

- Determine the variation of the function $f(x) = -2 \tan \frac{\pi}{6}(x+4) - 1$.

The function is decreasing over $\mathbb{R} \setminus \{-1 + 6n\}$ since $ab < 0$.

7.9 Trigonometric identities

ACTIVITY 1 Trigonometric identities

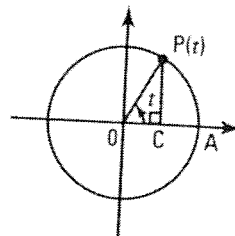
Consider the trigonometric point $P(t)$ and the right triangle OPC .

- a) Prove the identity $\sin^2 t + \cos^2 t = 1$.

Since $\overline{mOP} = 1$ (radius of the unit circle),

We have: $\sin t = \overline{mCP}$ and $\cos t = \overline{mOC}$.

Since $\overline{mCP}^2 + \overline{mOC}^2 = \overline{mOP}^2$ (Pythagorean theorem), we deduce that $\sin^2 t + \cos^2 t = 1$.



- b) Justify the steps proving the identity $\tan^2 t + 1 = \sec^2 t$.

Steps	Justifications
1. $\sin^2 t + \cos^2 t = 1$	See a)
2. $\frac{\sin^2 t + \cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$	Divide each side by $\cos^2 t$.
3. $\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$	Apply the property: $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.
4. $\tan^2 t + 1 = \sec^2 t$	$\tan t = \frac{\sin t}{\cos t}$ and $\sec t = \frac{1}{\cos t}$.

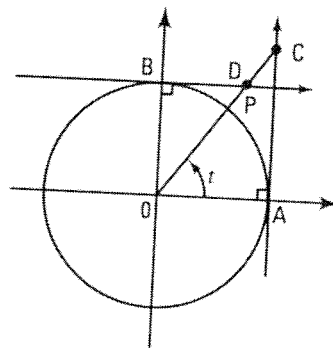
- c) Prove the identity: $1 + \cot^2 t = \csc^2 t$.

$$\sin^2 t + \cos^2 t = 1 \Rightarrow \frac{\sin^2 t + \cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \Rightarrow 1 + \cot^2 t = \csc^2 t.$$

- d) Consider the trigonometric circle on the right and the right triangle OAC .

Justify the steps showing that $\overline{mOC} = \sec t$.

Steps	Justifications
1. $\overline{mOC}^2 = \overline{mOA}^2 + \overline{mAC}^2$	Pythagorean theorem
2. $\overline{mOC}^2 = 1 + \tan^2 t$	$\overline{mOA} = 1$ and $\overline{mAC} = \tan t$
3. $\overline{mOC}^2 = \sec^2 t$	$1 + \tan^2 t = \sec^2 t$ (See b)
4. $\overline{mOC} = \sec t$	Consequence



- e) Refer to the trigonometric circle and right triangle OBD above to show that $\overline{mOD} = \csc t$.

$$\overline{mOD}^2 = \overline{mOB}^2 + \overline{mBD}^2 \quad (\text{Pythagorean theorem}).$$

$$\overline{mOD}^2 = 1^2 + \cot^2 t \quad (\overline{mOB} = 1 \text{ and } \overline{mBD} = \cot t)$$

$$\overline{mOD}^2 = \csc^2 t \quad (\text{See c})$$

$$\overline{mOD} = \csc t$$

3. Solve the following inequalities over

1. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

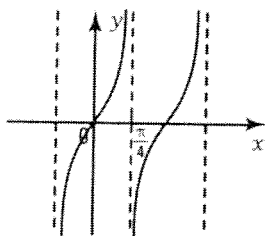
2. \mathbb{R}

a) $\tan x \geq \sqrt{3}$	1. $S = \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$	2. $S = \left[\frac{\pi}{3} + \pi n, \frac{\pi}{2} + \pi n\right]$
b) $\tan x \leq \frac{\sqrt{3}}{3}$	1. $S = \left[-\frac{\pi}{2}, \frac{\pi}{6}\right]$	2. $S = \left[-\frac{\pi}{2} + \pi n, \frac{\pi}{6} + \pi n\right]$
c) $\tan x \leq -1$	1. $S = \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]$	2. $S = \left[-\frac{\pi}{2} + \pi n, -\frac{\pi}{4} + \pi n\right]$
d) $\tan x \geq -\sqrt{3}$	1. $S = \left[-\frac{\pi}{3}, \frac{\pi}{2}\right]$	2. $S = \left[-\frac{\pi}{3} + \pi n, \frac{\pi}{2} + \pi n\right]$

ACTIVITY 3 Function $y = \tan bx$

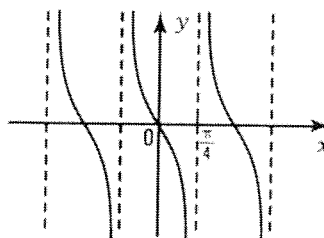
- a) For each of the following functions,
 1. determine the value of parameter b .
 2. find the period p .

a) $y = \tan 2x$



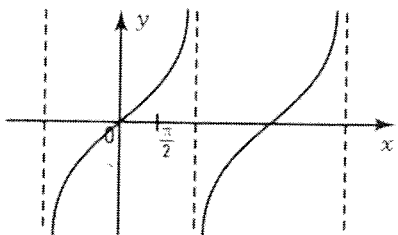
1. $b = 2$ 2. $p = \frac{\pi}{2}$

b) $y = \tan(-2x)$



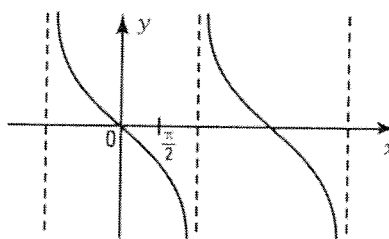
1. $b = -2$ 2. $p = \frac{\pi}{2}$

c) $y = \tan \frac{1}{2}x$



1. $b = \frac{1}{2}$ 2. $p = 2\pi$

d) $y = \tan\left(-\frac{1}{2}x\right)$



1. $b = -\frac{1}{2}$ 2. $p = 2\pi$

- b) What is the period p of the function $y = \tan bx$? $p = \frac{\pi}{|b|}$

ACTIVITY 4 Tangent function $y = a \tan b(x - h) + k$

a) Consider the function $y = 2 \tan \frac{1}{2}(x - \frac{\pi}{2}) + 1$.

$$a = 2, b = \frac{1}{2}, h = \frac{\pi}{2}, k = 1$$

1. Identify the parameters a, b, h and k .

$$p = \frac{\pi}{|b|} = 2\pi$$

2. What is the period p of the function?

b) To determine the equation of an asymptote, solve the equation $b(x - h) = \frac{\pi}{2}$.

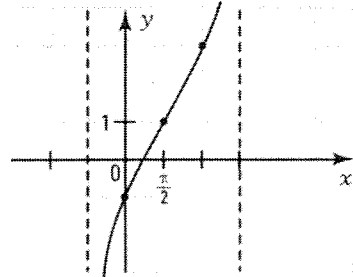
$$\frac{1}{2}(x - \frac{\pi}{2}) = \frac{\pi}{2}; \quad x = \frac{3\pi}{2}$$

1. Find the equation of an asymptote. **The period $p = 2\pi$.**

2. What distance separates two consecutive asymptotes?

c) In the Cartesian plane on the right, the point (h, k) has been located and 2 consecutive asymptotes are drawn. Complete the following table of values and draw one cycle of the function.

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
y		-1	1	3	



TANGENT FUNCTION $y = a \tan b(x - h) + k$

The graph of the function $y = a \tan b(x - h) + k$ is deduced from the graph of the basic function $y = \tan x$ by the transformation $(x, y) \rightarrow (\frac{x}{b} + h, ay + k)$.

- The function is periodic with period: $p = \frac{\pi}{|b|}$.

- The function has an infinite number of vertical asymptotes.

- To determine the equation of an asymptote, we solve the equation $b(x - h) = \frac{\pi}{2}$.

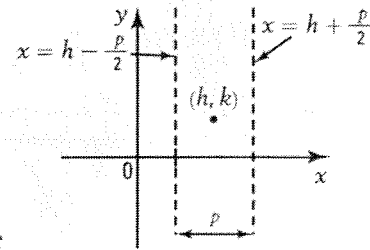
- Equation of an asymptote: $x = h - \frac{p}{2}$ or $x = h + \frac{p}{2}$.

- The distance separating two consecutive asymptotes is equal to the period p of the function.

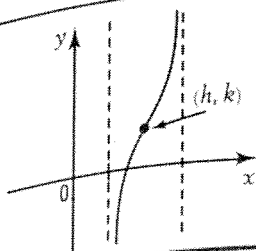
- To draw one cycle of the function $y = a \tan b(x - h) + k$,

- we locate the point (h, k) .
- we draw, equal distance from the point (h, k) , two consecutive asymptotes.

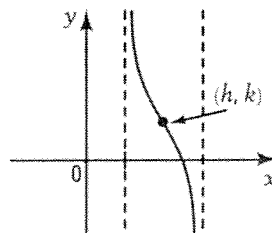
We have two possible situations depending on the signs of a and b .



$ab > 0$
Increasing function



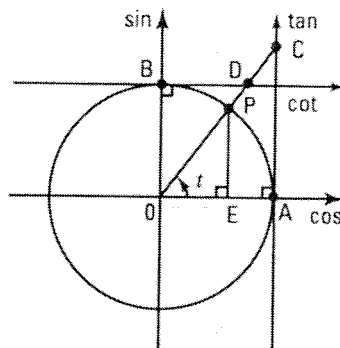
$ab < 0$
Decreasing function



BASIC TRIGONOMETRIC IDENTITIES

- If $P(t)$ is a trigonometric point, then
 - $m\overline{OE} = \cos t$, $m\overline{PE} = \sin t$.
 - $m\overline{AC} = \tan t$, $m\overline{BD} = \cot t$.
 - $m\overline{OC} = \sec t$, $m\overline{OD} = \csc t$.
- 1st basic identity:

$$\sin^2 t + \cos^2 t = 1$$



- Other basic identities:

$$1 + \tan^2 t = \sec^2 t$$

$$1 + \cot^2 t = \csc^2 t$$

1. Verify the three basic identities when $t = \frac{\pi}{6}$.

$$\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{6} = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$

$$1 + \tan^2 \frac{\pi}{6} = 1 + \left(\frac{\sqrt{3}}{3}\right)^2 = 1 + \frac{1}{3} = \frac{4}{3}; \quad \sec^2 \frac{\pi}{6} = \frac{1}{\cos^2 \frac{\pi}{6}} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

$$1 + \cot^2 \frac{\pi}{6} = 1 + \left(\frac{3}{\sqrt{3}}\right)^2 = 1 + 3 = 4; \quad \csc^2 \frac{\pi}{6} = \frac{1}{\sin^2 \frac{\pi}{6}} = \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{1}{4}} = 4.$$

2. Using an angle measure t of your choice, expressed in radians or in degrees, verify the three basic identities.

Various answers.

3. Use the appropriate basic identity to calculate

- a) $\sin t$, knowing that $\cos t = \frac{3}{5}$ and $270^\circ \leq t \leq 360^\circ$.

$$\sin^2 t = 1 - \cos^2 t = \frac{16}{25} \Rightarrow \sin t = \frac{-4}{5}$$

- b) $\cos t$, knowing that $\sin t = \frac{40}{41}$ and $90^\circ \leq t \leq 180^\circ$.

$$\cos^2 t = 1 - \sin^2 t = \frac{81}{1600} \Rightarrow \sec t = \frac{-9}{40}$$

- c) $\tan t$, knowing that $\sec t = \frac{-5}{4}$ and $180^\circ \leq t \leq 270^\circ$.

$$\tan^2 t = \sec^2 t - 1 = \frac{9}{16} \Rightarrow \tan t = \frac{3}{4}$$

- d) $\cotan t$, knowing that $\csc t = -\frac{13}{12}$ and $270^\circ \leq t \leq 360^\circ$.

$$\cot^2 t = \csc^2 t - 1 = \frac{25}{144} \Rightarrow \cot t = \frac{-5}{12}$$

- e) $\sec t$, knowing that $\tan t = \frac{7}{24}$ and $180^\circ \leq t \leq 270^\circ$.

$$\sec^2 t = 1 + \tan^2 t = \frac{625}{576} \Rightarrow \sec t = \frac{-25}{24}$$

- f) $\csc t$, knowing that $\cot t = \frac{4}{3}$ and $0^\circ \leq t \leq 90^\circ$.

$$\csc^2 t = 1 + \cot^2 t = \frac{25}{9} \Rightarrow \csc t = \frac{5}{3}$$

4. Reduce the following expressions* to a single term.

- | | | | |
|------------------------------|------------|-------------------------------------|------------|
| a) $1 - \sin^2 t$ | $\cos^2 t$ | b) $\sec^2 t - \tan^2 t$ | 1 |
| c) $\cot^2 t - \csc^2 t$ | -1 | d) $\sin t \sec t$ | $\tan t$ |
| e) $\tan x \cdot \csc x$ | $\sec x$ | f) $(1 - \sin^2 x) \sec^2 x$ | 1 |
| g) $(1 + \tan^2 x) \sin^2 x$ | $\tan^2 x$ | h) $\csc^2 x(1 - \cos^2 x)$ | 1 |
| i) $(\sec^2 x - 1) \cot^2 x$ | 1 | j) $\csc^2 x - \cot^2 x - \sin^2 x$ | $\cos^2 x$ |

5. Express each of the following trigonometric ratios in terms of $\sin x$ knowing that $0 \leq x \leq \frac{\pi}{2}$.

- | | | | |
|-------------|--------------------------------------|-------------|--------------------------------------|
| a) $\cos x$ | $\sqrt{1 - \sin^2 x}$ | b) $\tan x$ | $\frac{\sin x}{\sqrt{1 - \sin^2 x}}$ |
| c) $\cot x$ | $\frac{\sqrt{1 - \sin^2 x}}{\sin x}$ | d) $\sec x$ | $\frac{1}{\sqrt{1 - \sin^2 x}}$ |

6. If $\sin t = 0.6$ and $\frac{\pi}{2} \leq t \leq \pi$, deduce the other 5 trigonometric ratios.

$$\cos t = -0.8, \tan t = -\frac{3}{4}, \cot t = -\frac{4}{3}, \sec t = -\frac{5}{4}, \csc t = \frac{5}{3}.$$

7. If $\cos t = \frac{12}{13}$ and $\frac{3\pi}{2} \leq t \leq 2\pi$, deduce the other 5 trigonometric ratios.

$$\sin t = -\frac{5}{13}, \tan t = -\frac{5}{12}, \cot t = -\frac{12}{5}, \sec t = \frac{13}{12}, \csc t = -\frac{13}{5}.$$

8. If $\tan t = \frac{3}{4}$ and $0 \leq t \leq \frac{\pi}{2}$, deduce the other 5 trigonometric ratios.

$$\sec t = \frac{5}{4}, \cos t = \frac{4}{5}, \sin t = \frac{3}{5}, \cot t = \frac{4}{3}, \csc t = \frac{5}{3}.$$

9. If $\cot t = \frac{-5}{12}$ and $\frac{3\pi}{2} \leq t \leq 2\pi$, deduce the other 5 trigonometric ratios.

$$\csc t = -\frac{13}{12}, \sin t = -\frac{12}{13}, \cos t = \frac{5}{13}, \tan t = -\frac{12}{5}, \sec t = \frac{13}{5}.$$

10. Simplify the following expressions.

- | | | | |
|---|------------|--|------------|
| a) $\frac{\sin^2 x + \cos^2 x}{1 - \cos^2 x}$ | $\csc^2 x$ | b) $\frac{1 + \tan^2 x}{1 + \cot^2 x}$ | $\tan^2 x$ |
| c) $\frac{\sec^2 x - \tan^2 x}{1 - \sin^2 x}$ | $\sec^2 x$ | d) $\frac{\sec^2 x - 1}{\csc^2 x} \cdot \frac{\cot^2 x}{\sin^2 x}$ | 1 |

11. Simplify the following expressions.

- | | | | |
|--|-------------|---|------------|
| a) $\frac{1 - \cos^2 x}{1 - \sin^2 x}$ | $\tan^2 x$ | b) $\frac{1 + \tan^2 x}{1 + \cot^2 x}$ | $\tan^2 x$ |
| c) $(1 + \sec x)(1 - \sec x)$ | $-\tan^2 x$ | d) $\frac{\csc^2 x - \cot^2 x}{\cos^2 x}$ | $\sec^2 x$ |

12. Perform the following operations.

- | | |
|--|---------------------------------|
| a) $\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x}$ | $= \frac{2 \sec^2 x}{\cos^2 x}$ |
| b) $\frac{\cos x}{\sec x + 1} + \frac{\cos x}{\sec x - 1}$ | $= \frac{2}{\tan^2 x}$ |

ACTIVITY 2 Proving a trigonometric identity

Justify the steps which prove the identity: $\sin x + \cos x \cot x = \csc x$ according to the following procedures.

1st procedure: Transform the left side using algebraic manipulations to make it identical to the right side.

Steps	Justifications
$\sin x + \cos x \cot x = \csc x$	
$1. \sin x + \cos x \cdot \frac{\cos x}{\sin x} = \csc x$	$\cot x = \frac{\cos x}{\sin x}$
$2. \frac{\sin^2 x + \cos^2 x}{\sin x} = \csc x$	Reducing to a common denominator.
$3. \frac{1}{\sin x} = \csc x$	$\sin^2 x + \cos^2 x = 1$
$4. \csc x = \csc x$	$\frac{1}{\sin x} = \csc x$

2nd procedure: Transform the right side using algebraic manipulations to make it identical to the left side.

Steps	Justifications
$\sin x + \cos x \cot x = \csc x$	
$1. \sin x + \cos x \cot x = \frac{1}{\sin x}$	$\csc x = \frac{1}{\sin x}$
$2. \sin x + \cos x \cot x = \frac{\sin^2 x + \cos^2 x}{\sin x}$	$1 = \sin^2 x + \cos^2 x$
$3. \sin x + \cos x \cot x = \sin x + \frac{\cos^2 x}{\sin x}$	Divide each term by $\sin x$.
$4. \sin x + \cos x \cot x = \sin x + \cos x \cot x$	$\frac{\cos^2 x}{\sin x} = \cos x \cdot \frac{\cos x}{\sin x} \text{ and } \frac{\cos x}{\sin x} = \cot x$

PROVING A TRIGONOMETRIC IDENTITY

To prove an identity, we simplify one side (usually the more complex side) to make it identical to the other side. The algebraic manipulations used consist of

- substituting expressions by other known identities.
- using the definitions of the trigonometric ratios.
- multiplying or dividing by the same trigonometric expression.
- factoring.
- reducing to a common denominator.

...

Ex.: See activity 2.

13. Prove the following trigonometric identities.

a) $\cot^2 x \sin^2 x + \sin^2 x = 1$

$$\cot^2 x \sin^2 x + \sin^2 x = \frac{\cos^2 x}{\sin^2 x} \cdot \sin^2 x + \sin^2 x = \cos^2 x + \sin^2 x = 1$$

b) $\frac{1}{\csc^2 x} + \frac{1}{\sec^2 x} = 1$

$$\frac{1}{\csc^2 x} + \frac{1}{\sec^2 x} = \sin^2 x + \cos^2 x = 1$$

c) $\frac{1 + \tan^2 x}{\cot^2 x + 1} = \tan^2 x$

$$\frac{1 + \tan^2 x}{\cot^2 x + 1} = \frac{\sec^2 x}{\csc^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x$$

d) $\frac{\cos^2 x \tan x}{\cot x} - 1 = -\cos^2 x$

$$\frac{\cos^2 x \tan x}{\cot x} - 1 = \frac{\cos^2 x \cdot \sin x \cdot \sin x}{\cos x \cdot \cos x} - 1 = \sin^2 x - 1 = \sin^2 x - (\sin^2 x + \cos^2 x) = -\cos^2 x$$

e) $\frac{\sec x}{\cos x} - 1 = \tan^2 x$

$$\frac{\sec x}{\cos x} - 1 = \frac{1}{\cos^2 x} - 1 = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x$$

f) $(1 + \cot^2 x)(1 - \cos^2 x) = 1$

$$(1 + \cot^2 x)(1 - \cos^2 x) = \csc^2 x \cdot \sin^2 x = \frac{\sin^2 x}{\sin^2 x} = 1$$

g) $2 \cos^2 a - 1 = 1 - 2 \sin^2 a$

$$2 \cos^2 a - 1 = 2(1 - \sin^2 a) - 1 = 2 - 2 \sin^2 a - 1 = 1 - 2 \sin^2 a$$

h) $\tan x + \cot x = \sec x \csc x$

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} = \frac{1}{\cos x \sin x} = \frac{1}{\cos x} \cdot \frac{1}{\sin x} = \sec x \csc x$$

14. Prove the following trigonometric identities.

a) $\sec x - \cos x = \sin x \tan x$

$$\sec x - \cos x = \sin x \cdot \frac{\sin x}{\cos x} = \frac{\sin^2 x}{\cos x} = \frac{1 - \cos^2 x}{\cos x} = \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} = \sec x - \cos x$$

b) $\frac{1 + \sin x}{\cos x} = \frac{\cos x}{1 - \sin x}$

$$\frac{1 + \sin x}{\cos x} = \frac{(1 + \sin x)(1 - \sin x)}{\cos x(1 - \sin x)} = \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} = \frac{\cos^2 x}{\cos x(1 - \sin x)} = \frac{\cos x}{1 - \sin x}$$

c) $\frac{\sin^2 x}{1 - \cos x} = 1 + \cos x$

$$\frac{\sin^2 x}{1 - \cos x} = \frac{1 - \cos^2 x}{1 - \cos x} = \frac{(1 + \cos x)(1 - \cos x)}{1 - \cos x} = 1 + \cos x$$

d) $(1 + \tan x)^2 + (1 - \tan x)^2 = 2 \sec^2 x$

$$(1 + \tan x)^2 + (1 - \tan x)^2 = 1 + 2 \tan x + \tan^2 x + 1 - 2 \tan x + \tan^2 x = 2(1 + \tan^2 x) = 2 \sec^2 x$$

e) $(1 + \tan x)(1 - \tan x) - (1 + \cot x)(1 - \cot x) = \frac{1 - \tan^4 x}{\tan^2 x}$

$$(1 + \tan x)(1 - \tan x) - (1 + \cot x)(1 - \cot x) = 1 - \tan^2 x - (1 - \cot^2 x)$$

$$= \cot^2 x - \tan^2 x = \frac{1}{\tan^2 x} - \tan^2 x = \frac{1 - \tan^4 x}{\tan^2 x}$$

7.10 Trigonometric equations

ACTIVITY 1 Solving a trigonometric equation

A marble is hanging at the end of a spring that oscillates over a table. The height h (in cm) of the marble, relative to the table, as a function of elapsed time t (in sec) since the start of the movement is given by $h = 10 \sin \pi t + 15$.

a) What is the period of the function that describes the movement of the spring? $p = 2 \text{ s}$

b) Justify the steps in solving the equation that enables you to calculate at what moments, during the first period of the movement, the marble is located 20 cm above the table.

1. $10 \sin \pi t + 15 = 20$

Set up the equation.

2. $10 \sin \pi t = 5$

Subtract 15 from each side.

3. $\sin \pi t = 0.5$

Divide each side by 10.

4. $\pi t = \frac{\pi}{6}$ or $\pi t = \frac{5\pi}{6}$

Deduce the 2 possible values of πt .

5. $t = \frac{1}{6}$ or $t = \frac{5}{6}$

Deduce the 2 possible values of t .

c) Keeping the periodicity of the movement in mind, determine the moments during the first 5 seconds when the marble is located 20 cm above the table.

$\frac{1}{6} \text{ s}, \frac{5}{6} \text{ s}, \frac{13}{6} \text{ s}, \frac{17}{6} \text{ s}, \frac{25}{6} \text{ s}, \frac{29}{6} \text{ s}.$

ACTIVITY 2 Solving more complex trigonometric equations

Consider the equation $(2 \sin \theta + 1)(\sin \theta + 2) = 0$ where $\theta \in [0, 2\pi[$.

a) Apply the zero product principle to determine the possible values of $\sin \theta$.

$(2 \sin \theta + 1)(\sin \theta + 2) = 0$

$\Leftrightarrow 2 \sin \theta + 1 = 0$ or $\sin \theta + 2 = 0$

$\Leftrightarrow \sin \theta = -\frac{1}{2}$ or $\sin \theta = -2$

b) Of the two values found for $\sin \theta$, indicate which one must be rejected and explain why.

$\sin \theta = -2$ is rejected since $\forall \theta, -1 \leq \sin \theta \leq 1$

c) What are the solutions to the equation?

$\sin \theta = -\frac{1}{2} \Leftrightarrow \theta = \frac{7\pi}{6}$ or $\theta = \frac{11\pi}{6}$.

TRIGONOMETRIC EQUATIONS

Let us illustrate the procedure for solving: $(2 \sin \theta - 1)(\sin \theta + 2) = 0$.

1	$(2 \sin \theta - 1)(\sin \theta + 2) = 0$	1	Apply the zero product principle.
2	$2 \sin \theta - 1 = 0$ or $\sin \theta + 2 = 0$ $\sin \theta = \frac{1}{2}$ or $\sin \theta = -2$ (reject)	2	Deduce the values for $\sin \theta$ and reject some, if necessary.
3	$\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$	8	Deduce the value of θ over $[0, p[$ where $p = 2\pi$ is the period of $\sin \theta$.
4	$S = \left\{ \frac{\pi}{6} + 2\pi n \right\} \cup \left\{ \frac{5\pi}{6} + 2\pi n \right\}, n \in \mathbb{Z}$	4	Deduce the solution set S over \mathbb{R} , from the resulting solutions and period.

1. Solve the following trigonometric equations over $[0, 2\pi[$.

a) $\cos x \cdot (2 \sin x + 1) = 0$ $S = \left\{ \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6} \right\}$ b) $(\sin x - 2)(2 \cos x - 1) = 0$ $S = \left\{ \frac{\pi}{3}, \frac{5\pi}{3} \right\}$

2. Solve the following trigonometric equations over \mathbb{R} .

a) $(2 \sin x + 1)(\sin x - 1) = 0$ $S = \left\{ \frac{5\pi}{6} + 2\pi n \right\} \cup \left\{ \frac{7\pi}{6} + 2\pi n \right\} \cup \left\{ \frac{\pi}{2} + 2\pi n \right\}$

b) $(3 \sin x - 1) \cdot \cos x = 0$ $S = \{0, 34 + 2\pi n\} \cup \{2, 8 + 2\pi n\} \cup \left\{ \frac{\pi}{2} + 2\pi n \right\}$

3. Solve the following trigonometric equations over $[0, 2\pi[$.

a) $(2 \cos \theta + 1)(2 \cos \theta - 1) = 0$
 $S = \left\{ \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \right\}$

b) $(2 \sin \theta + 1)(\sin \theta - 1) = 0$
 $S = \left\{ \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6} \right\}$

c) $\tan^2 \theta - 3 = 0$
 $S = \left\{ \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \right\}$

d) $2 \sin^2 2x - \sin 2x = 0$
 $S = \left\{ 0, \frac{\pi}{12}, \frac{5\pi}{12}, \pi, \frac{13\pi}{12}, \frac{17\pi}{12} \right\}$

e) $(2 \cos \theta - 1)(\cos \theta + 1) = 0$
 $S = \left\{ \frac{\pi}{3}, \pi, \frac{5\pi}{3} \right\}$

f) $(\sec \theta + 2)(\sec \theta - 1) = 0$
 $S = \left\{ 0, \frac{2\pi}{3}, \frac{4\pi}{3} \right\}$

Evaluation 7

1. a) Convert $\frac{5\pi}{36}$ into degrees. 25°

b) Convert 75° into radians. $\frac{5\pi}{12}$ rd

2. A central angle measuring 54° sub-tends an arc on the circle.
If the circle has a radius of 12 cm, calculate the length of the sub-tended arc (round to the nearest tenth). 11.3 cm

3. What are the coordinates of the trigonometric point $P(t)$ located in the 3rd quadrant if $\cos t = -\frac{15}{17}$? $P(t) = \left(-\frac{15}{17}, -\frac{8}{17}\right)$

4. Find the coordinates of the point $P(t)$, located in the 3rd quadrant, if $\tan t = \frac{8}{15}$?
 $\sec^2 t = 1 + \tan^2 t = \frac{289}{225}$; $\sec t = \frac{-17}{15}$; $\cos t = \frac{-15}{17}$; $\sin t = \frac{-8}{17} \Rightarrow P(t) = \left(-\frac{15}{17}, -\frac{8}{17}\right)$

5. Find the exact Cartesian coordinates of the trigonometric point $P\left(-\frac{19\pi}{6}\right)$.
 $P\left(-\frac{19\pi}{6}\right) = P\left(\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

6. If $P\left(\frac{a+2}{5}, \frac{a+1}{5}\right)$ is a trigonometric point located in the 3rd quadrant, determine a .

$\left(\frac{a+2}{5}\right)^2 + \left(\frac{a+1}{5}\right)^2 = 1$; $2a^2 + 6a - 20 = 0$; $a = -5$ or $a = 2$.

Reject the solution $a = 2$ since the trigonometric point is located in the 3rd quadrant.

7. If $\sin t = \frac{15}{17}$ and $\frac{\pi}{2} \leq t \leq \pi$, find

a) $\cos t$ $\frac{-8}{17}$ b) $\tan t$ $\frac{-15}{8}$ c) $\cotan t$ $\frac{-8}{15}$ d) $\sec t$ $\frac{-17}{8}$ e) $\csc t$ $\frac{17}{15}$

8. The function f is given by the rule $f(x) = -4 \cos \frac{\pi}{6}(x - 2) + 2$. Determine

a) the amplitude. 4 b) the period of f . 12 c) $\text{ran } f$. $[-2, 6]$

d) the zeros of f . $\{4 + 12n\} \cup \{12 + 12n\}$

e) the sign of f over $[2, 14]$. $f(x) < 0 \Leftrightarrow x \in [2, 4[\cup]12, 14]$, $f(x) > 0 \Leftrightarrow x \in]4, 12[$

9. Determine the zeros of the following trigonometric functions over \mathbb{R} .

a) $f(x) = 2 \sin \frac{\pi}{2}(x - 3) - 1$ $\left\{\frac{10}{3} + 4n\right\} \cup \left\{\frac{14}{3} + 4n\right\}$

b) $f(x) = -6 \cos \frac{\pi}{3}(x + 1) + 3$ $\{0 + 6n\} \cup \{4 + 6n\}$

c) $f(x) = 2 \tan \frac{\pi}{4}(x - 1) + 2$ $\{4 + 8n\} \cup \{8 + 8n\}$

d) $f(x) = 5 \sin \frac{2\pi}{3}(x + 1) + 2$ $\{-1.2 + 3n\} \cup \{0.70 + 3n\}$

e) $f(x) = -2 \cos 2(x + 1) + 1$ $\{-0.48 + \pi n\} \cup \{1.62 + \pi n\}$