C. NASSIF

MATHEMATICS

Science

OPTION

Quebec Education Program

SOLUTIONS

Secondary 4

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Table of contents

Chapter 1

Algebraic Expressions

CHALLENGE 1

1. Write as a product of factors.

a) $6x^2 - 11x - 10$ ***(3x + 2)(2x −5)***

b) $16x^2 - 9$ ***(4x + 3)(4x − 3)***

c) $(2x + 3)^2 - (5x - 1)^2$ ***(7x + 2)(−3x + 4)***

d) $15a^2 - 9ac + 10ab - 6bc$ ***(3a + 2b)(5a − 3c)***

2. Simplify the rational expression $\frac{2x^2 - 5x - 3}{4x^2 - 1}$ after indicating the restrictions on the variable x.

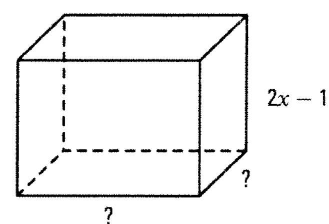
$\frac{2x - 3}{2x - 1}$

3. Calculate $\frac{3a + 2b}{3a - 2b} - \frac{3a - 2b}{3a + 2b}$.

$\frac{24ab}{9a^2 - b^2}$

4. The volume of the rectangular based prism on the right is given by $V = 12x^3 - 8x^2 - 3x + 2$. If the height of the prism is equal to $2x - 1$, determine the dimensions of the prism's base.

(2x + 1) and (3x − 2)



5. Determine the set of all real numbers greater than their square.

All real numbers between 0 and 1.

6. A projectile is thrown upward from a cliff bordering a river. The height h (in metres) of the projectile from the water level as a function of time t (in seconds) since it was thrown is given by $h = -t^2 + 10t + 11$.

a) At what point in time, during its descent, is the projectile at a height of 35 m?

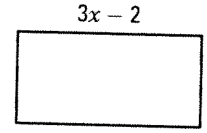
At t = 6 seconds

b) At what point in time does the projectile enter the water?

At t = 11 seconds

1.1 Polynomials

ACTIVITY 1 Perimeter and area of a rectangle

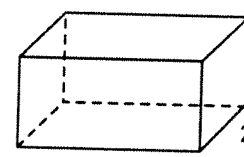


a) Let P(x) represent the perimeter of the rectangle on the right.
   1. Determine P(x). ***10x + 2***
   2. What is the degree of P(x)? ***1***

b) Let A(x) represent the area of the rectangle.
   1. Determine A(x). ***6x² + 5x – 6***
   2. What is the degree of A(x)? ***2***

c) Determine the perimeter and area of the rectangle when $x = 2$ cm,
   1. after calculating the dimensions of the rectangle. ***Perimeter = 22 cm; Area = 28 cm²***
   2. by evaluating the polynomials P(x) and A(x). ***P(2) = 20; A(2) = 28***

ACTIVITY 2 Total area and volume of a prism



a) Let A(x) represent the total area of the prism on the right.
   1. Determine A(x). ***16x² + 8x – 2***
   2. What is the degree of A(x)? ***2***

b) Let V(x) represent the volume of the prism.
   1. Determine V(x). ***4x³ + 4x² – x – 1***
   2. What is the degree of V(x)? ***3***

c) Determine the total area and volume of the prism when $x = 2$ cm,
   1. after calculating the dimensions of the prism. ***Total area = 78 cm²; Volume = 45 cm³***
   2. by evaluating the polynomials A(x) and V(x). ***A(2) = 78; V(2) = 45***

POLYNOMIALS

- A monomial in x is the product of a real number by a non-negative integer power of the variable x.
  Ex.: $-3x^2$ is a monomial with coefficient –3, variable x and exponent 2.
  $\frac{5}{x}$ is not a monomial, since $\frac{5}{x} = 5x^{-1}$, the exponent –1 being negative.
- The degree of a monomial ax^n is equal to the exponent n of the variable x.
  Ex.: The degree of $-3x^2$ is equal to 2.
- Certain monomials have more than one variable. In that case, the degree of a monomial with many variables is equal to the sum of the exponents of the variables.
  Ex.: The degree of the monomial $-3x^2y^3$ is equal to 5.

- A polynomial in x is monomial in x or a sum of monomials in x.
  Ex.: $P(x)= 3x^2 - \frac{5}{2}x + 4$ is a trinomial in x (sum of 3 monomials).
  $Q(x)= 5x^2 - 2x$ is a binomial in x (sum of 2 monomials).
  Generally, we order the monomials of a polynomial in decreasing order of the powers of the variable.
- The degree of a polynomial, once reduced, is equal to the degree of the monomial with the highest degree.
  Ex.: $P(x) = 3x^2 - 5x + 1$ is a second degree polynomial.
  $P(x, y) = 3x^2y - 2xy + 5x - 2$ is a polynomial of degree 2 in x, degree 1 in y, and degree 3 in total.
- Evaluating the polynomial $P(x) = 2x^3 + x^2 - 8x - 4$ for $x = -3$ consists of finding the numerical value of the polynomial when the variable x is replaced by the value -3.
  We get: $P(-3) = 2(-3)^3 + (-3)^2 - 8(-3) - 4 = -54 + 9 + 24 - 4 = -25$.
- The zero or root of a polynomial is any value of the variable which makes the polynomial zero. Thus, 2 is a zero of the polynomial $P(x) = 2x^3 + x^2 - 8x - 4$, since $P(2) = 0$.

1. Reduce each of the following expressions to a single monomial.

a) $2x^3 - 5x^3 + 7x^3$ ***$4x^3$***
b) $4x^2y - 6x^2y + x^2y$ ***$-x^2y$***
c) $\frac{3}{4}x^2 + \frac{2}{3}x^2 - x^2$ ***$\frac{5}{12}x^2$***
d) $-\frac{2}{3}xy^2 + \frac{3}{4}xy^2 - \frac{5}{6}xy^2$ ***$-\frac{3}{4}xy^2$***

2. Perform the following operations.

a) $-3x^2 \times 4x^3$ ***$-12x^5$***
b) $2x^2y^3 \times -3xy^2$ ***$-6x^3y^5$***
c) $-17x^2 \times -3x$ ***$51x^3$***
d) $-7x^2y \times 5x^2y^2$ ***$-35x^4y^3$***
e) $3x^2y \times -5xy \times -2xy^2$ ***$30x^4y^4$***
f) $20x^2y^2 \times -0.5x \times -1.2y^2$ ***$12x^3y^4$***
g) $\frac{3}{4}x^2y \times \frac{2}{5}xy^2 \times \frac{10}{9}x$ ***$\frac{1}{3}x^4y^3$***
h) $\frac{-3}{5}x^2y^3 \times \frac{2}{3}xy \times \frac{-5}{2}xy^2$ ***x^4y^6***

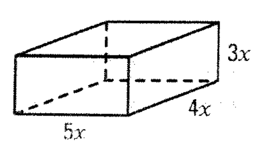
3. Perform the following operations.

a) $(3x^2y)^2$ ***$9x^4y^2$***
b) $(-2xy^3)^2$ ***$4x^2y^6$***
c) $(-2x^2y^3)^3$ ***$-8x^6y^9$***
d) $\left(-\frac{3}{4}x^2y^3\right)^2$ ***$\frac{9}{16}x^4y^6$***
e) $3x^2(-2x)^3$ ***$-24x^5$***
f) $(-2x^2)^2 \times (3x^3)^2$ ***$36x^{10}$***

4. Perform the following operations, and indicate if the result is a monomial.

a) $-12x^4 \div 3x^6$ ***$-4x^{-2}$, no***
b) $18x^6 \div 12x^4$ ***$\frac{3}{2}x^2$, yes***
c) $18x^6y^4 \div 9x^2y^2$ ***$2x^4y^2$, yes***
d) $-12x^2y^4 \div 6x^3y$ ***$-2x^{-1}y^3$, no***
e) $(-5x^3)^2 \div 10x^4$ ***$\frac{5}{2}x^2$, yes***
f) $(4x^2y^3)^3 \div (2xy^2)^4$ ***$4x^2y^3$, yes***

5. Consider the prism on the right.



a) Use a monomial to express:
1. the total area; ***$94x^2$***
2. the volume. ***$60x^3$***

b) If $x = 2$ cm, evaluate:
1. the total area; ***376 cm²***
2. the volume. ***480 cm³***

6. Using the following algebraic expressions, identify which ones are polynomials, order them in decreasing order of the powers of the variable and determine their degree.

a) $2x + 3x^2 - 1$ **Polynomial; $3x^2 + 2x - 1$; 2**

b) $2x^2 + 2\sqrt{x}$ **Not a polynomial.**

c) $3y - 2y^3 + \frac{1}{2}y^4 - 4y^2 + 1$ **Polynomial; $\frac{1}{2}y^4 - 2y^3 - 4y^2 + 3y + 1$; 4**

d) $x^{-1} + 5x + x^2$ **Not a polynomial.**

7. Reduce the following polynomials and determine their degree.

a) $P(x) = 3x^2 + 2x - 5x^2 - 3x + 1$ **$P(x) = -2x^2 - x + 1$; degree: 2**

b) $P(x, y) = 3x^3y - 2xy^2 + 4x^3y - xy^2$ **$P(x, y) = 7x^3y - 3xy^2$; degree: 4**

c) $P(z) = 4z^3 - 5z^2 + 8z^3 - z^2 + 4z - 5 + 6z^2 - 12z^3$ **$P(z) = 4z - 5$; degree: 1**

d) $P(x) = \frac{3}{2}x^2 + 5x^3 - \frac{2}{3}x^2 - \frac{3}{2}x^3 + \frac{3}{4}x - \frac{5}{2}x$ **$P(x) = \frac{7}{2}x^3 + \frac{5}{6}x^2 - \frac{7}{4}x$; degree: 3**

8. Evaluate the following polynomials.

a) $P(x) = 3x^2 + 5x$ for $x = -2$ **$P(-2) = 2$**

b) $P(x) = x^2 - 5x + 3$ for $x = 0$ **$P(0) = 3$**

c) $P(x) = 3x^2 + 2x - 5$ for $x = -1.5$ **$P(-1.5) = -1.25$**

d) $P(x) = 2x^2 - 7x - 15$ for $x = -\frac{3}{2}$ **$P\left(-\frac{3}{2}\right) = 0$**

9. Given the polynomial $P(x, y) = 3x^2y - 2xy^2 + xy - 2$. Calculate

a) $P(2, 3)$ **4**

b) $P(-3, 1)$ **28**

c) $P(-1, -2)$ **2**

d) $P\left(-\frac{2}{3}, \frac{3}{2}\right)$ **2**

10. Given the polynomial $P(x) = 2x^2 - x - 6$. Determine the zeros among the following real numbers.

a) $x = -1$ **$P(-1) = -3$, therefore, -1 is not a zero.**

b) $x = 2$ **$P(2) = 0$, therefore, 2 is a zero.**

c) $x = 0$ **$P(0) = -6$, therefore, 0 is not a zero.**

d) $x = -\frac{3}{2}$ **$P\left(-\frac{3}{2}\right) = 0$, therefore $-\frac{3}{2}$ is a zero.**

11. The following algebraic expressions are first degree binomials. Determine the zero of each of these binomials.

a) $P(x) = 3x - 2$ **$\frac{2}{3}$**

b) $P(y) = -2y + 5$ **$\frac{5}{2}$**

c) $P(z) = -2z + 1$ **$\frac{1}{2}$**

d) $P(x) = -3x + \frac{1}{2}$ **$\frac{1}{6}$**

e) $P(y) = -\frac{2}{3}x + 4$ **6**

f) $P(z) = -\frac{2}{5}x + \frac{3}{4}$ **$\frac{15}{8}$**

12. From the top of a bridge, Eric throws a stone straight down toward the water. The formula $d(t) = 4{,}9\ t^2 + 3t$ expresses the distance $d(t)$, in metres, traveled by the stone as a function of the elapsed time, in seconds, since it was thrown. If the stone enters the water after 3 seconds, what is the height of the bridge?

53.1 m

13. Neena buys x notebooks at \$2.50 each and y pens at \$1.75 each.

a) Express, using a polynomial, the total cost of these purchases. **$P(x, y) = 2.5x + 1.75y$**

b) Determine the total cost if she buys 6 notebooks and 2 pens. **$P(6, 2) = \$18.50$**

14. Given $P(x) = 4x^2 - 5x + 1$ and $Q(x) = -x^2 + 4x$.

a) Express, using a polynomial $S(x)$, the sum of the polynomials $P(x)$ and $Q(x)$. ***$S(x) = 3x^2 - x + 1$***

b) Verify that $S(-2) = P(-2) + Q(-2)$. ***$P(-2) = 27$, $Q(-2) = -12$, $S(-2) = 15$. We have: $15 = 27 + (-12)$.***

15. Given $P(x) = 3x^2 - 2x + 1$ and $Q(x) = 2x^2 - 3x$.

a) Express, using a polynomial $D(x)$, the difference of the polynomials $P(x)$ and $Q(x)$.
$D(x) = x^2 + x + 1$

b) Verify that $D(1) = P(1) - Q(1)$. ***$P(1) = 2$, $Q(1) = -1$, $D(1) = 3$. We have: $3 = 2 - (-1)$.***

16. Given $P = 3x^2 - 2x + 1$, $Q = -x^2 - 3x + 2$ and $R = -2x + 5$. Determine:

a) $P + Q + R$ ***$2x^2 - 7x + 8$***

b) $P - Q + R$ ***$4x^2 - x + 4$***

c) $P - Q - R$ ***$4x^2 + 3x - 6$***

d) $-P + Q - R$ ***$-4x^2 + x - 4$***

17. Given $P = \frac{2}{3}x^2 - \frac{3}{2}x + 1$, $Q = \frac{3}{2}x^2 + \frac{5}{6}x - \frac{1}{3}$ and $R = \frac{3}{2}x - \frac{1}{6}$. Determine:

a) $P + Q + R$ ***$\frac{13}{5}x^2 + \frac{5}{6}x + \frac{1}{2}$***

b) $P - Q + R$ ***$-\frac{5}{6}x^2 - \frac{5}{2}x + \frac{7}{6}$***

c) $P - Q - R$ ***$-\frac{5}{6}x^2 - \frac{23}{6}x + \frac{3}{2}$***

d) $-P + Q - R$ ***$\frac{5}{6}x^2 + \frac{5}{2}x - \frac{7}{6}$***

18. Perform the following operations.

a) $(4x^2 - 8x + 1) - (2x^2 - 3x + 5)$ ***$2x^2 - 5x - 4$***

b) $(3x^2 - 2xy^2 + 3xy) + (2x^2 + 3x^2y - 5xy)$ ***$5x^2 + xy^2 - 2xy$***

c) $(3a^2b - 5ab^2) - (2a^2b + 3ab^2)$ ***$a^2b - 8ab^2$***

d) $(3x^2y - 2xy + 4xy^2) - (-3xy^2 + 4xy) + 2x^2y$ ***$5x^2y - 6xy + 7xy^2$***

e) $(3x^3 - 5x^2 - 4x - 1) - [(x^3 - 5x^2) - (x^2 - 4x + 1)]$ ***$2x^3 + x^2 - 8x$***

19. Given $A(x) = 3x^2 - 2x + 1$ and $B(x) = 2x - 5$.

a) Express, using a polynomial $P(x)$, the product of the polynomials $A(x)$ and $B(x)$.
$P(x) = 6x^3 - 19x^2 + 12x - 5$

b) Verify that $P(2) = A(2) \times B(2)$. ***$P(2) = -9$; $A(2) = 9$; $B(2) = -1$. We have: $-9 = 9 \times -1$.***

20. Determine the following products.

a) $3x^2(2x - 5)$ ***$6x^3 - 15x^2$***

b) $-3y(y^2 - 2y)$ ***$-3y^3 + 6y^2$***

c) $-2x^2(3xy^2 + 5x^2y)$ ***$-6x^3y^2 - 10x^4y$***

d) $(2xy - 5x)(-3x^2y)$ ***$-6x^3y^2 + 15x^3y$***

e) $\frac{3}{4}x^2\left(\frac{2}{3}x - 8x^2\right)$ ***$\frac{1}{2}x^3 - 6x^4$***

f) $(4x^2 - 8x + 12)\left(-\frac{3}{4}x\right)$ ***$-3x^3 + 6x^2 - 9x$***

21. Determine the following products.

a) $(x + 3)(x - 2)$ ***$x^2 + x - 6$***

b) $(x - 5)(3 - x)$ ***$-x^2 + 8x - 15$***

c) $(2a + b)(3a - 2b)$ ***$6a^2 - ab - 2b^2$***

d) $(5 - 2x)(3x - 4)$ ***$-6x^2 + 23x - 20$***

e) $(-2x + 5)(3x - 2)$ ***$-6x^2 + 19x - 10$***

f) $(-5x - 3)(-2x + 4)$ ***$10x^2 - 14x - 12$***

22. Perform the following operations.

a) $5x(3x + 2y) - 3x(2x - 4y)$ ___ ***$9x^2 + 22xy$***

b) $(2x - 3y)(3x + 2y) + (5x - y)(2x + 3y)$ ___ ***$16x^2 + 8xy - 9y^2$***

c) $(3a - 2b)(2a + b) - (2a + 3b)(3a - b)$ ___ ***$b^2 - 8ab$***

d) $(2a + 3)(3a - 2)(2a - 5)$ ___ ***$12a^3 - 20a^2 - 37a + 30$***

e) $(2x + 3)(2x - 5)(3x - 4)$ ___ ***$12x^3 - 28x^2 - 29x + 60$***

23. Simplify the following expressions.

a) $(x - 1)(x + 1) + x(x - 1)$ ___ ***$2x^2 - x - 1$***

b) $(3x + 5y)(3x + 5y) - x(2x + 3y) + y(3x - 2y)$ ___ ***$7x^2 + 30xy + 23y^2$***

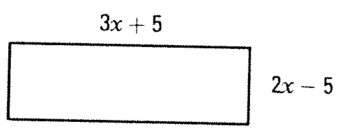
c) $a(a - b) + b(a + b) + (a + b)(a - b)$ ___ ***$2a^2$***

d) $(4a + 3b)(4a - 3b) - (4a + b)(4a + 3b)$ ___ ***$-16ab - 12b^2$***

e) $m(m + 2) + 3(m^2 - 2m - 1) - (m + 2)(m - 2)$ ___ ***$3m^2 - 4m + 1$***

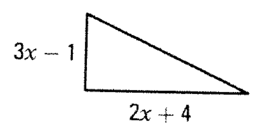
24. Use a polynomial to express the area of the following figures.

a)



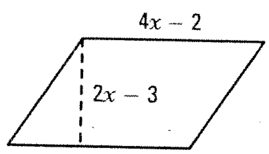
$6x^2 - 5x - 25$

b)



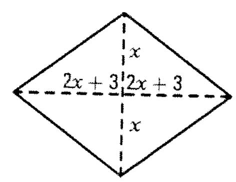
$3x^2 + 5x - 2$

c)



$8x^2 - 16x + 6$

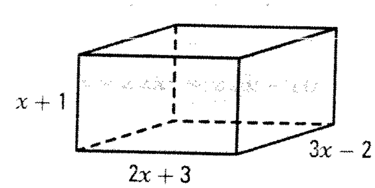
d)



$4x^2 + 6x$

25. a) Use a polynomial to express the total area A of the following right prism.

$A = 22x^2 + 22x - 10$



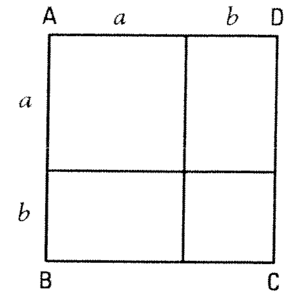
b) Use a polynomial to express the volume V of the prism.

$V = 6x^3 + 11x^2 - x - 6$

c) Evaluate the total area and volume when $x = 2$ cm. ***Total area: 122 cm²; Volume: = 84 cm³***

1.2 Remarkable identities

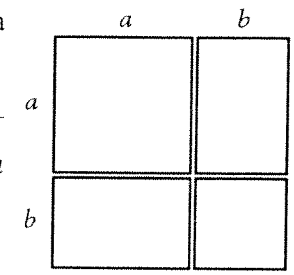
Activity 1 Squaring a sum



Consider the square ABCD on the right with a side length of $(a + b)$.
The area of this square is equal to $(a + b)^2$.

a) The square ABCD is split into four regions as shown on the right. We, therefore, can calculate the area of square ABCD by adding the area of the four regions. What is the sum of the areas of the four regions?

   $a^2 + 2ab + b^2$

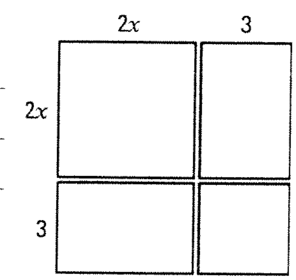


b) The equality $(a + b)^2 = a^2 + 2ab + b^2$ that is true for any real numbers a and b is called an identity.
   Verify this identity by expanding the product $(a + b)(a + b)$.

   $(a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

c) Show that $(2x + 3)^2 = 4x^2 + 12x + 9$ in three ways:

   1. by expanding the product $(2x + 3)(2x + 3)$. ***$4x^2 + 12x + 9$***
   2. by considering the figure on the right. ***$4x^2 + 12x + 9$***
   3. by using the identity: $(a + b)^2 = a^2 + 2ab + b^2$. ***$4x^2 + 12x + 9$***



Activity 2 Squaring a difference

a) Prove the identity $(a - b)^2 = a^2 - 2ab + b^2$ by expanding the product $(a - b)(a - b)$.

   $(a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$

b) Show that $(3x - 2)^2 = 9x^2 - 12x + 4$:

   1. by expanding the product $(3x - 2)(3x - 2)$. ***$9x^2 - 12x + 4$***
   2. by using the identity: $(a - b)^2 = a^2 - 2ab + b^2$. ***$9x^2 - 12x + 4$***

Activity 3 Product of a sum and a difference

a) Expand the product $(a + b)(a - b)$ to prove the identity $(a + b)(a - b) = a^2 - b^2$.

   $(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2$

b) Using this identity, determine the following products.

   1. $(2x + 3)(2x - 3)$ ***$4x^2 - 9$*** 2. $(3x - 4)(3x + 4)$ ***$9x^2 - 16$***

REMARKABLE IDENTITIES

Square of a sum

$$(a + b)^2 = a^2 + 2ab + b^2$$

We say that $a^2 + 2ab + b^2$ is a perfect square trinomial.

Ex.: $(2x + 5y)^2 = (2x)^2 + 2(2x)(5y) + (5y)^2$
$= 4x^2 + 20xy + 25y^2$

Square of a difference

$$(a - b)^2 = a^2 - 2ab + b^2$$

We say that $a^2 - 2ab + b^2$ is a perfect square trinomial.

Ex.: $(3x - 2y)^2 = (3x)^2 - 2(3x)(2y) + (2y)^2$
$= 9x^2 - 12xy + 4y^2$

Product of a sum and a difference

$$(a + b)(a - b) = a^2 - b^2$$

The product of a sum by a difference is equal to a difference of two squares.

Ex.: $(3x + 5y)(3x - 5y) = (3x)^2 - (5y)^2$
$= 9x^2 - 25y^2$

1. a) Calculate the following products using the distributive property.

1. $(3x + 5)^2$ **$9x^2 + 30x + 25$** 2. $(2x - 7)^2$ **$4x^2 - 28x + 49$** 3. $(2x + 5)(2x - 5)$ **$4x^2 - 25$**

b) Calculate the following products using the appropriate remarkable identity.

1. $(3x + 5)^2$ **$9x^2 + 30x + 25$** 2. $(2x - 7)^2$ **$4x^2 - 28x + 49$** 3. $(2x + 5)(2x - 5)$ **$4x^2 - 25$**

2. Calculate the following products using the identity $(a + b)^2 = a^2 + 2ab + b^2$.

a) $(x + 5)^2$ **$x^2 + 10x + 25$**
b) $(3x + 4)^2$ **$9x^2 + 24x + 16$**
c) $(2x + 5y)^2$ **$4x^2 + 20xy + 25y^2$**
d) $\left(\frac{1}{2}x + 7\right)^2$ **$\frac{1}{4}x^2 + 7x + 49$**
e) $(3x^2y + xy^2)^2$ **$9x^4y^2 + 6x^3y^3 + x^2y^4$**
f) $\left(\frac{3}{4}x^2 + \frac{2}{9}y^2\right)^2$ **$\frac{9}{16}x^4 + \frac{1}{3}x^2y^2 + \frac{4}{81}y^4$**

3. Calculate the following products using the identity $(a - b)^2 = a^2 - 2ab + b^2$.

a) $(2x - 7)^2$ **$4x^2 - 28x + 49$**
b) $(4a - b)^2$ **$16a^2 - 8ab + b^2$**
c) $(3x^2 - 2x)^2$ **$9x^4 - 12x^3 + 4x^2$**
d) $(-3x - 4y)^2$ **$9x^2 + 24xy + 16y^2$**
e) $\left(\frac{3}{4}x - \frac{5}{6}\right)^2$ **$\frac{9}{16}x^2 - \frac{5}{4}x + \frac{25}{36}$**
f) $\left(\frac{2}{3}x^2 - \frac{3}{4}x\right)^2$ **$\frac{4}{9}x^4 - x^3 + \frac{9}{16}x^2$**

4. Calculate the following products using the identity $(a + b)(a - b) = a^2 - b^2$.

a) $(2x + 7)(2x - 7)$ **$4x^2 - 49$**
b) $(4x + 3y)(4x - 3y)$ **$16x^2 - 9y^2$**
c) $(2 - 3x)(2 + 3x)$ **$4 - 9x^2$**
d) $(-2x + 3y)(-2x - 3y)$ **$4x^2 - 9y^2$**
e) $\left(\frac{2}{3}x + \frac{3}{4}\right)\left(\frac{2}{3}x - \frac{3}{4}\right)$ **$\frac{4}{9}x^2 - \frac{9}{16}$**
f) $(5x^2y + 3x^3)(5x^2y - 3x^3)$ **$25x^4y^2 - 9x^6$**

5. Expand using the appropriate identity.

a) $(3x^2 - 2y)^2$ $9x^4 - 12x^2y + 4y^2$

b) $(2x^3 + 3y^2)(2x^3 - 3y^2)$ $4x^6 - 9y^4$

c) $(3x^4 + 2y^2)^2$ $9x^8 + 12x^4y^2 + 4y^4$

d) $(-3x^2 + 5x)^2$ $9x^4 - 30x^3 + 25x^2$

e) $(-2x - 3y)^2$ $4x^2 + 12xy + 9y^2$

f) $(5x^2 - 3y)(5x^2 + 3y)$ $25x^4 - 9y^2$

6. Expand, and simplify the resulting expressions.

a) $(3x + 2y)^2 + (2x - 3y)^2$ $13x^2 + 13y^2$

b) $(3x + 5y)^2 - (3x - 5y)^2$ $60xy$

c) $(2x + 3)(4x^2 + 9)(2x - 3)$ $16x^4 - 81$

d) $3x(2x - 3)^2 - 2x(3x + 2)^2$ $-6x^3 - 60x^2 + 19x$

e) $(x + 1)(x^2 - 1)(x - 1)$ $x^4 - 2x^2 + 1$

f) $(2x + 3)^2 + (2x - 3)^2 + (2x + 3)(2x - 3)$ $12x^2 + 9$

g) $(3x + 5)^2 - (3x - 5)^2 + (3x + 5)(3x - 5)$ $9x^2 + 60x - 25$

h) $(4x + 3y)^2 - (4x + 3y)(4x - 3y)$ $18y^2 + 24xy$

i) $(4x - 1)^2 - (3x + 5)^2$ $7x^2 - 38x - 24$

7. Use the remarkable identities to calculate mentally.

a) 101^2 10 201

b) 99^2 9 801

c) 101×99 9 999

8. If x represents Karen's age, use a reduced polynomial to represent

a) the square of Karen's age 3 years from now; $x^2 + 6x + 9$

b) the square of Karen's age 5 years ago; $x^2 - 10x + 25$

c) the product of Karen's age 4 years from now with her age 4 years ago; $x^2 - 16$

d) the difference between the square of Karen's age 1 year from now and the square of her age 1 year ago; $4x$

9. Complete the following trinomials to obtain a perfect square trinomial.

a) $x^2 + 6x +$ 9

b) $4x^2 - 20x +$ 25

c) $9x^2$ $+ 12x + 4$

d) $9x^2$ $- 6x + 1$

e) $9x^2 +$ $24x$ $+ 16$

f) $16x^2 -$ $24x$ $+ 9$

g) $9x^4 + 12x^2y +$ $4y^2$

h) $4x^2$ $+ 12x^3 + 9x^4$

i) $25x^4 +$ $20x^2y^2$ $+ 4y^4$

10. Prove the following identities.

a) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

b) $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

c) $(a + b)(a^2 - ab + b^2) = a^3 + b^3$

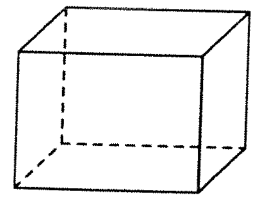
d) $(a - b)(a^2 + ab + b^2) = a^3 - b^3$

1.3 Polynomial division

ACTIVITY 1 Dividing a polynomial by a monomial

a) The prism on the right has the volume $V(x) = 12x^3 + 10x^2 - 2x$. Determine the area of the base knowing that the height of the prism is equal to $2x$.

$6x^2 + 5x - 1$



b) Explain how to divide a polynomial by a monomial.

Divide each term of the polynomial by the monomial.

DIVISION BY A MONOMIAL

- To divide a polynomial by a monomial, we divide each term of the polynomial by the monomial.

  Ex.: The quotient $Q(x)$ of the polynomial $A(x) = 18x^4 - 9x^3 + 12x^2$ by the monomial $B(x) = 6x^2$ is:

$$\begin{aligned} Q(x) &= A(x) \div B(x) \\ &= (18x^4 - 9x^3 + 12x^2) \div 6x^2 \\ &= (18x^4 \div 6x^2) - (9x^3 \div 6x^2) + (12x^2 \div 6x^2) \\ &= 3x^2 - \frac{3}{2}x + 2. \end{aligned}$$

- The quotient of two polynomials is not always a polynomial.

  Ex.: $(12x^3 - 6x) \div (3x^2) = 4x - 2x^{-1}$ which is not a polynomial.

1. Perform the following divisions.

a) $(24x^4 + 12x^3 - 18x^2) \div 6x^2$ ***$4x^2 + 2x - 3$***

b) $(36x^2y^4 + 27x^3y^2 - 9x^2y^2) \div 9x^2y$ ***$4y^3 + 3xy - y$***

c) $(3x^2 + 2x)(4x + 6) \div 2x$ ***$6x^2 + 13x + 6$***

d) $(3x + 6)^2 \div 3x$ ***$3x + 12 + 12x^{-1}$***

e) $(4x^2 + 2x)(4x^2 - 2x) \div 4x^2$ ***$4x^2 - 1$***

EUCLIDEAN DIVISION

- Consider the polynomials $A(x) = 6x^2 + 5x - 4$ and $B(x) = 3x - 2$.

  To divide $A(x)$ (the dividend) by $B(x)$ (the divisor), we proceed in the following manner to determine the quotient $Q(x)$ and the remainder $R(x)$:

When writing the division, make sure $A(x)$ and $B(x)$ are in decreasing order of the exponents. →

$$B(x)\,\overline{\big)\,A(x)\,}\quad Q(x)$$
$$R(x)$$

1° Divide the term in the dividend with the highest degree ($6x^2$) by the term in the divisor with the highest degree ($3x$). We get ($2x$) which represents the term in the quotient $Q(x)$ with the highest degree.

$$\begin{array}{r|l} & 2x \\ \hline 3x - 2 & 6x^2 + 5x - 4 \end{array}$$

2° Calculate the product between the divisor ($3x - 2$) and the 1st term 2x obtained in the first step. Align the resulting product ($6x^2 - 4x$) under the dividend.

$$\begin{array}{r|l} & 2x \\ \hline 3x - 2 & 6x^2 + 5x - 4 \\ & 6x^2 - 4x \end{array}$$

3° Calculate the first remainder by subtracting the product obtained in 2nd step ($6x^2 - 4x$) from the dividend. We then get ($9x - 4$).

$$\begin{array}{r|l} & 2x \\ \hline 3x - 2 & 6x^2 + 5x - 4 \\ & \underline{6x^2 - 4x} \\ & \quad\quad 9x - 4 \end{array}$$

Repeat the process...

Stop the division when the degree of the remainder is less than the degree of the divisor.

We, therefore, get the quotient $Q(x) = 2x + 3$ and the remainder $R(x) = 2$.

$$\begin{array}{r|l} & 2x + 3 \leftarrow \text{quotient} \\ \hline 3x - 2 & 6x^2 + 5x - 4 \\ & \underline{6x^2 - 4x} \\ & \quad\quad 9x - 4 \\ & - \underline{\quad 9x - 6} \\ \text{remainder} \longrightarrow & \quad\quad\quad\quad 2 \end{array}$$

- The dividend $A(x)$, the divisor $B(x)$, the quotient $Q(x)$ and the remainder $R(x)$ verify the following Euclidean relation:

$$\boxed{A(x) = B(x) \cdot Q(x) + R(x)} \quad \text{where} \quad \deg R(x) < \deg B(x).$$

In fact, $6x^2 + 5x - 4 = (3x - 2)(2x + 3) + 2$.

2. Determine the quotient $Q(x)$ and the remainder $R(x)$ in the division of $A(x) = 2x^2 + 5x - 3$ by $B(x) = x - 1$. ***$Q(x) = 2x + 7$; $R(x) = 4$*** ***$(x - 1)(2x + 7) + 4 = 2x^2 + 5x - 3$***

3. In each of the following cases, determine the quotient $Q(x)$ and the remainder $R(x)$ in the division of $A(x)$ by $B(x)$.

a) $A(x) = 2x^2 - x - 6$; $B(x) = 2x + 3$ ***$Q(x) = x - 2$; $R(x) = 0$***

b) $A(x) = 3x^2 - 2x + 1$; $B(x) = x - 2$ ***$Q(x) = 3x + 4$; $R(x) = 9$***

c) $A(x) = 2x^3 + 3x^2 + 2x + 4$; $B(x) = x + 1$ ***$Q(x) = 2x^2 + x + 1$; $R(x) = 3$***

d) $A(x) = x^3 - 2x + 1$; $B(x) = x - 1$ ***$Q(x) = x^2 + x - 1$; $R(x) = 0$***

e) $A(x) = x^4 - 1$; $B(x) = x + 1$ ***$Q(x) = x^3 - x^2 + x - 1$; $R(x) = 0$***

f) $A(x) = x^3 + 27$; $B(x) = x + 3$ ***$Q(x) = x^2 - 3x + 9$; $R(x) = 0$***

ACTIVITY 2 Remainder in Euclidean division

a) Given $P(x) = 3x^2 - 5x + 1$.

1. Calculate $P(2)$. ***3***
2. Verify that the remainder in the division of $P(x)$ by $(x - 2)$ is equal to $P(2)$.
3. Calculate $P(-2)$. ***23***
4. Verify that the remainder in the division of $P(x)$ by $(x + 2)$ is equal to $P(-2)$.

b) Given $P(x) = x^2 + 2x - 15$.

A polynomial $A(x)$ is **divisible** by a polynomial $B(x)$ when the remainder in Euclidean division of $A(x)$ by $B(x)$ is 0.

1. Show that $P(x)$ is divisible by $(x - 3)$.
2. Verify that $P(3) = 0$.
3. Show that $P(x)$ is divisible by $(x + 5)$.
4. Verify that $P(-5) = 0$.

REMAINDER THEOREM

- The remainder in the division of a polynomial $P(x)$ by $(x - a)$ is equal to $P(a)$.
- A polynomial $P(x)$ is divisible by a polynomial $Q(x)$ if and only if the remainder in the division of $P(x)$ by $Q(x)$ is equal to 0.

Consequently,

$$P(x) \text{ is divisible by } (x - a) \Leftrightarrow P(a) = 0$$

4. Given $P(x) = 2x^3 + 3x^2 - 4x - 1$.

Determine the remainder in the division of $P(x)$ by:

1. $(x - 2)$ ***19*** 2. $(x + 2)$ ***3*** 3. $(x - 1)$ ***0***

5. Given $P(x) = x^3 + 2x^2 - 5x - 6$.

a) Show that $P(x)$ is divisible by:

1. $(x + 3)$ ***P(−3) = 0*** 2. $(x - 2)$ ***P(2) = 0*** 3. $(x + 1)$ ***P(−1) = 0***

b) Show that $P(x)$ is not divisible by:

1. $(x - 1)$ ***P(1) ≠ 0*** 2. $(x + 2)$ ***P(−2) ≠ 0*** 3. $(x - 3)$ ***P(3) ≠ 0***

6. The area of a parallelogram is $A(x) = 10x^2 + 19x - 15$.

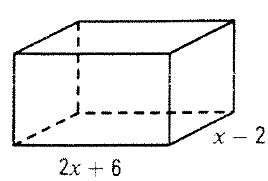
The height is represented by the binomial $5x - 3$. Use a polynomial to express the parallelogram's base. ***2x + 5***

7. The area of a rectangle is given by the polynomial $A(x) = 6x^2 - 13x - 5$.

The width is represented by the binomial $3x + 1$. Use a polynomial to express the perimeter of this rectangle.

10x − 8

8. The following right prism has the volume $V(x) = 2x^3 + 4x^2 - 10x - 12$.
The dimensions of the prism's base are $(2x + 6)$ and $(x - 2)$.



a) Determine the height of the prism. ***x + 1***

b) Use a polynomial to express the total area of the prism. ***10x² + 18x − 16***

9. The area of a triangle is $A(x) = x^2 + x - 6$. The base is represented by the binomial $2x + 6$. Use a polynomial to express the height of this triangle.

x − 2

1.4 Factoring a polynomial

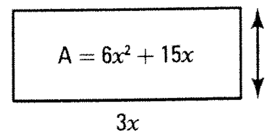
ACTIVITY 1 Removing the common factor

a) The sum $ab + ac$ has two terms ab and ac. Each term is a product of factors.

1. What is the common factor to both terms? **a**
2. Write the sum $ab + ac$ as a product of two factors. **a (b + c)**

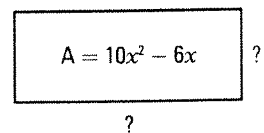
b) The rectangle on the right has the area $A = 6x^2 + 15x$. If one of the dimensions is $3x$, find the other dimension.

(2x + 5)



c) The rectangle on the right has the area $A = 10x^2 - 6x$. What could the dimensions of this rectangle be?

Varied answers. For example, 2x and (5x − 3)



d) Given the polynomial $P(x) = 6x^3 - 15x^2$. In each of the following statements, a common factor to both terms of the polynomial $P(x)$ has been removed. Complete the writing of $P(x)$ as a product of two factors.

1. $6x^3 - 15x^2 = 3$ **$(2x^3 - 5x^2)$**
2. $6x^3 - 15x^2 = x$ **$(6x^2 - 15x)$**
3. $6x^3 - 15x^2 = x^2$ **$(6x - 15)$**
4. $6x^3 - 15x^2 = 3x$ **$(2x^2 - 5x)$**
5. $6x^3 - 15x^2 = 3x^2$ **$(2x - 5)$**

REMOVING THE COMMON FACTOR

- Factoring a polynomial means writing the polynomial as a product of factors.
- Removing a common factor is a method which can be used to factor a polynomial composed of monomials which all have a common factor. To factor, you need to apply the distributive property of multiplication over addition.

factor →

$$ab + ac = a(b + c)$$

← expand

Ex.: Factor: $P(x) = 6x^4 + 15x^3 - 18x^2$

$= 3x^2(2x^2 + 5x - 6)$.

Notice that the **greatest common factor** ($3x^2$) to all three terms is factored out.

1. Find the greatest common factor of the following algebraic expressions.

a) $18x^4$; $24x^3$; $12x^5$ **$6x^3$**

b) $18x^3y^2z^4$; $24x^4y^3z^4$; $36x^2y^4z^3$ **$6x^2y^2z^3$**

c) $15x^2(a + b)^3$; $18x^3(a + b)^2$ **$3x^2(a + b)^2$**

d) $24x^3y^2(a - b)^3$; $36x^2y^4(a - b)^2$ **$12x^2y^2(a - b)^2$**

2. Factor the following polynomials.

a) $5x - 10$ $5(x - 2)$

b) $18x + 24y - 12z$ $6(3x + 4y - 2z)$

c) $4x^2 + 6x$ $2x(2x + 3)$

d) $12x + x^2 - 5x^3$ $x(12 + x - 5x^2)$

e) $12a^2b + 18a^2b^2$ $6a^2b(2 + 3b)$

f) $-3x^4 + 6x^3 - 9x^2$ $-3x^2(x^2 - 2x + 3)$

g) $a^2 + ab + a$ $a(a + b + 1)$

h) $x^4 - x^3y - x^2$ $x^2(x^2 - xy - 1)$

i) $24x^3y^2 - 16x^2y^3 + 28x^3y^4$ $4x^2y^2(6x - 4y + 7xy^2)$

j) $21x^3y^2z - 14x^2y^3z^2 + 28x^2y^2z^2$ $7x^2y^2z(3x - 2y + 4z)$

3. Factor the following polynomials.

a) $12x^3 - 16x^2$ $4x^2(3x - 4)$

b) $6x^2y^3 + 4x^3y^2$ $2x^2y^2(3y + 2x)$

c) $18x^2y^3z^4 - 12xy^4z^3$ $6xy^3z^3(3xz - 2y)$

d) $12x^2y^3 - 18x^3y^2 + 24x^2y^2$ $6x^2y^2(2y - 3x + 4)$

e) $-14x^3 + 21x^2 - 7x$ $-7x(2x^2 - 3x + 1)$

f) $-25m^4n^3 + 50m^3n^4$ $-25m^3n^3(m - 2n)$

g) $2x(x + 1) + 3y(x + 1)$ $(x + 1)(2x + 3y)$

h) $2(x - 3) - x(x - 3)$ $(x - 3)(2 - x)$

4. Factor the following polynomials.

a) $x(x + 2) + 5(x + 2)$ $(x + 2)(x + 5)$

b) $3(x - 2) - x(x - 2)$ $(x - 2)(3 - x)$

c) $a(b + c) - d(b + c)$ $(b + c)(a - d)$

d) $x(3 - y) + y(3 - y)$ $(3 - y)(x + y)$

e) $(x + 3)(x + 2) + (x + 3)(x - 1)$ $(x + 3)(2x + 1)$

f) $(x + y)(x - 2) - (x + y)(2x - 3)$ $(x + y)(-x + 1)$

g) $(x + y)^2 + x(x + y)$ $(x + y)(2x + y)$

h) $(x - y)^2 + (x - y)(x + y)$ $2x(x - y)$

5. Factor the following polynomials.

a) $x(x - 1) - 3(1 - x)$ $(x - 1)(x + 3)$

b) $x(x + 3) + 2(-x - 3)$ $(x + 3)(x - 2)$

c) $(x - 5)^2 - 2(5 - x)$ $(5 - x)(3 - x)$

d) $(2x + 1)(2x - 1) + (1 - 2x)^2$ $4x(2x - 1)$

e) $(2x + 3y)(x + y) + (4x + 6y)(x - y)$ $(2x + 3y)(3x - y)$

f) $(x + 1)(2x + 6) - (x - 2)(3x + 9)$ $(x + 3)(-x + 8)$

ACTIVITY 2 Factoring by grouping

a) The sum $ac + ad + bc + bd$ is composed of 4 terms. Each term is the product of two factors. Can we find a common factor to all 4 of these terms? **No**

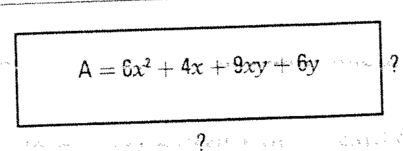
b) Justify the steps which enable you to factor the sum $ac + ad + bc + bd$.

$ac + ad + bc + bd = (ac + ad) + (bc + bd)$ **Addition is associative**

$= a(c + d) + b(c + d)$ **Remove the common factor**

$= (c + d)(a + b)$ **Remove the common factor**

c) The rectangle on the right has an area of $A = 6x^2 + 4x + 9xy + 6y$. What could the dimensions of this rectangle be?



$(6x^2 + 4x) + (9xy + 6y) = 2x(3x + 2) + 3y(3x + 2)$

$= (3x + 2)(2x + 3y)$

The dimensions of the rectangle could be $(3x + 2)$ and $(2x + 3y)$.

FACTORING BY GROUPING

- Factoring by grouping is a method which enables you to factor polynomials by grouping the terms which contain a common factor.

You then remove the common factor in each of the groupings:

$$\begin{aligned} ac + ad + bc + bd &= (ac + ad) + (bc + bd) \\ &= a(c + d) + b(c + d) \\ &= (c + d)(a + b). \end{aligned}$$

Ex.: Factor the following expression using factoring by grouping.

$P(x) = \underbrace{9x^2 - 12xy^2} + \underbrace{6xy - 8y^2}$ ← Group the terms containing a common factor.

$= 3x(3x - 4y^2) + 2y(3x - 4y)$ ← Remove the common factor in each grouping.

$= (3x - 4y^2)(3x + 2y)$ ← Remove the common factor a 2nd time.

6. Factor the following polynomials.

a) $x^2 + 5xy + 3x + 15y$ **$(x + 3)(x + 5y)$**

b) $2x^2 + 3xy - 10x - 15y$ **$(x - 5)(2x + 3y)$**

c) $6a^2 - 15a + 2ab - 5b$ **$(3a + b)(2a - 5)$**

d) $6x^2 - 8xy - 9xy + 12y^2$ **$(2x - 3y)(3x - 4)$** ← * Mista in bo Doesn't work other

e) $10xy + 2x + 15y + 3$ **$(2x + 3)(5y + 1)$**

f) $x^3 - x^2 + x - 1$ **$(x^2 + 1)(x - 1)$**

7. Factor the following polynomials.

a) $2x^2y + 3x^2 + 10y + 15$ **$(x^2 + 5)(2y + 3)$**

b) $15x^4y^2 + 35x^2y^2 - 9x^2 - 21$ **$(3x^2 + 7)(5x^2y^2 - 3)$**

c) $2x^3 + 4x^2y - 2x^2 - 4xy$ **$2x(x - 1)(x + 2y)$**

d) $3x^3y - 9x^3 + 6x^2y - 18x^2$ **$3x^2(x + 2)(y - 3)$**

e) $30x^4y - 10x^3y^2 + 15x^3y - 5x^2y^2$ **$5x^2y(2x + 1)(3x - y)$**

f) $2x^4 - 2x^3 + 6x^2 - 6x$ **$2x(x^2 + 3)(x - 1)$**

8. Factor the following polynomials.

a) $ax - ay + bx - by + cx - cy$ **$(a + b + c)(x - y)$**

b) $6ax - 3ay + 10bx - 5by - 4x + 2y$ **$(3a + 5b - 2)(2x - y)$**

c) $a^3 - 2ab + ac^2 - a^2b + 2b^2 - bc^2 + a^2c - 2bc + c^3$ **$(a^2 - 2b + c^2)(a - b + c)$**

d) $ab(x^2 + y^2) - xy(a^2 + b^2)$ **$(ax - by)(bx - ay)$**

ACTIVITY 3 **Difference of squares**

The following remarkable identity enables you to factor a difference of squares.

Product of a sum by a difference → $(a + b)(a - b) = a^2 - b^2$ ← Difference of squares

(expand →, ← factor)

a) Factor the following differences of squares.

1. $x^2 - 25$ ***(x + 5)(x − 5)***
2. $4x^2 - 9y^2$ ***(2x + 3y)(2x − 3y)***
3. $x^2 - 7$ $(x + \sqrt{7})(x - \sqrt{7})$
4. $-x^2 + 9$ ***(3 + x)(3 − x)***
5. $(3x + 1)^2 - 4x^2$ $[(3x + 1) + 2x][(3x + 1) - 2x] = (5x + 1)(x + 1)$

b) The rectangle on the right has an area of $A = 16x^2 - 9$. What could be the possible dimensions of this rectangle?

(4x + 3) and (4x − 3)

$A = 16x^2 - 9$? ?

DIFFERENCE OF SQUARES

- A difference of squares is an algebraic expression of the form $a^2 - b^2$.
- Every difference of squares is factorable. You simply need to apply the remarkable identity:

$$a^2 - b^2 = (a + b)(a - b)$$

Ex.: Factor:

- $9x^2 - 4y^2 = (3x)^2 - (2y)^2$ ← Write in the form $a^2 - b^2$.
  $= (3x + 2y)(3x - 2y)$ ← Apply the remarkable identity.
- $(2x + 1)^2 - 36 = (2x + 1)^2 - 6^2$
  $= [(2x + 1) + 6][(2x + 1) - 6]$
  $= (2x + 7)(2x - 5)$
- $(3x + 5)^2 - (2x + 1)^2 = [(3x + 5) + (2x + 1)][(3x + 5) - (2x + 1)]$
  $= (5x + 6)(x + 4)$

- A sum of squares is not factorable.

9. Factor the following differences of squares.

a) $x^2 - 25$ ***(x + 5)(x − 5)***

b) $16x^2 - 9$ ***(4x + 3)(4x − 3)***

c) $49x^2 - 36y^2$ ***(7x + 6y)(7x − 6y)***

d) $36x^4 - 25y^6$ $(6x^2 + 5y^3)(6x^2 - 5y^3)$

e) $100 - x^2$ ***(10 + x)(10 − x)***

f) $\frac{x^2}{16} - \frac{y^2}{9}$ $\left(\frac{x}{4} + \frac{y}{3}\right)\left(\frac{x}{4} - \frac{y}{3}\right)$

g) $x^2 - 3$ $(x + \sqrt{3})(x - \sqrt{3})$

h) $x^2 - 1$ ***(x + 1)(x − 1)***

i) $16x^2 - \frac{1}{9}$ $\left(4x + \frac{1}{3}\right)\left(4x - \frac{1}{3}\right)$

j) $\frac{25}{16}x^2y^4 - \frac{4}{9}z^6$ $\left(\frac{5}{4}xy^2 + \frac{2}{3}z^3\right)\left(\frac{5}{4}xy^2 - \frac{2}{3}z^3\right)$

10. Factor the following differences of squares.

a) $(3x-1)^2-9$ **$(3x+2)(3x-4)$**

b) $(x+1)^2-4$ **$(x+3)(x-1)$**

c) $(2x+5)^2-16x^2$ **$(6x+5)(-2x+5)$**

d) $25x^2-(2x-5)^2$ **$(7x-5)(3x+5)$**

e) $16x^2-(3x+2)^2$ **$(7x+2)(x-2)$**

f) $36x^2-(2-x)^2$ **$(5x+2)(7x-2)$**

g) $(x+3)^2-(2x+5)^2$ **$(3x+8)(-x-2)$**

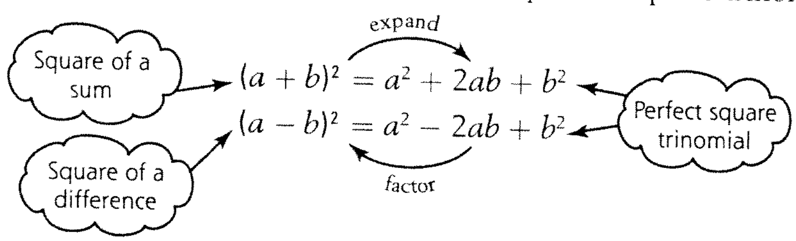
h) $(3x-5y)^2-(2x-3y)^2$ **$(5x-8y)(x-2y)$**

i) $4(x+5)^2-1$ **$(2x+11)(2x+9)$**

j) $25(x-3)^2-9(2x+1)^2$ **$(11x-12)(-x-18)$**

ACTIVITY 4 Perfect square trinomials

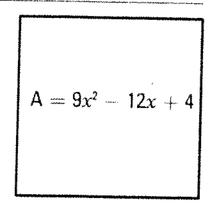
The following remarkable identities enable you to factor a perfect square trinomial.



a) Use the remarkable identities to factor the following perfect square trinomials.

1. x^2+6x+9 **$(x+3)^2$**
2. $x^2-8x+16$ **$(x-4)^2$**
3. $4x^2+20x+25$ **$(2x+5)^2$**
4. $9x^2-12xy+4y^2$ **$(3x-2y)^2$**

b) The square on the right has an area of $A = 9x^2-12x+4$.
What is the length of each side? **$3x-2$**



c) 1. Explain why $4x^2+13x+9$ is not a prefect square trinomial.
$4x^2=(2x)^2$; $9=(3)^2$ but $13x \neq 2(2x)(3)$

2. Explain why $9x^2+30x+25$ is a perfect square trinomial and factor it.
$9x^2=(3x)^2$; $25=(5)^2$ and $30x=2(3x)(5)$. $9x^2+30x+25=(3x+5)^2$.

PERFECT SQUARE TRINOMIALS

- A perfect square trinomial is an algebraic expression of the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$. A trinomial is a perfect square when the middle term is equal to twice the product of the square roots of the end terms.
- Every perfect square trinomial is factorable. You simply need to apply one of the remarkable identities below, depending on the sign of the middle term.

$$a^2+2ab+b^2=(a+b)^2 \quad \text{or} \quad a^2-2ab+b^2=(a-b)^2$$

Ex.: Factor:

- $4x^2+12x+9 = (2x)^2+2(2x)(3)+3^2$ ← We write in the form: $a^2+2ab+b^2$.
  $= (2x+3)^2$ ← We apply the remarkable identity.
- $4x^2-12x+9 = (2x)^2-2(2x)(3)+3^2$ ← We write in the form: $a^2-2ab+b^2$.
  $= (2x-3)^2$ ← We apply the remarkable identity.

11. Factor the following perfect square trinomials.

a) $x^2 + 10x + 25$ **$(x + 5)^2$**

b) $x^2 - 14x + 49$ **$(x - 7)^2$**

c) $4x^2 + 12xy + 9y^2$ **$(2x + 3y)^2$**

d) $25x^2 - 20xy + 4y^2$ **$(5x - 2y)^2$**

e) $9x^4 - 30x^2 + 25$ **$(3x^2 - 5)^2$**

f) $25x^4 + 30x^2y^3 + 9y^6$ **$(5x^2 + 3y^3)^2$**

g) $x^2 - x + \frac{1}{4}$ **$\left(x - \frac{1}{2}\right)^2$**

h) $\frac{9}{16}x^2 + x + \frac{4}{9}$ **$\left(\frac{3}{4}x + \frac{2}{3}\right)^2$**

12. Explain why the following trinomials are not perfect squares.

a) $4x^2 + 6x + 9$ **$6x \neq 2 \times 2x \times 3$**

b) $4x^2 + 12x - 9$ **The term -9 is negative.**

c) $-4x^2 + 12x + 9$ **The term $-4x^2$ is negative.**

d) $9x^2 - 15x + 25$ **$15x \neq 2 \times 3x \times 5$**

13. Complete the trinomials to obtain perfect square trinomials and factor them.

a) $x^2 +$ **[6x]** $+ 9$ **$(x + 3)^2$**

b) $4x^2 -$ **[12x]** $+ 9$ **$(2x - 3)^2$**

c) $9x^2 + 30x +$ **[25]** **$(3x + 5)^2$**

d) **[$25x^2$]** $+ 20x + 4$ **$(5x + 2)^2$**

e) $4x^2 - 28x +$ **[49]** **$(2x - 7)^2$**

f) **[$9x^2$]** $- 6x + 1$ **$(3x - 1)^2$**

g) $x^2 + \frac{2}{3}x +$ **[$\frac{1}{9}$]** **$\left(x + \frac{1}{3}\right)^2$**

h) $x^2 -$ **[7x]** $+ \frac{49}{4}$ **$\left(x - \frac{7}{2}\right)^2$**

14. Complete the factoring of the trinomials of the form $x^2 + bx + c$ (the coefficient of x^2 is equal to 1).

a) $x^2 + 7x + 12 = (x$ **+ 3**$)(x$ **+ 4**$)$

b) $x^2 - 7x + 10 = (x$ **- 2**$)(x$ **- 5**$)$

c) $x^2 + 2x - 15 = (x$ **+ 5**$)(x$ **- 3**$)$

d) $x^2 - 5x - 14 = (x$ **+ 2**$)(x$ **- 7**$)$

e) $x^2 + 14x + 48 = (x$ **+ 8**$)(x$ **+ 6**$)$

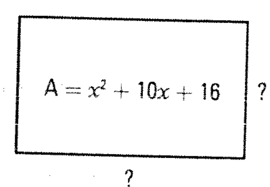
f) $x^2 - 15x + 36 = (x$ **- 12**$)(x$ **- 3**$)$

g) $x^2 - 8x - 33 = (x$ **- 11**$)(x$ **+ 3**$)$

h) $x^2 + 2x - 63 = (x$ **+ 9**$)(x$ **- 7**$)$

ACTIVITY 5 Second degree trinomials $ax^2 + bx + c$

a) The rectangle on the right has an area of $A = x^2 + 10x + 16$.



1. Explain why $x^2 + 10x + 16$ is not a perfect square trinomial.

   $10x \neq 2(x)(4)$

2. Find a method for factoring this trinomial. What could be the possible dimensions of this rectangle?

   $x^2 + 10x + 16 = (x + 2)(x + 8)$. Possible dimensions: $(x + 2)$ and $(x + 8)$

b) The trinomial $2x^2 + 9x + 20$ is not a perfect square. Find a method for factoring this trinomial and factor it.

$2x^2 + 9x + 10 = 2x^2 + 4x + 5x + 10$

$= 2x(x + 2) + 5(x + 2)$

$= (x + 2)(2x + 5)$

SECOND DEGREE TRINOMIALS: $ax^2 + bx + c$

- The "product and sum" method enables you to factor a second degree trinomial. Let us illustrate this method by factoring $P(x) = 2x^2 + 7x + 6$.

1. **Identify** the coefficients a, b and c:	1. $a = 2; b = 7; c = 6$
2. **Find** two integers m and n such that $\begin{cases} m \cdot n = ac \leftarrow \text{product of the end coefficients} \\ m + n = b \leftarrow \text{middle coefficient.} \end{cases}$	2. $\begin{cases} mn = 12 \\ m + n = 7 \end{cases}$ $m = 4, n = 3$
3. Write: $ax^2 + bx + c = ax^2 + mx + nx + c$ and **factor** by **grouping**.	3. $2x^2 + 7x + 6 = 2x^2 + 4x + 3x + 6$ $= 2x(x + 2) + 3(x + 2)$ $= (x + 2)(2x + 3)$

15. Factor the following trinomials using the "product and sum" method.

a) $2x^2 + 9x + 4$ ***(2x + 1)(x + 4)***
b) $6x^2 - 19x + 10$ ***(3x − 2)(2x − 5)***
c) $4x^2 - 5x - 21$ ***(4x + 7)(x − 3)***
d) $5x^2 - 32x - 21$ ***(5x + 3)(x − 7)***
e) $12x^2 + 13x + 3$ ***(3x + 1)(4x + 3)***
f) $16x^2 - 26x + 3$ ***(8x − 1)(2x − 3)***
g) $6x^2 + 11x - 10$ ***(3x − 2)(2x + 5)***
h) $8x^2 + 2x - 15$ ***(2x + 3)(4x − 5)***
i) $x^2 + 10x + 24$ ***(x + 6)(x + 4)***
j) $x^2 - 11x + 30$ ***(x − 6)(x − 5)***

16. Factor the following trinomials.

a) $x^2 - 10x + 21$ ***(x − 3)(x − 7)***
b) $x^2 - 5x - 14$ ***(x − 7)(x + 2)***
c) $x^2 - 7x + 12$ ***(x − 3)(x − 4)***
d) $x^2 - 9x + 20$ ***(x − 5)(x − 4)***
e) $2x^2 + 7x + 3$ ***(2x + 1)(x + 3)***
f) $3x^2 + 5x - 2$ ***(3x − 1)(x + 2)***
g) $6x^2 + x - 2$ ***(2x − 1)(3x + 2)***
h) $10x^2 - 19x + 6$ ***(5x − 2)(2x − 3)***

17. Factor the following trinomials.

a) $x^2 + 8x + 15$ ***(x + 3)(x + 5)***
b) $x^2 - 8x + 15$ ***(x − 3)(x − 5)***
c) $x^2 + 5x - 14$ ***(x + 7)(x − 2)***
d) $x^2 + 9x + 14$ ***(x + 7)(x + 2)***
e) $6x^2 + 19x + 15$ ***(2x + 3)(3x + 5)***
f) $2x^2 - 7x - 15$ ***(2x + 3)(x − 5)***
g) $3x^2 - x - 4$ ***(3x − 4)(x + 1)***
h) $5x^2 - 17x + 6$ ***(5x − 2)(x − 3)***

ACTIVITY 6 Multi-step factoring

Explain the steps in factoring the following polynomials.

a) $2x^3 - 18x = 2x(x^2 - 9)$ ***Remove the common factor***

$= 2x(x + 3)(x - 3)$ ***Difference of two squares***

b) $12x^3 - 12x^2 + 3x = 3x(4x^2 - 4x + 1)$ ***Remove the common factor***

$= 3x(2x - 1)^2$ ***Perfect square trinomial***

c) $4x^3 - 4x^2 - 8x = 4x(x^2 - x - 2)$ ***Remove the common factor***

$= 4x(x + 1)(x - 2)$ ***Non-perfect square trinomial***

d) $x^4 - 16 = (x^2 + 4)(x^2 - 4)$ ***Difference of two squares***

$= (x^2 + 4)(x + 2)(x - 2)$ ***Difference of two squares***

e) $x^4 - 8x^2 + 16 = (x^2 - 4)^2$ ***Perfect square trinomial***

$= [(x + 2)(x - 2)]^2$ ***Difference of two squares***

$= (x + 2)^2(x - 2)^2$ ***Property $(ab)^2 = a^2b^2$***

MULTI-STEP FACTORING

Many steps are sometimes necessary to completely factor a polynomial.

Ex.: $2x^3 - 18x = 2x(x^2 - 9)$ ← remove the common factor

$= 2x(x + 3)(x - 3)$ ← difference of squares

$4x(2x + 3) + 4x^2 - 9 = 4x(2x + 3) + (2x + 3)(2x - 3)$ ← difference of squares

$= (2x + 3)[4x + (2x - 3)]$ ← remove the common factor

$= (2x + 3)(6x - 3)$ ← reduce

$= (2x + 3)\cdot 3(2x - 1)$ ← remove the common factor

$= 3(2x + 3)(2x - 1)$ ← commutative property of multiplication

18. Completely factor the following polynomials.

a) $x^3 + 3x^2 + 2x =$ ***$x(x + 1)(x + 2)$***

b) $x^3 - 2x^2 + x =$ ***$x(x - 1)^2$***

c) $x^3 - 16x =$ ***$x(x + 4)(x - 4)$***

d) $x^4 - 2x^2 + 1 =$ ***$(x + 1)^2(x - 1)^2$***

e) $x^4 - 1 =$ ***$(x^2 + 1)(x + 1)(x - 1)$***

19. Completely factor the following polynomials.

a) $6x^4 + 9x^3 + 3x^2 =$ ***$3x^2(x + 1)(2x + 1)$***

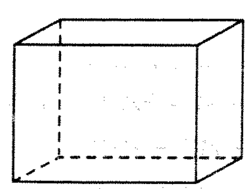
b) $3x^3 - 12x =$ ***$3x(x + 2)(x - 2)$***

c) $16x^4 - 8x^2 + 1 =$ ***$(2x + 1)^2 \cdot (2x - 1)^2$***

d) $-x^2 + 6x - 9 =$ ***$-(x - 3)^2$***

e) $(x^2 - 1) + (x - 1)^2 =$ ***$2x(x - 1)$***

20. The prism on the right has a volume of $V = 2x^3 + 6x^2 + 4x$. What could the dimensions of this prism be?



$2x^3 + 6x^2 + 4x = 2x(x^2 + 3x + 2)$

$= 2x(x + 1)(x + 2)$

The dimensions are: $2x$, $(x + 1)$ and $(x + 2)$.

21. The area of a rectangle is expressed by the polynomial $A(x) = 6x^2 + 17x + 12$. What could be the perimeter of this rectangle?

Perimeter: $10x + 14$; possible dimensions: $(3x + 4)$ and $(2x + 3)$

22. The area of a square is expressed by the polynomial $9x^2 + 12x + 4$. What is the perimeter of this square?

$12x + 8$

23. The volume of a right rectangular prism is expressed by $V(x) = x^3 + 2x^2 - x - 2$. What could the dimensions of the prism be?

$(x + 1)$, $(x - 1)$ and $(x + 2)$

24. The total area of a cube is expressed by $A(x) = 24x^2 + 24x + 6$. What is the volume of this cube?

$8x^3 + 12x^2 + 6x + 1$

25. The volume of a right rectangular prism is expressed by $V(x) = x^3 + 4x^2 + x - 6$. If $(x + 3)$ represents the height of the prism, find two binomials that could express the dimensions of the prism's base.

$(x + 2)$ and $(x - 1)$

26. A sum of two cubes and a difference of two cubes can be factored.

a) Show that

1. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$. ***Expand the right side.***
2. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. ***Expand the right side.***

b) Factor

1. $x^3 + 64$ ***$(x + 2)(x^2 - 2x + 4)$***
2. $8x^3 - 27$ ***$(2x - 3)(4x^2 + 6x + 9)$***
3. $27x^3 - 8y^3$ ***$(3x - 2y)(9x^2 + 6xy + 4y^2)$***

1.5 Rational expressions

ACTIVITY 1 Reducing a rational expression

a) A fraction is undefined when the denominator is zero. What conditions must the variable x respect for the fraction $\frac{x^2-8x+15}{x^2-9}$ to be defined?

$x \neq -3$ and $x \neq 3$

b) Given that the variable x respects the conditions established in a), justify the steps enabling you to reduce the fraction.

1. $\frac{x^2-8x+15}{x^2-9} = \frac{(x-5)(x-3)}{(x+3)(x-3)}$ ***Factor the numerator and denominator.***

$= \frac{x-5}{x+3}$ ***Divide the numerator and denominator by the common factor $(x - 3)$.***

RATIONAL EXPRESSIONS

- A rational expression is an algebraic expression of the form $\frac{P(x)}{Q(x)}$, where P(x) is a polynomial and Q(x) is a non-zero polynomial.

  Ex.: $\frac{5x^2}{2}$; $\frac{2x^2+1}{5x}$; $\frac{x^2-5x+6}{2x-1}$ are rational expressions.

- A rational expression is defined for any value of the variable that does not make the denominator zero.

  Ex.: $\frac{5x^2}{2}$ Restriction: None

  $\frac{2x^2+1}{5x}$ Restriction: $x \neq 0$

  $\frac{x^2-5x+6}{2x-1}$ Restriction: $x \neq \frac{1}{2}$

- To simplify (or reduce) a rational expression, we proceed in the following manner:

  Ex.: Simplify $\frac{x^2-x-2}{x^2-4}$.

  1° State the restrictions: $x \neq -2$; $x \neq 2$.

  2° Factor the numerator and denominator.

  $$\frac{x^2-x-2}{x^2-4} = \frac{(x+1)(x-2)}{(x+2)(x-2)}.$$

  3° Simplify, if possible, considering all restrictions.

  $$\frac{x^2-x-2}{x^2-4} = \frac{(x+1)(x-2)}{(x+2)(x-2)} = \frac{x+1}{x+2}.$$

1. For each of the following rational expressions, indicate the restrictions and simplify if possible.

a) $\frac{5x^2}{20x^3}$ $\frac{1}{4x}$ $(x \neq 0)$

b) $\frac{4x^2-6x}{3x^2+6x}$ $\frac{4x-6}{3x+6}$ $(x \neq 0 \text{ and } x \neq -2)$

c) $\frac{5x+10y}{5x-10y}$ $\frac{x+2y}{x-2y}$ $(x \neq 2y)$

d) $\frac{2x^2+6x}{6x^2+10x}$ $\frac{x+3}{3x+5}$ $\left(x \neq 0 \text{ and } x \neq \frac{-5}{3}\right)$

e) $\frac{6x^3+4x^2}{9x^2+6x}$ $\frac{2x}{3}$ $\left(x \neq 0 \text{ and } x \neq \frac{-2}{3}\right)$

f) $\frac{x^2+3x+2}{x^2+x-2}$ $\frac{x+1}{x-1}$ $(x \neq -2 \text{ and } x \neq 1)$

g) $\frac{2x^2-x-6}{2x^2+5x+3}$ $\frac{x-2}{x+1}$ $\left(x \neq \frac{-3}{2} \text{ and } x \neq -1\right)$

h) $\frac{x^2-9}{x^2+6x+9}$ $\frac{x-3}{x+3}$ $(x \neq -3)$

2. For each of the following rational expressions, indicate the restrictions and simplify if possible.

a) $\frac{x^2-5x}{x^2-25}$ $\frac{x}{x+5}$ $(x \neq 5 \text{ and } x \neq -5)$

b) $\frac{x^4-1}{x^3-x}$ $\frac{x^2+1}{x}$ $(x \neq -1, x \neq 1 \text{ and } x \neq 0)$

c) $\frac{(x+2)^2-9}{x^2-25}$ $\frac{x-1}{x-5}$ $(x \neq -5 \text{ and } x \neq 5)$

d) $\frac{x^2+2x-15}{x^2-9}$ $\frac{x+5}{x+3}$ $(x \neq 3 \text{ and } x \neq -3)$

e) $\frac{2x^2+7x+3}{4x^2-1}$ $\frac{x+3}{2x-1}$ $\left(x \neq \frac{-1}{2} \text{ and } x \neq \frac{1}{2}\right)$

f) $\frac{x^2+5x+6}{x^2+x-2}$ $\frac{x+3}{x-1}$ $(x \neq -2 \text{ and } x \neq 1)$

g) $\frac{x^2-x-6}{2x^2-5x-3}$ $\frac{x+2}{2x+1}$ $\left(x \neq 3 \text{ and } x \neq \frac{-1}{2}\right)$

h) $\frac{x^2-2x+1}{x^2-1}$ $\frac{x-1}{x+1}$ $(x \neq 1 \text{ and } x \neq -1)$

ACTIVITY 2 Multiplying and dividing rational expressions

Explain the steps in the following operations
In each case, the variable does not take values which make the denominators zero.

a) $\frac{x^2-9}{x^2-2x+1} \cdot \frac{x^2-3x+2}{x+3} = \frac{(x+3)(x-3)}{(x-1)^2} \cdot \frac{(x-1)(x-2)}{(x+3)}$ *Factor the numerators and denominators.*

$= \frac{(x+3)(x-3)(x-1)(x-2)}{(x-1)^2(x+3)}$ *Apply the product rule:* $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

$= \frac{(x-3)(x-2)}{(x-1)}$ *Reduce the fraction.*

b) $\frac{x^2-2x+1}{x^2-3x} \div \frac{x-1}{x-3} = \frac{x^2-2x+1}{x^2-3x} \cdot \frac{x-3}{x-1}$ *Apply the rule:* $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

$= \frac{(x-1)^2}{x(x-3)} \cdot \frac{(x-3)}{(x-1)}$ *Factor the numerators and denominators.*

$= \frac{(x-1)^2(x-3)}{x(x-3)(x-1)}$ *Apply the rule:* $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

$= \frac{x-1}{x}$ *Reduce the fraction.*

MULTIPLYING AND DIVIDING RATIONAL EXPRESSIONS

Apply the rules: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ and $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$.

In each of the following operations, the variable does not take values which make the denominators zero.

Ex.: $\frac{x^2 - x}{x + 3} \cdot \frac{x^2 + 5x + 6}{x^2 - 1} = \frac{x(x - 1)}{(x + 3)} \cdot \frac{(x + 2)(x + 3)}{(x + 1)(x - 1)}$

$= \frac{x(x - 1)\,(x + 2)(x + 3)}{(x + 3)(x + 1)(x - 1)}$

$= \frac{x(x + 2)}{x + 1}$

Ex.: $\frac{x^2 - 5x}{x^2 - 6x + 9} \div \frac{x - 5}{x - 3} = \frac{x^2 - 5x}{x^2 - 6x + 9} \cdot \frac{x - 3}{x - 5}$

$= \frac{x(x - 5)}{(x - 3)^2} \cdot \frac{(x - 3)}{(x - 5)}$

$= \frac{x(x - 5)(x - 3)}{(x - 3)^2(x - 5)}$

$= \frac{x}{x - 3}$

3. Perform the following multiplications, given that the variable does not take values which make the denominators zero.

a) $\frac{2x - 4}{x + 3} \times \frac{x^2 + 6x + 9}{x^2 - 4}$ **$\frac{2x + 6}{x + 2}$**

b) $\frac{x^2 - 1}{x + 3} \times \frac{x - 3}{x^2 - 4x + 3}$ **$\frac{x + 1}{x + 3}$**

c) $\frac{2x + 3}{x - 1} \times \frac{x^2 + 2x - 3}{2x^2 - x - 6}$ **$\frac{x + 3}{x - 2}$**

d) $\frac{2x^2 + 6x}{x + 4} \times \frac{x^2 + 8x + 16}{5x^2 + 15x}$ **$\frac{2x + 8}{5}$**

e) $\frac{x^2 + x - 6}{x^2 - 4x - 5} \times \frac{x^2 + 3x + 2}{x^2 - 6x + 8}$ **$\frac{x^2 + 5x + 6}{x^2 - 9x + 20}$**

f) $\frac{2x^2 - 3x - 2}{x^2 - 1} \times \frac{x - 1}{2x + 1}$ **$\frac{x - 2}{x + 1}$**

4. Perform the following divisions, given that the variable does not take values which make the denominators zero.

a) $\frac{x^2 - 1}{x + 2} \div \frac{x - 1}{3x + 6}$ **$3x + 3$**

b) $\frac{x^2 - x - 2}{x^2 - x - 6} \div \frac{x + 1}{x + 2}$ **$\frac{x - 2}{x - 3}$**

c) $\frac{3x^2 + 8x - 3}{x^2 + x - 6} \div \frac{2x + 1}{x - 2}$ **$\frac{3x - 1}{2x + 1}$**

d) $\frac{2x^2 + 2x}{x + 5} \div \frac{2x^3 - 2x}{x^2 + 10x + 25}$ **$\frac{x + 5}{x - 1}$**

e) $\frac{2x-4}{x^2+6x+9} \div \frac{x^2-4}{x^2-9}$ **$\frac{2x-6}{x^2+5x+6}$**

f) $\frac{x^4-1}{x^2+1} \div \frac{x^2-1}{x+2}$ **$x+2$**

ACTIVITY 3 Adding and subtracting rational expressions

a) 1. Justify the steps in the following operations.

$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}$ ***Reducing to a common denominator.***

$=\frac{ad+bc}{bd}$ ***Addition of two fractions that have a common denominator.***

2. Perform the following addition, given that $x \neq -3$ and $x \neq 1$.

$$\frac{2}{x+3}+\frac{3}{x-1}=\frac{2(x-1)}{(x+3)(x-1)}+\frac{3(x+3)}{(x+3)(x-1)}$$
$$=\frac{2x-2+3x+9}{(x+3)(x-2)}$$
$$=\frac{5x+7}{(x+3)(x-1)}$$

b) 1. Justify the steps in the following operations.

$\frac{a}{b}+c=\frac{a}{b}+\frac{bc}{b}$ ***Reducing to a common denominator.***

$=\frac{a+bc}{b}$ ***Adding two fractions that have a common denominator.***

2. Perform the following subtraction, given that $x \neq -2$.

$$\frac{5}{(x+2)}-2=\frac{5}{(x+2)}-\frac{2(x+2)}{(x+2)}$$
$$=\frac{5-2(x+1)}{(x+2)}$$
$$=\frac{1-2x}{x+2}$$

c) Justify the steps in the following operations, given that $x \neq -1$, $x \neq 0$ and $x \neq 1$.

$\frac{2}{x^2-1}-\frac{3}{x^2-x}=\frac{2}{(x+1)(x-1)}-\frac{3}{x(x-1)}$ ***Factor the denominators***

$=\frac{2x}{x(x+1)(x-1)}-\frac{3(x+1)}{x(x+1)(x-1)}$ ***Reduce to a common denominator.***

$=\frac{2x-3(x+1)}{x(x+1)(x-1)}$ ***Subtraction of two fractions that have a common denominator.***

$=\frac{-x-3}{x(x+1)(x-1)}$ ***Reduce.***

ADDING AND SUBTRACTING RATIONAL EXPRESSIONS

Denominators that have no common factor

$$\frac{A(x)}{B(x)} \pm \frac{C(x)}{D(x)} = \frac{A(x)\,D(x) \pm B(x)\,C(x)}{B(x)\,D(x)}$$

Ex.: $\frac{2}{x-1} + \frac{3}{x+3} = \frac{2(x+3)+3(x-1)}{(x-1)(x+3)}$

$= \frac{2x+6+3x-3}{(x-1)(x+3)}$

$= \frac{5x+3}{(x-1)(x+3)}$

Here, the product of the denominators is the common denominator.

Ex.: $\frac{2x}{x+1} - \frac{3x}{x-1} = \frac{2x(x-1)-3x(x+1)}{(x+1)(x-1)}$

$= \frac{2x^2-2x-3x^2-3x}{(x+1)(x-1)}$

$= \frac{-x^2-5x}{(x+1)(x-1)}$

The denominators have a common factor.

Ex.: $\frac{4}{x^2-9} - \frac{2}{(x+3)^2} = \frac{4}{(x+3)(x-3)} - \frac{2}{(x+3)^2}$ Factor the denominators.

$= \frac{4(x+3)}{(x+3)^2(x-3)} - \frac{2(x-3)}{(x+3)^2(x-3)}$ Find the common denominator using the least number of factors.

$= \frac{4(x+3)-2(x-3)}{(x+3)^2(x-3)}$ Reduce to a common denominator.

$= \frac{2x+18}{(x+3)^2(x-3)}$ Simplify.

$(x+3)^2(x-3)$ is the lowest common denominator.

In each of the following operations, the variable does not take values which make the denominators zero.

Perform the following operations.

$\frac{2}{x} + \frac{5}{y}$ $\frac{2y+5x}{xy}$

$\frac{3}{x} - \frac{2}{x^2}$ $\frac{3x-2}{x^2}$

$\frac{x+y}{x} - \frac{x+y}{y}$ $\frac{y^2-x^2}{xy}$

$\frac{2x+3y}{9x} + \frac{2x-3y}{6y}$ $\frac{6x^2+6y^2-5xy}{18xv}$

Perform the following operations.

$\frac{2}{x+1} + \frac{3}{x-2}$ $\frac{5x-1}{(x+1)(x-2)}$

$\frac{3}{x+2} - \frac{2}{x-2}$ $\frac{x-10}{x^2-4}$

$\frac{3x}{x-1} + \frac{2x}{x+1}$ $\frac{5x^2+x}{x^2-1}$

$\frac{x-2}{x+3} - \frac{x-3}{x+2}$ $\frac{5}{(x+3)(x+2)}$

Perform the following operations and simplify your answer.

$\frac{x}{x^2-9} - \frac{1}{2x-6}$ $\frac{1}{2x+6}$

$\frac{1}{x+5} + \frac{1}{x-5}$ $\frac{2x}{x^2-25}$

$\frac{1}{x^2-2x+1} + \frac{1}{x-1}$ $\frac{x}{(x-1)^2}$

$\frac{2a}{3a-15} + \frac{4a}{2a-10}$ $\frac{8a}{3a-15}$

Perform the following additions and subtractions.

$\frac{2x-1}{2} - \frac{3x-1}{5}$ $\frac{4x+7}{10}$

$\frac{5}{x-2} + \frac{3}{x+3}$ $\frac{8x+9}{(x-2)(x+3)}$

$\frac{x+1}{x-2} - \frac{x+2}{x-1}$ $\frac{3}{(x-2)(x-1)}$

$\frac{x+3}{x-3} - \frac{x-3}{x+3}$ $\frac{12x}{x^2-9}$

$\frac{x+1}{x^2-4} - \frac{x-2}{x+2}$ $\frac{-x^2+5x-3}{x^2-4}$

$\frac{x+1}{x^2-2x+1} + \frac{1}{x^2-1}$ $\frac{x^2+3x}{(x-1)^2(x+1)}$

1.6 Second degree equations

Activity 1 Zero product

a) Solve the following 1st degree equations.

1. $2x - 6 =$ ***S = {3}***
2. $3x + 6 = 0$ ***S = {-2}***
3. $5x = 0$ ***S = {0}***
4. $2x - 3 = 0$ ***S =*** $\left\{\frac{3}{2}\right\}$

b) Given the real numbers a and b such that $ab = 0$.
What can be said about the numbers a and b? ***a = 0 or b = 0***

c) For what values of x is the product $(x - 2)(x + 3)$ equal to zero? ***x = 2 or x = -3***

ZERO PRODUCT PRINCIPLE

A product of factors is zero if and only if at least one of the factors is zero.

$$ab = 0 \Leftrightarrow a = 0 \text{ or } b = 0$$

Ex.: $(x - 5)(x + 2) = 0$
$\Leftrightarrow x - 5 = 0 \text{ or } x + 2 = 0$
$\Leftrightarrow x = 5 \text{ or } x = -2$

Thus, the product of the factors $(x - 5)(x + 2)$ is zero if and only if $x = 5$ or $x = -2$.

1. For which values of x are the following products zero?

a) $(x - 1)(x + 3)$ ***1 or -3***
b) $(2x - 1)(3x + 6)$ $\frac{1}{2}$ ***or -2***
c) $5(x + 1)$ ***-1***
d) $2x(x - 5)$ ***0 or 5***
e) $(x + 2)(x - 1)(x - 4)$ ***-2, 1 or 4***
f) $(2x + 1)(3x - 2)(2x + 5)$ $-\frac{1}{2}, \frac{2}{3}$ ***or*** $-\frac{5}{2}$

Activity 2 Solving quadratic equations by factoring

a) Justify the steps which enable you to solve the quadratic equation $x^2 - 8x + 15 = 0$.

$x^2 - 8x + 15 = 0$	
$\Leftrightarrow (x - 3)(x - 5) = 0$	***Factor the non-zero side.***
$\Leftrightarrow x - 3 = 0 \text{ or } x - 5 = 0$	***Apply the zero product principle.***
$\Leftrightarrow x = 3 \text{ or } x = 5$	***Solve each 1st degree equation.***
Thus, $S = \{3, 5\}$.	***Establish the solution set.***

b) Justify the steps which enable you to solve the quadratic equation $2x^2 - 5x = 3$.

$2x^2 - 5x = 3$	
$\Leftrightarrow 2x^2 - 5x - 3 = 0$	***Subtract 3 from each side.***
$\Leftrightarrow (2x + 1)(x - 3) = 0$	***Factor the non-zero side.***
$\Leftrightarrow 2x + 1 = 0$ or $x - 3 = 0$	***Apply the zero product principle.***
$\Leftrightarrow \quad x = -\frac{1}{2}$ or $\quad x = 3$	***Solve each 1st degree equation.***
Thus, $S = \left\{-\frac{1}{2}, 3\right\}$.	***Establish the solution set.***

SOLVING QUADRATIC EQUATIONS BY FACTORING

- We call a second degree equation or quadratic equation in the variable x any equation that can be written in the form: $ax^2 + bx + c = 0, \ a \neq 0$.

  Ex.: $2x^2 + 7x + 3 = 0$ $\quad (a = 2; b = 7; c = 3)$
  $3x^2 - 5x = 0$ $\quad (a = 3; b = -5; c = 0)$
  $x^2 - 4 = 0$ $\quad (a = 1; b = 0; c = -4)$

  are quadratic equations.
- The zero product principle makes it possible to solve quadratic equations by factoring.

1. Write the equation in the general form: $ax^2 + bx + c = 0$.
2. Factor the non-zero side.
3. Apply the zero product principle.
4. Solve each 1st degree equation.
5. Write the solution set.

Ex.: Solve: $x(5 - x) = 6$

$\Leftrightarrow \quad x^2 - 5x + 6 = 0$

$\Leftrightarrow \quad (x - 2)(x - 3) = 0$

$\Leftrightarrow \quad x - 2 = 0$ or $x - 3 = 0$

$x = 2$ or $x = 3$

$S = \{2, 3\}$

2. Solve the following equations by factoring.

a) $x^2 - 10x = 0$ ***S = {0, 10}***

b) $2x^2 + 5x = 0$ ***S = $\left\{0, -\frac{5}{2}\right\}$***

c) $3x^2 = 2x$ ***S = $\left\{0, \frac{2}{3}\right\}$***

d) $x^2 - 9 = 0$ ***S = {-3, 3}***

e) $2x^2 - 50 = 0$ ***S = {-5, 5}***

f) $4x^2 = 1$ ***S = $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$***

3. Solve the following equations by factoring.

a) $x^2 - 2x - 15 = 0$ ***S = {5, -3}***

b) $x(x - 7) = -10$ ***S = {2, 5}***

c) $x^2 = 5x$ ***S = {0, 5}***

d) $4x(x - 3) = -9$ ***S = $\left\{\frac{3}{2}\right\}$***

e) $x(6x + 5) = 6$ ***S = $\left\{-\frac{3}{2}, \frac{2}{3}\right\}$***

f) $16x^2 = 25$ ***S = $\left\{-\frac{5}{4}, \frac{5}{4}\right\}$***

ACTIVITY 3 Form $x^2 = k$

a) Solve the equations

1. $x^2 = 25$ **$S = \{-5, 5\}$** 2. $x^2 = 7$ **$S = \{-\sqrt{7}, \sqrt{7}\}$** 3. $x^2 = -1$ **$S = \emptyset$**

b) What are the solutions to the equation $x^2 = k$ when

1. $k > 0$. **$S = \{-\sqrt{k}, \sqrt{k}\}$** 2. $k = 0$. **$S = \{0\}$** 3. $k < 0$. **$S = \emptyset$**

FORM $x^2 = k$

The equation $x^2 = k$ has the solution set in $\mathbb{R}$

- $S = \emptyset$ if $k < 0$.
- $S = \{0\}$ if $k = 0$.
- $S = \{-\sqrt{k}, \sqrt{k}\}$ if $k > 0$.

4. Solve the following equations referring to the preceding box.

a) $x^2 = 9$ **$S = \{-3, 3\}$** b) $x^2 = 5$ **$S = \{-\sqrt{5}, \sqrt{5}\}$** c) $x^2 = -4$ **$S = \emptyset$**

5. Write the following equations in the form $x^2 = k$ and then solve them.

a) $x^2 - 16 = 0$ **$S = \{-4, 4\}$** b) $x^2 + 9 = 0$ **$S = \emptyset$**

c) $x^2 - 2 = 0$ **$S = \{-\sqrt{2}, \sqrt{2}\}$** d) $2x^2 - 18 = 0$ **$S = \{-3, 3\}$**

e) $3x^2 - 15 = 0$ **$S = \{-\sqrt{5}, \sqrt{5}\}$** f) $2x^2 + 50 = 0$ **$S = \emptyset$**

g) $4x^2 - 9 = 0$ **$S = \left\{-\frac{3}{2}, \frac{3}{2}\right\}$** h) $2x^2 - 1 = 0$ **$S = \left\{-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\}$**

ACTIVITY 4 Form $a(x - h)^2 + k = 0$

a) Justify the steps in the solving of the equation $(x - 3)^2 - 16 = 0$.

$(x - 3)^2 - 16 = 0$ ***Add 16 to each side.***

$\Leftrightarrow (x - 3)^2 = 16$ ***Apply the form $x^2 = k$.***

$\Leftrightarrow x - 3 = -4$ or $x - 3 = 4$ ***Solve each 1st degree equation.***

$\Leftrightarrow x = -1$ or $x = 7$ ***Establish the solution set.***

Thus, $S = \{-1, 7\}$.

b) Justify the steps in the solving of the equation $2(x + 1)^2 - 18 = 0$.

$2(x + 1)^2 - 18 = 0$ ***Add 18 to each side.***

$\Leftrightarrow 2(x + 1)^2 = 18$ ***Divide each side by 2.***

$\Leftrightarrow (x + 1)^2 = 9$ ***Apply the form $x^2 = k$.***

$\Leftrightarrow x + 1 = -3$ or $x + 1 = 3$ ***Solve each 1st degree equation.***

$\Leftrightarrow x = -4$ or $x = 2$ ***Establish the solution set.***

Thus; $S = \{-4, 2\}$.

c) Justify the steps enabling you to conclude that the following equation has no solution.

$3(x+1)^2 + 12 = 0$	***Subtract 12 from each side.***
$\Leftrightarrow 3(x+1)^2 = -12$	***Divide each side by 3.***
$\Leftrightarrow (x+1)^2 = -4$	***The square of a real number cannot be negative.***

Thus $S = \varnothing$.

FORM $a(x-h)^2 + k = 0$

- We use the following method consisting of isolating the variable to solve such an equation.

$2(x+1)^2 - 10 = 0$	
$\Leftrightarrow 2(x-1)^2 = 10$	Add 10 to each side.
$\Leftrightarrow (x-1)^2 = 5$	Divide each side by 2.
$\Leftrightarrow x - 1 = -\sqrt{5}$ or $x - 1 = \sqrt{5}$	Apply the form $x^2 = k$.
$\Leftrightarrow x = 1 - \sqrt{5}$ or $x = 1 + \sqrt{5}$	Solve each 1st degree equation.
Thus, $S = \{-1 - \sqrt{5}, 1 + \sqrt{5}\}$.	Establish the solution set S.

- Note that the equation $a(x-h)^2 + k = 0$ has no solution if a and k have the same sign.

6. Solve the following equations by isolating the variable.

a) $(x-3)^2 = 16$ ***S = {-1, 7}***

b) $(2x+1)^2 = 9$ ***S = {-2, 1}***

c) $(2x-3)^2 - 1 = 0$ ***S = {1, 2}***

d) $2(x+1)^2 - 8 = 0$ ***S = {-3, 1}***

e) $-2(x-1)^2 + 18 = 0$ ***S = {-2, 4}***

7. Show that the equation $a(x-h)^2 + k = 0$ has the solution set

a) $S = \left\{h - \sqrt{\frac{-k}{a}}, h + \sqrt{\frac{-k}{a}}\right\}$ if $\frac{-k}{a} > 0$.

$a(x-h)^2 + k = 0 \Leftrightarrow a(x-h)^2 = -k \Leftrightarrow (x-h)^2 = -\frac{k}{a}$

$\Leftrightarrow x - h = -\sqrt{-\frac{k}{a}}$ or $x - h = +\sqrt{-\frac{k}{a}} \Leftrightarrow x = h - \sqrt{-\frac{k}{a}}$ or $x = h + \sqrt{-\frac{k}{a}}$

b) $S = \{h\}$ if $k = 0$.

The solutions obtained in a) are equal to h when k = 0.

c) $S = \varnothing$ if $\frac{-k}{a} < 0$. $\sqrt{-\frac{k}{a}} \notin \mathbb{R}$ ***when*** $-\frac{k}{a} < 0$. ***Therefore, no real solutions exist.***

SOLVING THE EQUATION $a(x-h)^2 + k = 0$

The equation $a(x-h)^2 + k = 0$ has the solution set

- $S = \left\{h - \sqrt{\frac{-k}{a}}, h + \sqrt{\frac{-k}{a}}\right\}$ if $\frac{-k}{a} > 0$.
- $S = \{h\}$ if $k = 0$.
- $S = \varnothing$ if $\frac{-k}{a} < 0$.

8. Solve the following equations using the formulas stated in the theory box on page 32.

a) $(x-1)^2 + 9 = 0$ **$S = \varnothing$**

b) $2(x-3)^2 = 0$ **$S = \{3\}$**

c) $(2x-5)^2 - 25 = 0$ **$S = \{0, 5\}$**

d) $2(x+1)^2 - 14 = 0$ **$S = \{-1-\sqrt{7}, -1+\sqrt{7}\}$**

e) $-2(2x+1)^2 + 32 = 0$ **$S = \left\{\frac{5}{2}, -\frac{3}{2}\right\}$**

f) $3(x-2)^2 + 27 = 0$ **$S = \varnothing$**

SOLVING QUADRATIC EQUATIONS: THE DISCRIMINANT METHOD

- The discriminant, noted Δ (read delta), of the quadratic equation $ax^2 + bx + c = 0$ is the real number:

$$\Delta = b^2 - 4ac$$

- The existence and number of real solutions depend on the sign of the discriminant Δ.

Sign of Δ	Number of solutions	Solutions
$\Delta > 0$	2 solutions	$x_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b+\sqrt{\Delta}}{2a}$
$\Delta = 0$	1 solution	$x = -\frac{b}{2a}$
$\Delta < 0$	no real solution	

$3x^2 - 11x - 4 = 0$
- $a = 3, b = -11, c = -4$
- $\Delta = b^2 - 4ac$
  $\Delta = (-11)^2 - 4 \times 3 \times -4$
  $\Delta = 169$
- $x_1 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{11-13}{6} = -\frac{1}{3}$
  $x_2 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{11+13}{6} = 4$

Thus, $S = \left\{-\frac{1}{3}, 4\right\}$.

$x^2 - 6x + 9 = 0$
- $a = 1, b = -6, c = 9$
- $\Delta = b^2 - 4ac$
  $\Delta = (-6)^2 - 4 \times 1 \times 9$
  $\Delta = 0$
- $x = \frac{-b}{2a} = \frac{6}{2} = 3$

Thus, $S = \{3\}$.

$x^2 - x + 1 = 0$
- $a = 1, b = -1, c = 1$
- $\Delta = b^2 - 4ac$
  $\Delta = (-1)^2 - 4 \times 1 \times 1$
  $\Delta = -3$

Thus, $S = \varnothing$.

9. Using the sign of the discriminant, indicate the number of solutions to the following equations.

a) $2x^2 + 3x - 2 = 0$ **$\Delta = 25$; 2 solutions**

b) $-2x^2 - 5x + 3 = 0$ **$\Delta = 49$, 2 solutions**

c) $4x^2 + 12x + 9 = 0$ **$\Delta = 0$; 1 solution**

d) $-x^2 + x - 1 = 0$ **$\Delta = -3$; 0 solution**

e) $x^2 - 6x = 0$ **$\Delta = 36$; 2 solutions**

f) $2x^2 - 8 = 0$ **$\Delta = 64$; 2 solutions**

g) $x(x-3) = -2$ **$\Delta = 1$; 2 solutions**

h) $-x^2 + 2x - 1 = 0$ **$\Delta = 0$; 1 solution**

10. Solve the equations of the preceding exercise using the discriminant method.

a) **$S = \left\{-2, \frac{1}{2}\right\}$**

b) **$S = \left\{\frac{1}{2}, -3\right\}$**

c) **$S = \left\{-\frac{3}{2}\right\}$**

d) **$S = \varnothing$**

e) **$S = \{0, 6\}$**

f) **$S = \{-2, 2\}$**

g) **$S = \{1, 2\}$**

h) **$S = \{1\}$**

11. Solve the following equations using the most appropriate method.

a) $2x^2 - x - 10 = 0$ ***$S = \left\{-2, \frac{5}{2}\right\}$***

b) $x^2 - 8x + 15 = 0$ ***$S = \{3, 5\}$***

c) $x^2 = x$ ***$S = \{0, 1\}$***

d) $x^2 = 9$ ***$S = \{-3, 3\}$***

e) $2x^2 + 5x = 0$ ***$S = \left\{0, -\frac{5}{2}\right\}$***

f) $x^2 - 6x = -5$ ***$S = \{1, 5\}$***

g) $(2x + 1)^2 = 9$ ***$S = \{-2, 1\}$***

h) $\frac{3}{2}x^2 - 9x + 12 = 0$ ***$S = \{2, 4\}$***

i) $3(x - 1)^2 - 12 = 0$ ***$S = \{-1, 3\}$***

j) $-2(x + 1)^2 + 6 = 0$ ***$S = \{-1 - \sqrt{3}, -1 + \sqrt{3}\}$***

12. Show that when the solutions to the quadratic equation $ax^2 + bx + c = 0$ exist,

a) the sum of the solutions is equal to $-\frac{b}{a}$.

$\frac{-b - \sqrt{\Delta}}{2a} + \frac{-b + \sqrt{\Delta}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$

b) the product of the solutions is equal to $\frac{c}{a}$.

$\left(\frac{-b - \sqrt{\Delta}}{2a}\right)\left(\frac{-b + \sqrt{\Delta}}{2a}\right) = \frac{b^2 - \Delta}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$

13. Solve the following equations.

a) $(2x + 1)^2 = (x - 3)^2$ ***$S = \left\{-4, \frac{2}{3}\right\}$***

b) $(2x + 3)(x - 2) = 9$ ***$S = \left\{-\frac{5}{2}, 3\right\}$***

c) $x^2 + 2x - 1 = 0$ ***$S = \{-1 - \sqrt{2}, -1 + \sqrt{2}\}$***

d) $x^3 + 2x^2 - 3x = 0$ ***$S = \{0, 1, -3\}$***

14. The length of a rectangular field measures 5 m more than twice its width. If the total area is equal to 250 m^2, what is the perimeter of the field? ***70 m***

15. The height $h(t)$ of a projectile, measured from ground level, is given by $h(t) = -t^2 + 8t$, where t represents the elapsed time in seconds since it was launched.

a) Can the projectile hit a target located at a height of 20 m?

No, the equation $-t^2 + 8t = 20$ has no solution, since $\Delta = -16$.

b) At what instant does the projectile hit the ground? ***At $t = 8$ s.***

c) At what instant, during the projectile's ascent, is a target hit if it is located 15 m above the ground? ***At $t = 3$ s.***

16. The value $v(t)$, in cents, of a share is given by $v(t) = 2t^2 - 16t + 40$ where t represents the number of weeks since the share's purchase.

a) After how many weeks is the share worth 58 ¢? ***After 9 weeks***

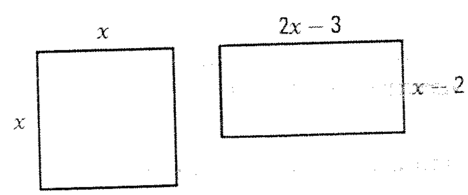
b) Can the share reach a value of 6 ¢? Justify your answer.

No, the equation $2t^2 - 16t + 34 = 0$ has no solution since $\Delta = -16$.

17. A mother is presently 5 years older than twice her daughter's age. Ten years ago, the product of the mother and daughter's ages was equal to 125. What is the mother's present age?

35 years old

18. The square and the rectangle on the right have the same area. What is the numerical value of the rectangle's perimeter?



x = 6; Perimeter = 26 units

19. In a triangle, the height relative to its base is 2 cm less than that base. Determine this height if the area of the triangle is equal to 12 cm^2.

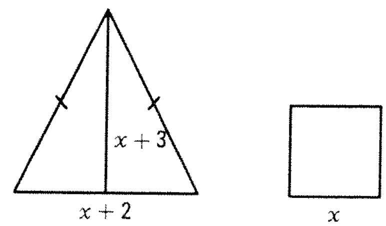
Height: 4 cm

20. The height of a cylinder measures 4 cm more than its radius. Find the height of this cylinder if its total area is equal to 140 π cm^2.

Height: 9 cm

21. An isosceles triangle and a square have the same area.
What is the numerical value of the triangle's perimeter?

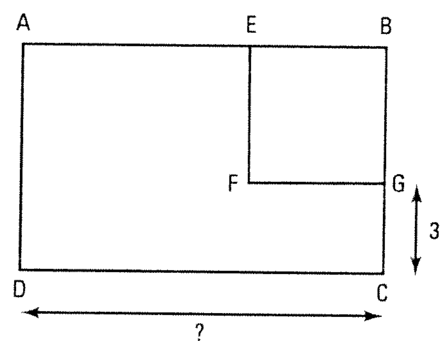
27.7 u



22. Consider the figure on the right. The area of square EBGF is represented by the polynomial $x^2 + 8x + 16$.

If the area of rectangle ABCD is represented by the polynomial $2x^2 + 13x - 7$, what is the length of the rectangle?

2x - 1



1.7 Second degree inequalities

ACTIVITY 1 Sign of a 1st degree binomial

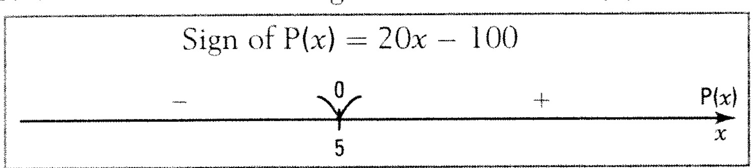
Frank rents a booth at a flea market to sell watches. The rental fees are \$100 per day. Frank makes a \$20 profit per watch sold.

a) Determine the binomial $P(x)$ which gives the day's profit as a function of the number x of watches sold. ***$P(x) = 20x - 100$***

b) What must be the number x of watches sold for the day's profit $P(x)$ to be

1. zero. ***$x = 5$*** 2. positive. ***$x > 5$*** 3. negative. ***$x < 5$***

c) It is useful to show the sign of the binomial $P(x)$ on the real number line.



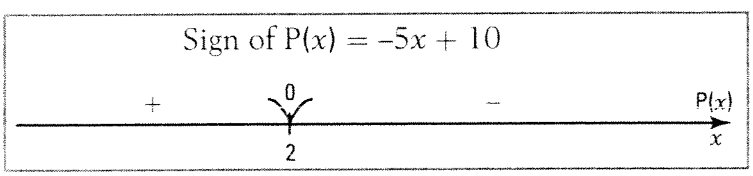
We place 0 above 5 to indicate that $P(x)$ is zero when $x = 5$.

- Under the axis, we write the zero 5 which divides the real number line into two intervals $]-\infty, 5[$ and $]5, +\infty[$.
- Above the axis, we write the sign of the binomial in each interval. Thus, the profit is strictly negative fo $x < 5$, zero for $x = 5$ and strictly positive for $x > 5$.

d) Consider the binomial $P(x) = -5x + 10$.

1. What is the zero of $P(x)$? ***2***
2. In what interval is

   $P(x) > 0$? ***$]-\infty, 2[$*** $P(x) < 0$? ***$]2, +\infty[$***
3. Show the sign of the binomial $P(x)$ on the real number line.



ACTIVITY 2 Sign of a second degree trinomial

The manager of a company has established that the production cost $C(x)$ and the revenue $R(x)$ from selling a roll of fabric with a length of x metres is:

$$C(x) = 5x + 6 \qquad R(x) = x^2 + 10$$

a) Determine the trinomial $P(x)$ which gives the profit from selling a length of x metres of this roll of fabric.

$P(x) = R(x) - C(x) = x^2 - 5x + 4$

b) What must be the length of fabric from this roll for the profit $P(x)$ to be zero?

$P(x) = 0 \Leftrightarrow x^2 - 5x + 4 = 0 \Leftrightarrow x = 1$ or $x = 4$ The roll must have a length of 1 m or 4 m.

c) The manager will have a gain or a loss depending on the sign of the trinomial P(x). Since it is easier to study the sign of a product than that of a sum, factor the trinomial P(x).

$P(x) = x^2 - 5x + 4 = (x - 1)(x - 4)$

d) The following table shows the sign of $P(x) = x^2 - 5x + 4$.

x	$-\infty$		1		4		$+\infty$
$x - 1$		$-$	0	$+$		$+$	
$x - 4$		$-$		$-$	0	$+$	
$x^2 - 5x + 4$		$+$	0	$-$	0	$+$	

The product of the signs of the 1st line with the signs of the 2nd line gives the sign of the trinomial in each interval on the 3rd line.

- The zeros 1 and 4 of the trinomial P(x) divide the real number line in 3 intervals: $]-\infty, 1[$, $]1, 4[$ and $]4, +\infty[$.
- In the interval $]-\infty, 1[$, we notice that the factor $(x - 1)$ is strictly negative, the factor $(x - 4)$ is strictly negative and therefore P(x) is strictly positive.

Complete the following statements

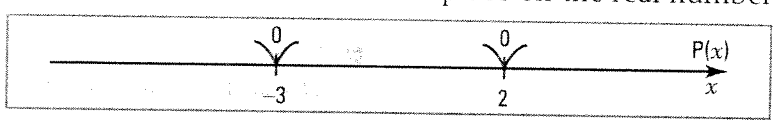
1. the profit is zero if $x =$ ***1*** or if $x =$ ***4***.
2. the profit is positive if $x \leq$ ***1*** or $x \geq$ ***4***.
3. the profit is negative if ***1*** $\leq x \leq$ ***4***.

The preceding table shows that the sign of a polynomial does not change in each of the intervals delimited by the zeros of the polynomial.

SIGN OF A SECOND DEGREE TRINOMIAL – NUMBER LINE METHOD

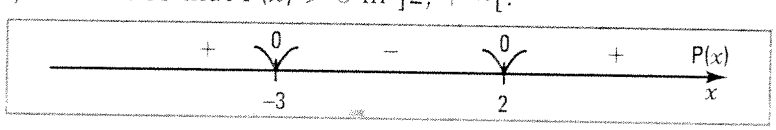
To determine the sign of the trinomial $P(x) = x^2 + x - 6$.

1. We determine the zeros of the trinomial that we place on the real number line.



The zeros of the trinomial thus delimit the 3 intervals $]-\infty, -3[$, $]-3, 2[$ and $]2, +\infty[$.

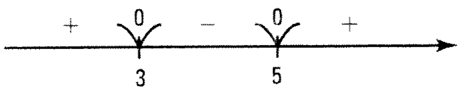
2. Since the sign of the trinomial does not change (see activity 2) in each of these intervals, we determine the sign of the trinomial in each interval by evaluating the trinomial for a value of x randomly chosen from each interval.
   - Since $P(-4) = 6$, we deduce that $P(x) > 0$ in $]-\infty, -3[$.
   - Since $P(0) = -6$, we deduce that $P(x) < 0$ in $]-3, 2[$.
   - Since $P(3) = 6$, we deduce that $P(x) > 0$ in $]2, +\infty[$.



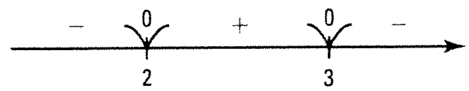
Thus, $P(x) \geq 0$ if $x \in]-\infty, -3] \cup [2, +\infty[$ and $P(x) \leq 0$ if $x \in [-3, 2]$.

1. Determine, using the number line method, the sign of the following trinomials.

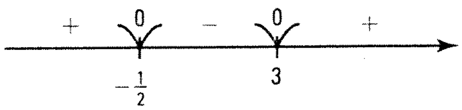
a) $P(x) = x^2 - 8x + 15$



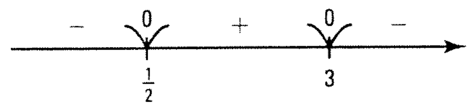
b) $P(x) = -x^2 + 5x - 6$



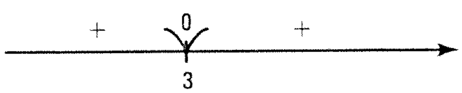
c) $P(x) = 2x^2 - 5x - 3$



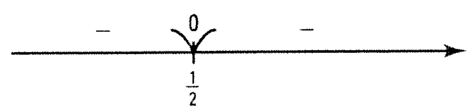
d) $P(x) = -2x^2 + 7x - 3$



e) $P(x) = x^2 - 6x + 9$



f) $P(x) = -4x^2 + 4x - 1$



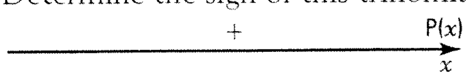
2. The profit $P(x)$, in millions of dollars, of a company depends on the number x of units sold in the month. This profit is given by $P(x) = -0.5x^2 + 40x - 750$. Determine the interval in which the number of units sold must be for the company's profit to be strictly positive.

***x* ∈]30, 50[**

ACTIVITY 3 Sign of a trinomial that has no zeros

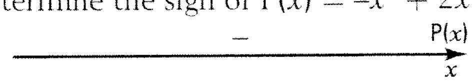
a) 1. Explain why the trinomial $P(x) = x^2 - x + 1$ has no zeros. **Δ = −3**

2. Since the trinomial $x^2 - x + 1$ has no zeros, the real number line is therefore considered as one single interval in which the sign of the trinomial $x^2 - x + 1$ does not change. Determine the sign of this trinomial.



The trinomial is therefore strictly positive for all x.

b) Determine the sign of $P(x) = -x^2 + 2x - 3$.



Δ = −8; P(0) = −3 The trinomial is therefore strictly negative for all x.

ACTIVITY 4 Second degree inequalities

a) Solving the second degree inequality $x^2 - 6x + 8 < 0$ involves finding the values of x for which the trinomial $P(x) = x^2 - 6x + 8$ is strictly negative.

1. Study the sign of the trinomial $P(x) = x^2 - 6x + 8$.

2. Deduce the solution set S. **S =]2, 4[**

b) Referring to the sign of the trinomial $P(x) = x^2 - 6x + 8$ studied in question a), solve

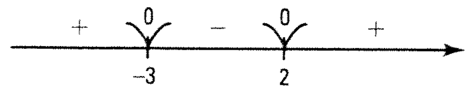
1. $x^2 - 6x + 8 \leq 0$. **S = [2, 4]**

2. $x^2 - 6x + 8 > 0$. **S =]−∞, 2 [∪] 4, +∞[**

3. $x^2 - 6x + 8 \geq 0$. **S =]−∞, 2] ∪ [4, +∞[**

SOLVING A SECOND DEGREE INEQUALITY – NUMBER LINE METHOD

- Solving the inequality $x^2 + x - 6 < 0$ consists of finding the values of x for which the trinomial $x^2 + x - 6$ is strictly negative.
  1. We study the sign of the trinomial using a number line.
  2. We deduce the solution set of the inequality.



S =]–3,2[

- Note that
  $x^2 + x - 6 \leqslant 0$ has the solution set S = [–3, 2].
  $x^2 + x - 6 > 0$ has the solution set S =]–∞, –3[U]2, +∞[.
  $x^2 + x - 6 \geqslant 0$ has the solution set S =]–∞, –3] U [2, +∞[.

3. Solve the following inequalities.

a) $2x^2 - x - 6 > 0$]–∞, $\frac{-3}{2}$[U] 2, +∞[

b) $-6x^2 + 11x - 4 \geqslant 0$ [$\frac{1}{2}$, $\frac{4}{3}$]

c) $x^2 + x - 3 > 0$ ℝ

d) $x^2 - 5x < 0$]0, 5[

4. Solve the following inequalities.

a) $x^2 \leqslant x$ [0, 1]

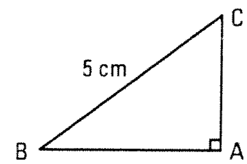
b) $x^2 \geqslant 16$]–∞, –4] U [4, +∞[

c) $x(x - 3) + 3 > x$]–∞, 1 [U] 3, +∞[

d) $x(x - 2) > 3$]–∞, –1 [U] 3, +∞[

5. The hypotenuse of the right triangle measures less than 5 cm. If side AB measures 1 cm more than side AC, find the possible measures for side AC.

The side AC must measure more than 3 cm.



6. The product of two consecutive numbers is at most equal to 12. In what interval is the smaller number located? [–4, 3]

7. From the top of a seaside cliff, a stone is thrown upward. The polynomial $h(t) = -2t^2 + 12t + 10$ represents the height h (in m) of the stone relative to sea level as a function of elapsed time t in seconds. Between what instants is the height of the stone over 26 m relative to the sea level?

Between the instants t = 2s and t = 4s.

8. The value V(x), in dollars, of a share x months after its purchase is given by $V(x) = -0.1x^2 + x + 1.90$. Determine the interval of time for which the value of the share is greater or equal to \$3.50.

x ∈ [2, 8]. Between 2 and 8 months after its purchase.

9. For which values of m does the equation $x^2 + mx + m = 0$ have

a) two solutions? $m \in$]–∞, 0 [U] 4, +∞[

b) one unique solution? $m = 0$ or $m = 4$

c) no solution? $m \in$]0, 4[

Evaluation 1

1. Expand the following expressions using the appropriate identity.

a) $(3x^2 + 2x)^2$ **$9x^4 + 12x^3 + 4x^2$**

b) $(2x^3 - 3y^2)^2$ **$4x^6 - 12x^3y^2 + 9y^4$**

c) $(3a^2 + 2b)(3a^2 - 2b)$ **$9a^4 - 4b^2$**

d) $(2x + 1)(2x - 1)(4x^2 + 1)$ **$16x^4 - 1$**

2. A rectangular base prism has the volume $V(x) = 4x^3 + 6x^2 - 16x - 24$. The dimensions of the prism's base are $(2x + 3)$ and $(x - 2)$. Determine the height of this prism. **$2x + 4$**

3. Factor the following polynomials.

a) $12a^3b^2 - 18a^2b^4$ **$6a^2b^2(2a - 3b^2)$**

b) $(2x + 3)(x - 2) - (2x + 3)(2x + 1)$ **$(2x + 3)(-x -3)$**

c) $(2x + 1)^2 + (2x + 1)(x - 5)$ **$(2x + 1)(3x - 4)$**

d) $6a^2 - 4ac + 9ab - 6bc$ **$(2a + 3b)(3a - 2c)$**

e) $16x^4 - 25y^2$ **$(4x^2 + 5y)(4x^2 - 5y)$**

4. Factor the following trinomials.

a) $x^2 + 2x - 15$ **$(x - 3)(x + 5)$**

b) $2x^2 - x - 6$ **$(2x + 3)(x - 2)$**

c) $9x^2 - 30xy + 25y^2$ **$(3x - 5y)^2$**

d) $4x^4 + 16x^3 + 16x^2$ **$4x^2 (x + 2)^2$**

5. Factor the following expressions.

a) $16x^2 - (2x - 1)^2$ **$(6x - 1)(2x + 1)$**

b) $x^4 - 18x^2 + 81$ **$(x + 3)^2 \cdot (x - 3)^2$**

c) $x^4 - 81$ **$(x + 3)(x - 3)(x^2 + 9)$**

d) $6x^3 - 4x^2 - 2x$ **$2x(3x + 1)(x - 1)$**

6. Simplify the following rational expressions after indicating the restrictions on the variable.

a) $\dfrac{(x + 1)^2 - 16}{x^2 + 8x + 15}$ **$\dfrac{x - 3}{x + 3}$, $x \neq -5$ and $x \neq -3$**

b) $\dfrac{2xy + 10x + 3y + 15}{y^2 - 25}$ **$\dfrac{2x + 3}{y - 5}$, $y \neq -5$ and $y \neq 5$**

c) $\dfrac{8ab - 6a + 8b - 12}{4b - 6}$ **$a + 2$, $b \neq \frac{3}{2}$**

d) $\dfrac{2x^3 + 4x^2 - 6x}{2x^2 + 6x}$ **$x - 1$, $x \neq 0$; $x \neq -3$**

7. Perform the following operations given that variables satisfy the restrictions.

a) $\dfrac{x}{x - 3} + \dfrac{2x - 6}{x^2 - 6x + 9} =$ **$\dfrac{x + 2}{x - 3}$**

b) $\dfrac{x^2 + 10x + 25}{2x^2 + 9x - 5} \times \dfrac{3}{3x + 15} =$ **$\dfrac{1}{2x - 1}$**

c) $\dfrac{4x^2 - 12x + 9}{3x^2 - 5x} \div \dfrac{2x - 3}{3x - 5} =$ **$\dfrac{2x - 3}{x}$**

d) $\dfrac{x^2}{2y - 2} \div \dfrac{x}{y^2 - 1} =$ **$\dfrac{x(y + 1)}{2}$**

8. Solve the following equations.

a) $(2x-1)(x-3)=0$ **$S=\left\{\frac{1}{2}, 3\right\}$**

b) $x^2 = x$ **$S=\{0, 1\}$**

c) $4x^2 - 9 = 0$ **$S=\left\{-\frac{3}{2}, \frac{3}{2}\right\}$**

d) $x^2 + 1 = 0$ **$S=\varnothing$**

e) $(x-3)^2 - 4 = 0$ **$S=\{1, 5\}$**

f) $2(x-1)^2 - 8 = 0$ **$S=\{3, -1\}$**

9. Solve the following equations.

a) $2x^2 - 9x - 5 = 0$ **$S=\left\{-\frac{1}{2}, 5\right\}$**

b) $4x^2 - 6x + 9 = 0$ **$S=\left\{\frac{3}{2}\right\}$**

c) $x^2 - 2x + 3 = 0$ **$S=\varnothing$**

d) $(x+1)(x+2) = 6$ **$S=\{-4, 1\}$**

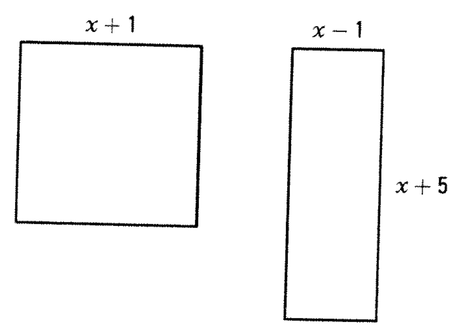
e) $(2x+3)^2 = (x-2)^2$ **$S=\left\{-\frac{1}{3}, -5\right\}$**

f) $2x^3 + 6x^2 + 4x = 0$ **$S=\{0, -1, -2\}$**

10. The square and the rectangle on the right have the same area. Determine the perimeter of the rectangle.

$(x+1)^2 = (x-1)(x+5)$

$x = 3 \Rightarrow$ Perimeter of the rectangle = 20 units



11. Solve the following inequalities.

a) $x^2 \geq x$ **$]-\infty, 0] \cup [1, +\infty[$**

b) $x^2 \geq 9$ **$]-\infty, -3] \cup [3, +\infty[$**

c) $x^2 + 3x - 4 \geq 0$ **$]-\infty, -4] \cup [1, +\infty[$**

d) $-x^2 + 5x - 6 \geq 0$ **$[2, 3]$**

e) $x^2 - 6x + 9 \leq 0$ **$\{3\}$**

f) $x^2 - x + 1 \geq 0$ $\mathbb{R}$

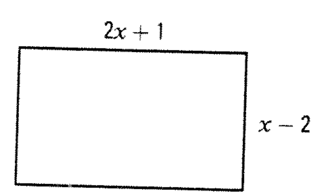
12. The polynomial $P(x) = -x^2 + 6x + 4$ enables you to calculate the price $P(x)$ of a share x months after its purchase.

a) What is the share's purchase value? **$4**

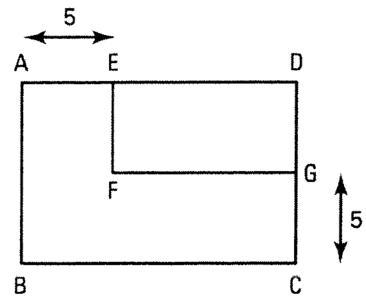
b) If the value of the share is greater than $9, in what interval must the number of elapsed months since the share's purchase be? **$]1, 5[$**

13. Determine the interval in which the variable x must be located for the area of the rectangle on the right to be greater than $18\ u^2$?

$x \in]4, +\infty[$



14. Consider the rectangles ABCD and DEFG represented on the right. Knowing that $m\overline{AE} = m\overline{CG} = 5$ units and that the polynomial $6x^2 + 5x - 25$ represents the area of rectangle ABCD, determine the polynomial which represents the area of rectangle DEFG.



$6x^2 + 5x - 25 = (3x - 5)(2x + 5)$

Dimensions of DEFG: $(3x - 10)$ and $2x$

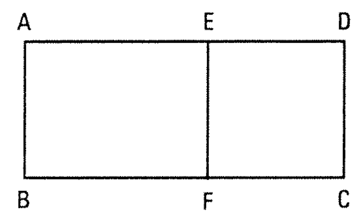
Area of rectangle DEFG = $6x^2 - 20x$

15. The polynomial $2x^3 + 11x^2 + 15x$ represents the volume of a rectangular base prism. The dimensions of the prism's base are x and $(x + 3)$.

What binomial represents the height of the prism? ***$2x + 5$***

16. Consider the rectangle ABFE and the square CDEF on the right. Segment AE measures 2 units more than segment ED.

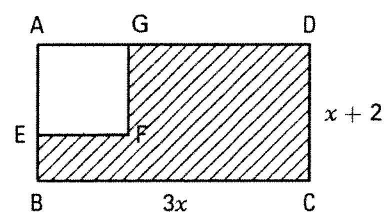
What is the numerical value of the area of rectangle ABFE if the area of rectangle ABCD is equal to 40 cm^2?



• $m\overline{ED} = x$; $m\overline{AE} = x + 2$; Area of ABCD = $x(2x + 2) = 2x^2 + 2x$

$2x^2 + 2x = 40 \Rightarrow x = 4 \Rightarrow$ Area of ABFE = 24 cm^2

17. The dimensions of the rectangle ABCD on the right are $3x$ and $(x + 2)$. The quadrilateral AEFG is a square with an area of 16 cm^2. The area of the shaded region is equal to $(2x^2 + 6x)$ cm^2. What is the numerical value of the area of rectangle ABCD?



$3x(x + 2) = 2x^2 + 6x + 16 \Rightarrow x = 4$

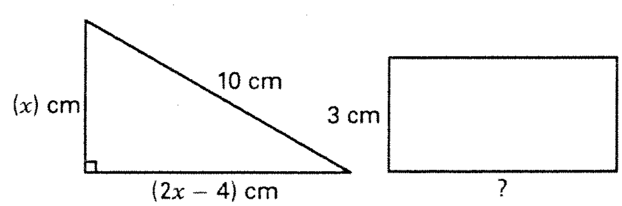
Area of rectangle ABCD = $12 \times 6 = 72$ cm^2

18. The polynomial $h(t) = -t^2 + 6t + 6$ enables you to calculate the height $h(t)$, in metres, of an object t seconds after it is launched. Between what instants after its launch does the object reach a height greater than 14 m?

Between the instants $t = 2$s and $t = 4$s.

19. The given right triangle and rectangle have the same area.

What is the numerical value of the rectangle's length?



$x = 6$; length = 8 cm