

7.5 Solving Rational Equations

7.5 OBJECTIVES

1. Rewrite a rational equation by clearing the fractions
2. Solve an equation that contains a rational expression
3. Find the zeros of a rational function

Applications of your work in algebra will often result in equations involving rational expressions. Our objective in this section is to develop methods to find the solutions of such equations.

The usual technique is to multiply both sides of the equation by the lowest common denominator (LCD) of all the rational expressions appearing in the equation. The resulting equation will be cleared of fractions, and we can then proceed to solve the equation as before. Example 1 illustrates the process.

Example 1

Clearing Equations of Fractions

Solve.

$$\frac{2x}{3} + \frac{x}{5} = 13$$

NOTE The LCM for 3 and 5 is 15.

The LCD for $\frac{2x}{3}$ and $\frac{x}{5}$ is 15.

The LCM for 3 and 5 is 15. Multiplying both sides of the equation by 15, we have

$$15\left(\frac{2x}{3} + \frac{x}{5}\right) = 15 \cdot 13 \quad \text{Distribute 15 on the left.}$$

$$15 \cdot \frac{2x}{3} + 15 \cdot \frac{x}{5} = 15 \cdot 13$$

$$10x + 3x = 195 \quad \text{Simplify. The equation is now cleared of fractions.}$$

$$13x = 195$$

$$x = 15$$

To check, substitute 15 in the original equation.

$$\frac{2 \cdot 15}{3} + \frac{15}{5} \stackrel{?}{=} 13$$

$$10 + 3 \stackrel{?}{=} 13$$

$$13 = 13 \quad \text{A true statement.}$$

So 15 is the solution for the equation.

Be Careful! A common mistake is to confuse an *equation* such as

$$\frac{2x}{3} + \frac{x}{5} = 13$$

and an *expression* such as

$$\frac{2x}{3} + \frac{x}{5}$$



CAUTION

Let's compare.

Equation: $\frac{2x}{3} + \frac{x}{5} = 13$

Here we want to *solve the equation for x*, as in Example 1. We multiply both sides by the LCD to clear fractions and proceed as before.

Expression: $\frac{2x}{3} + \frac{x}{5}$

Here we want to find a *third fraction* that is equivalent to the given expression. We write each fraction as an equivalent fraction with the LCD as a common denominator.

$$\begin{aligned}\frac{2x}{3} + \frac{x}{5} &= \frac{2x \cdot 5}{3 \cdot 5} + \frac{x \cdot 3}{5 \cdot 3} \\ &= \frac{10x}{15} + \frac{3x}{15} = \frac{10x + 3x}{15} \\ &= \frac{13x}{15}\end{aligned}$$



CHECK YOURSELF 1

Solve.

$$\frac{3x}{2} - \frac{x}{3} = 7$$

The process is similar when variables are in the denominators. Consider Example 2.

Example 2

Solving an Equation Involving Rational Expressions

Solve.

$$\frac{7}{4x} - \frac{3}{x^2} = \frac{1}{2x^2}$$

NOTE We assume that x cannot have the value 0. Do you see why?

The LCM of $4x$, x^2 , and $2x^2$ is $4x^2$. So, the LCD for the equation is $4x^2$. Multiplying both sides by $4x^2$, we have

$$4x^2 \left(\frac{7}{4x} - \frac{3}{x^2} \right) = 4x^2 \cdot \frac{1}{2x^2} \quad \text{Distribute } 4x^2 \text{ on the left side.}$$

$$4x^2 \cdot \frac{7}{4x} - 4x^2 \cdot \frac{3}{x^2} = 4x^2 \cdot \frac{1}{2x^2} \quad \text{Simplify.}$$

$$7x - 12 = 2$$

$$7x = 14$$

$$x = 2$$

We leave the check of the solution, $x = 2$, to you. Be sure to return to the original equation and substitute 2 for x .

**CHECK YOURSELF 2**

Solve.

$$\frac{5}{2x} - \frac{4}{x^2} = \frac{7}{2x^2}$$

Example 3 illustrates the same solution process when there are binomials in the denominators.

Example 3**Solving an Equation Involving Rational Expressions**

Solve.

NOTE Here we assume that x cannot have the value -2 or 3 .

$$\frac{4}{x+2} + 3 = \frac{3x}{x-3}$$

The LCD is $(x+2)(x-3)$. Multiplying by that LCD, we have

NOTE Notice that multiplying each term by the LCD is the same as multiplying both sides of the equation by the LCD.

$$(x+2)(x-3)\left(\frac{4}{x+2}\right) + (x+2)(x-3)(3) = (x+2)(x-3)\left(\frac{3x}{x-3}\right)$$

Or, simplifying each term, we have

$$4(x-3) + 3(x+2)(x-3) = 3x(x+2)$$

We now clear the parentheses and proceed as before.

$$4x - 12 + 3x^2 - 3x - 18 = 3x^2 + 6x$$

$$3x^2 + x - 30 = 3x^2 + 6x$$

$$x - 30 = 6x$$

$$-5x = 30$$

$$x = -6$$

Again, we leave the check of this solution to you.

**CHECK YOURSELF 3**

Solve.

$$\frac{5}{x-4} + 2 = \frac{2x}{x-3}$$

Factoring plays an important role in solving equations containing rational expressions.

Example 4**Solving an Equation Involving Rational Expressions**

Solve.

$$\frac{3}{x-3} - \frac{7}{x+3} = \frac{2}{x^2-9}$$

In factored form, the denominator on the right side is $(x-3)(x+3)$, which forms the LCD, and we multiply each term by that LCD.

$$(x-3)(x+3)\left(\frac{3}{x-3}\right) - (x-3)(x+3)\left(\frac{7}{x+3}\right) = (x-3)(x+3)\left[\frac{2}{(x-3)(x+3)}\right]$$

Again, simplifying each term on the right and left sides, we have

$$3(x+3) - 7(x-3) = 2$$

$$3x + 9 - 7x + 21 = 2$$

$$-4x = -28$$

$$x = 7$$

Be sure to check this result by substitution in the original equation.

**CHECK YOURSELF 4**

$$\text{Solve } \frac{4}{x-4} - \frac{3}{x+1} = \frac{5}{x^2-3x-4}$$

Whenever we multiply both sides of an equation by an expression containing a variable, there is the possibility that a proposed solution may make that multiplier 0. As we pointed out earlier, multiplying by 0 does not give an equivalent equation, and therefore verifying solutions by substitution serves not only as a check of our work but also as a check for extraneous solutions. Consider Example 5.

Example 5**Solving an Equation Involving Rational Expressions**

Solve.

$$\frac{x}{x-2} - 7 = \frac{2}{x-2}$$

The LCD is $x-2$, and multiplying, we have

$$\left(\frac{x}{x-2}\right)(x-2) - 7(x-2) = \left(\frac{2}{x-2}\right)(x-2)$$

Simplifying yields

$$x - 7(x-2) = 2$$

$$x - 7x + 14 = 2$$

$$-6x = -12$$

$$x = 2$$

NOTE Notice that we must assume that $x \neq 2$.

NOTE Notice that each of the three terms gets multiplied by $(x-2)$.

**CAUTION**

Because division by 0 is undefined, we conclude that 2 is *not a solution* for the original equation. It is an extraneous solution. The original equation has no solution.

To check this result, by substituting 2 for x , we have

$$\frac{2}{2-2} - 7 \stackrel{?}{=} \frac{2}{2-2}$$

$$\frac{2}{0} - 7 \stackrel{?}{=} \frac{2}{0}$$

**CHECK YOURSELF 5**

Solve $\frac{x-3}{x-4} = 4 + \frac{1}{x-4}$.

Equations involving rational expressions may also lead to quadratic equations, as illustrated in Example 6.

Example 6**Solving an Equation Involving Rational Expressions**

Solve.

NOTE Assume $x \neq 3$ and $x \neq 4$.

$$\frac{x}{x-4} = \frac{15}{x-3} - \frac{2x}{x^2-7x+12}$$

After factoring the trinomial denominator on the right, the LCD of $x-3$, $x-4$, and $x^2-7x+12$ is $(x-3)(x-4)$. Multiplying by that LCD, we have

$$(x-3)(x-4)\left(\frac{x}{x-4}\right) = (x-3)(x-4)\left(\frac{15}{x-3}\right) - (x-3)(x-4)\left[\frac{2x}{(x-3)(x-4)}\right]$$

Simplifying yields

$$x(x-3) = 15(x-4) - 2x \quad \text{Remove the parentheses.}$$

$$x^2 - 3x = 15x - 60 - 2x \quad \text{Write in standard form and factor.}$$

$$x^2 - 16x + 60 = 0$$

$$(x-6)(x-10) = 0$$

So

$$x = 6 \quad \text{or} \quad x = 10$$

Verify that 6 and 10 are both solutions for the original equation.

**CHECK YOURSELF 6**

Solve $\frac{3x}{x+2} - \frac{2}{x+3} = \frac{36}{x^2+5x+6}$.

The following algorithm summarizes our work in solving equations containing rational expressions.

Step by Step: Solving Equations Containing Rational Expressions

- Step 1** Clear the equation of fractions by multiplying both sides of the equation by the LCD of all the fractions that appear.
- Step 2** Solve the equation resulting from step 1.
- Step 3** Check all solutions by substitution in the original equation.

The techniques we have just discussed can also be used to find the zeros of rational functions. Remember that a zero of a function is a value of x for which $f(x) = 0$.

Example 7

Finding the Zeros of a Function

Find the zeros of

$$f(x) = \frac{1}{x} - \frac{3}{7x} - \frac{4}{21}$$

Set the function equal to 0, and solve the resulting equation for x .

$$f(x) = \frac{1}{x} - \frac{3}{7x} - \frac{4}{21} = 0$$

The LCM for x , $7x$, and 21 is $21x$. Multiplying both sides by $21x$, we have

$$21x\left(\frac{1}{x} - \frac{3}{7x} - \frac{4}{21}\right) = 21x \cdot 0 \quad \text{Distribute } 21x \text{ on the left side.}$$

$$21 - 9 - 4x = 0 \quad \text{Simplify.}$$

$$12 - 4x = 0$$

$$12 = 4x$$

$$3 = x$$

So 3 is the value of x for which $f(x) = 0$, that is, 3 is a zero of $f(x)$.



CAUTION

This is different from finding values for which the rational function does not exist. In that case, we found x values that caused division by zero. Here, we set the function equal to zero and find the x values that make the statement true.



CHECK YOURSELF 7

Find the zeros of the function.

$$f(x) = \frac{5x + 2}{x - 6} - \frac{11}{4}$$

CHECK YOURSELF ANSWERS

1. $\{6\}$ 2. $\{3\}$ 3. $\{9\}$ 4. $\{-11\}$ 5. No solution 6. $\left\{-5, \frac{8}{3}\right\}$
7. $\left\{-\frac{74}{9}\right\}$

7.5

Exercises

Name _____

Section _____ Date _____

In exercises 1 to 8, decide whether each of the following is an expression or an equation. If it is an equation, solve it. If it is an expression, write it as a single fraction.

1. $\frac{x}{2} - \frac{x}{3} = 6$

2. $\frac{x}{4} - \frac{x}{7} = 3$

3. $\frac{x}{2} - \frac{x}{5}$

4. $\frac{x}{6} - \frac{x}{8}$

5. $\frac{3x + 1}{4} = x - 1$

6. $\frac{3x - 1}{2} - \frac{x}{5} - \frac{x + 3}{4}$

7. $\frac{x}{4} = \frac{x}{12} + \frac{1}{2}$

8. $\frac{2x - 1}{3} + \frac{x}{2}$

In exercises 9 to 50, solve each equation.

9. $\frac{x}{3} + \frac{3}{2} = \frac{x}{6} + \frac{7}{3}$

10. $\frac{x}{10} - \frac{1}{5} = \frac{x}{5} + \frac{1}{2}$

11. $\frac{4}{x} + \frac{3}{4} = \frac{10}{x}$

12. $\frac{3}{x} = \frac{5}{3} - \frac{7}{x}$

13. $\frac{5}{4x} - \frac{1}{2} = \frac{1}{2x}$

14. $\frac{7}{6x} - \frac{1}{3} = \frac{1}{2x}$

15. $\frac{3}{x + 4} = \frac{2}{x + 3}$

16. $\frac{5}{x - 2} = \frac{4}{x - 1}$

17. $\frac{9}{x} + 2 = \frac{2x}{x + 3}$

18. $\frac{6}{x} + 3 = \frac{3x}{x + 1}$

ANSWERS

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ANSWERS

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$$19. \frac{3}{x+2} - \frac{5}{x} = \frac{13}{x+2}$$

$$21. \frac{3}{2} + \frac{2}{2x-4} = \frac{1}{x-2}$$

$$23. \frac{x}{3x+12} + \frac{x-1}{x+4} = \frac{5}{3}$$

$$25. \frac{x-1}{x+3} - \frac{x-3}{x} = \frac{3}{x^2+3x}$$

$$27. \frac{1}{x-2} - \frac{2}{x+2} = \frac{2}{x^2-4}$$

$$29. \frac{7}{x+5} - \frac{1}{x-5} = \frac{x}{x^2-25}$$

$$31. \frac{11}{x+2} - \frac{5}{x^2-x-6} = \frac{1}{x-3}$$

$$33. \frac{5}{x-2} - \frac{3}{x+3} = \frac{24}{x^2+x-6}$$

$$35. \frac{x}{x-3} - 2 = \frac{3}{x-3}$$

$$37. \frac{2}{x^2-3x} - \frac{1}{x^2+2x} = \frac{2}{x^2-x-6}$$

$$39. \frac{2}{x^2-4x+3} - \frac{3}{x^2-9} = \frac{2}{x^2+2x-3}$$

$$20. \frac{7}{x} - \frac{2}{x-3} = \frac{6}{x}$$

$$22. \frac{2}{x-1} + \frac{5}{2x-2} = \frac{3}{4}$$

$$24. \frac{x}{4x-12} - \frac{x-4}{x-3} = \frac{1}{8}$$

$$26. \frac{x+1}{x-2} - \frac{x+3}{x} = \frac{6}{x^2-2x}$$

$$28. \frac{1}{x+4} + \frac{1}{x-4} = \frac{12}{x^2-16}$$

$$30. \frac{2}{x-2} = \frac{3}{x+2} + \frac{x}{x^2-4}$$

$$32. \frac{5}{x-4} = \frac{1}{x+2} - \frac{2}{x^2-2x-8}$$

$$34. \frac{3}{x+1} - \frac{5}{x+6} = \frac{2}{x^2+7x+6}$$

$$36. \frac{x}{x-5} + 2 = \frac{5}{x-5}$$

$$38. \frac{2}{x^2-x} - \frac{4}{x^2+5x-6} = \frac{3}{x^2+6x}$$

ANSWERS

$$40. \frac{2}{x^2 - 4} - \frac{1}{x^2 + x - 2} = \frac{3}{x^2 - 3x + 2}$$

$$41. 2 - \frac{6}{x^2} = \frac{1}{x}$$

$$42. 3 - \frac{7}{x} - \frac{6}{x^2} = 0$$

$$43. 1 - \frac{7}{x - 2} + \frac{12}{(x - 2)^2} = 0$$

$$44. 1 + \frac{3}{x + 1} = \frac{10}{(x + 1)^2}$$

$$45. 1 + \frac{3}{x^2 - 9} = \frac{10}{x + 3}$$

$$46. 3 - \frac{7}{x^2 - x - 6} = \frac{5}{x - 3}$$

$$47. \frac{2x}{x - 3} + \frac{2}{x - 5} = \frac{3x}{x^2 - 8x + 15}$$

$$48. \frac{x}{x - 4} = \frac{5x}{x^2 - x - 12} - \frac{3}{x + 3}$$

$$49. \frac{2x}{x + 2} = \frac{5}{x^2 - x - 6} - \frac{1}{x - 3}$$

$$50. \frac{3x}{x - 1} = \frac{2}{x - 2} - \frac{2}{x^2 - 3x + 2}$$

In exercises 51 to 58, find the zeros of each function.

$$51. f(x) = \frac{x}{10} - \frac{12}{5}$$

$$52. f(x) = \frac{4x}{3} - \frac{x}{6}$$

$$53. f(x) = \frac{12}{x + 5} - \frac{5}{x}$$

$$54. f(x) = \frac{1}{x - 2} - \frac{3}{x}$$

$$55. f(x) = \frac{1}{x - 3} + \frac{2}{x} - \frac{5}{3x}$$

$$56. f(x) = \frac{2}{x} - \frac{1}{x + 1} - \frac{3}{x^2 + x}$$

$$57. f(x) = 1 + \frac{39}{x^2} - \frac{16}{x}$$

$$58. f(x) = x - \frac{72}{x} + 1$$

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58. _____

Answers

1. Equation, $\{36\}$ 3. Expression, $\frac{3x}{10}$ 5. Equation, $\{5\}$
7. Equation, $\{3\}$ 9. $\{5\}$ 11. $\{8\}$ 13. $\left\{\frac{3}{2}\right\}$ 15. $\{-1\}$
17. $\left\{-\frac{9}{5}\right\}$ 19. $\left\{-\frac{2}{3}\right\}$ 21. No solution 23. $\{-23\}$ 25. $\{6\}$
27. $\{4\}$ 29. $\{8\}$ 31. $\{4\}$ 33. $\left\{\frac{3}{2}\right\}$ 35. No solution 37. $\{7\}$
39. $\{5\}$ 41. $\left\{2, -\frac{3}{2}\right\}$ 43. $\{5, 6\}$ 45. $\{4, 6\}$ 47. $\left\{-\frac{1}{2}, 6\right\}$
49. $\left\{-\frac{1}{2}\right\}$ 51. 24 53. $\frac{25}{7}$ 55. $\frac{3}{4}$ 57. 3, 13