Chapter 2

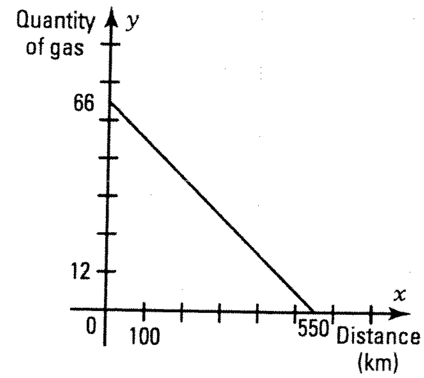
Functions

CHALLENGE 2

EVALUATION 2

CHALLENGE 2

1. At the beginning of a car ride, the gas tank contains 66 litres. The car consumes 12 litres per 100 km.



a) What is the rule of the function f which gives the remaining quantity of gas in the tank as a function of the number of kilometers traveled?
y = −0.12x + 66

b) Represent function f in the Cartesian plane.

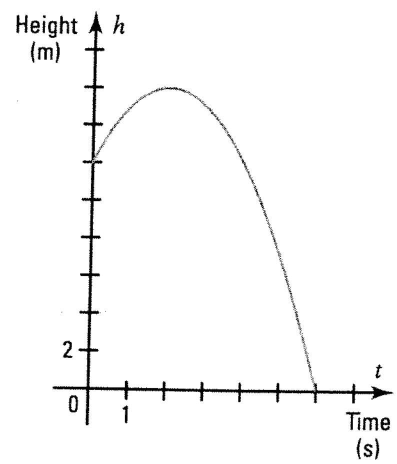
c) Determine and interpret the zero of function f.
550. After 550 km, the tank is empty.

d) Determine
1. the domain of f. ***[0, 550]***
2. the range of f. ***[0, 66]***

e) Is function f increasing or decreasing? ***Decreasing***

f) Determine
1. the maximum of f. ***66***
2. the minimum of f. ***0***

2. A stone is thrown from the top of a seaside cliff. On the right, we have graphed the function f which gives the height h (in metres) of the stone as a function of the time (in seconds) since it was thrown until it hits the water.



a) Determine
1. the domain of f. ***[0, 6]***
2. the range of f. ***[0, 16]***

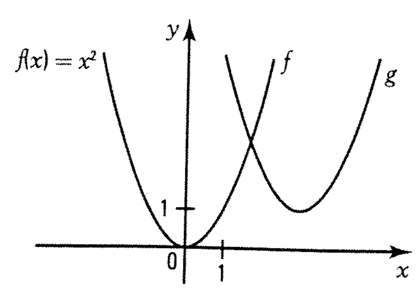
b) Determine and interpret the zero of f. ***6***
The stone hits the water at the instant t = 6 sec.

c) Determine and interpret the initial value of f. ***12***
Initially, the stone is thrown from a height of 12 m.

d) Determine and interpret the maximum of f. ***Max f = 16***
The maximum height reached by the stone is 16 m.

e) Describe the variation of f. ***f is increasing over [0, 2]; f is decreasing over [2, 6]***

3. The function f on the right has the rule $f(x) = x^2$.



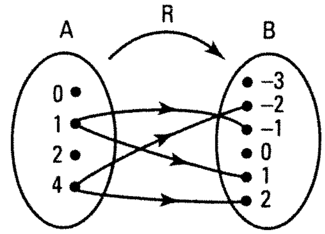
a) Which transformation will associate the graph of function f to the graph of function g?
$(x, y) \rightarrow (x + 3, y + 1)$

b) What is the rule of function g?
$g(x) = (x - 3)^2 + 1$

2.1 Properties of functions

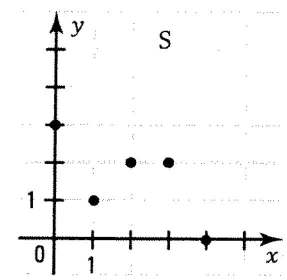
ACTIVITY 1 Recognizing a function

a) Consider the mapping diagram of the relation R represented on the right.



1. What is the source set? ***A = {0, 1, 2, 4}***
2. What is the target set? ***B = {−3, −2, −1, 0, 1, 2}***
3. Describe the rule of the relation R. ***An element x from set A is in relation with an element of y from set B if x is the square of y.***
4. Is there an element from the source set that is in relation with more than one element from the target set? ***Yes***
5. Is this relation a function? Justify your answer.
No, since 1 is in relation with two elements.

b) Consider the Cartesian graph of the relation S represented on the right.



1. What is the image of 3? ***2***
2. What is the antecedent of 3? ***0***
3. Is there an element from the source set that is in relation with more than one element from the target set? ***No***
4. Is this relation a function? Justify your answer.
Yes, since each element from the source set has at most one image.

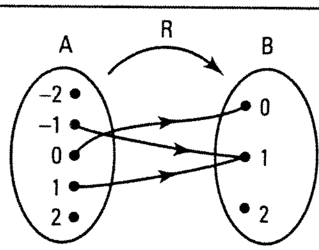
FUNCTION

A relation from source set A to a target set B is a function if each element from A is associated with at most one element from B.

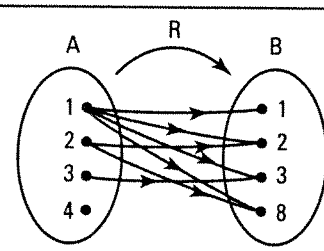
- **Mapping diagram**

  Given the mapping diagram of a relation, this relation is a function if, from each element of the source set, at most one arrow is drawn.

Ex.:



The relation "is the square of" is a function.
The ordered pair (−1, 1) verifies this relation.
We say that 1 is the image of −1 and we write: $f(-1) = 1$. We also say that −1 is the antecedent of 1.

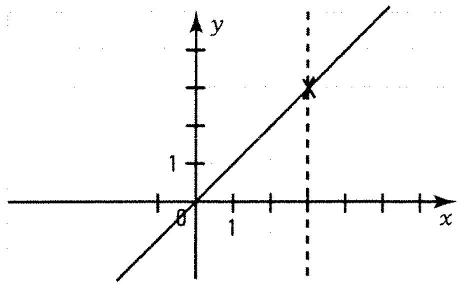


The relation "is a divisor of" is not a function.
The element 2 from source set A is in relation with 2 elements 2 and 8 from target set B.

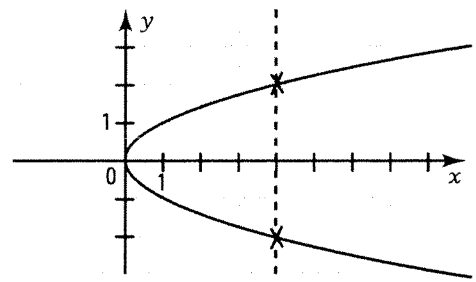
- **Cartesian graph**

  Given the Cartesian graph of a relation, this relation is a function if any vertical line intersects the graph of this relation in at most one point.

  Ex.:



The relation $y = x$ is a function



The relation $x = y^2$ is not a function.

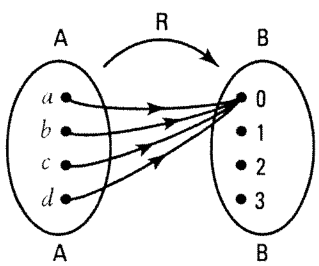
- **Set of ordered pairs**

  Given a relation's set of ordered pairs, this relation is a function if the first coordinate of each pair verifying the relation appears only once.

  Ex.: The relation whose set of ordered pairs is: $\{(0, 1), (1, 1), (2, 8), (3, 27)\}$ is a function.
  The relation whose set of ordered pairs is: $\{(0, 0), (1, -1), (1, 1\ \}$ is not a function.

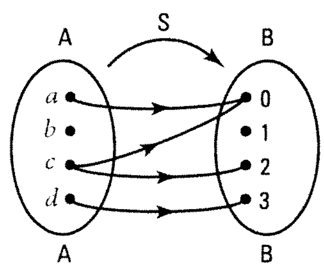
1. In each of the following cases, indicate if the relation is a function. If it is not, explain why.

a)



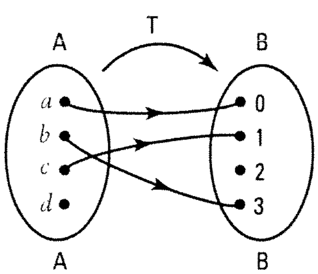
Yes

b)



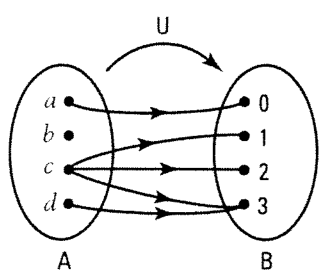
No, since c is in relation with 2 elements: 0 and 2.

c)



Yes

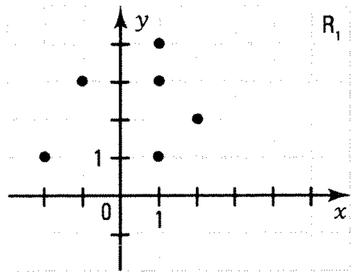
d)



No, since c is in relation with 3 elements: 1, 2 and 3.

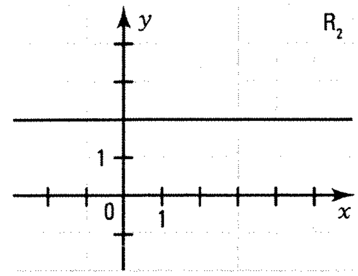
2. Determine if the following relations are functions.

a)



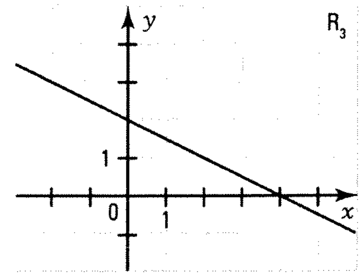
No

b)



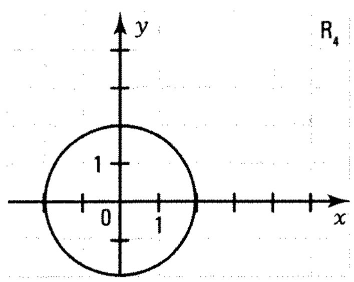
Yes

c)



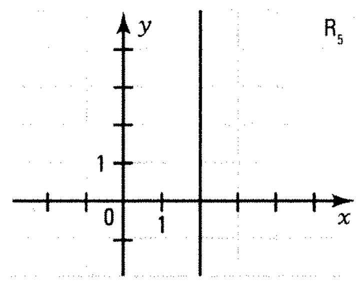
Yes

d)



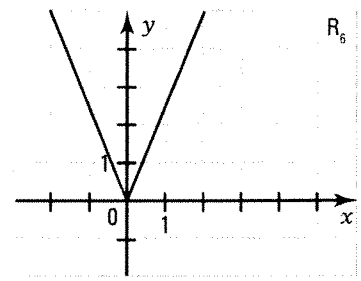
No

e)



No

f)



Yes

3. In each of the following cases, determine the relations that are not functions. Justify your answer.

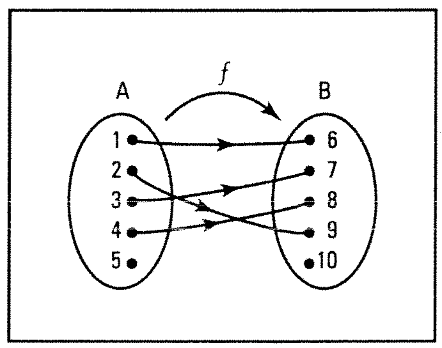
a) $R_1 = \{(0, a), (2, b), (3, a), (4, b)\}$ ***Yes***

b) $R_2 = \{(-1, 4), (0, 2), (0, 7), (5, 8)\}$ ***No, since 0 is in relation with 2 elements: 2 and 7.***

c) $R_3 = \{(5, 8), (6, 8), (7, 8), (8, 8)\}$ ***Yes***

d) $R_4 = \{(4, -2), (1, -1), (0, 0), (1, 1)\}$ ***No, since 1 is in relation with 2 elements: –1 and 1.***

4. True or false?



a) 7 is the image of 3 by f. ***True***

b) 2 is the image of 9 by f. ***False***

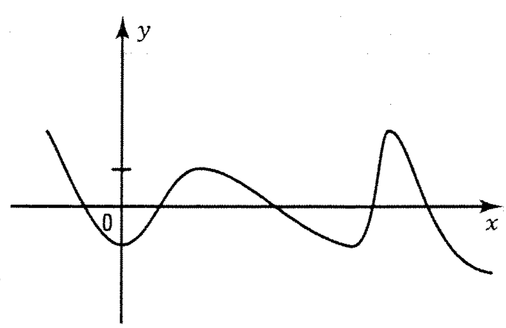
c) $f(1) = 6$. ***True***

d) $f(8) = 4$. ***False***

e) $f(2)$ is undefined. ***False***

f) The image of 5 by f does not exist. ***True***

5. Consider the function represented on the right.



a) Determine

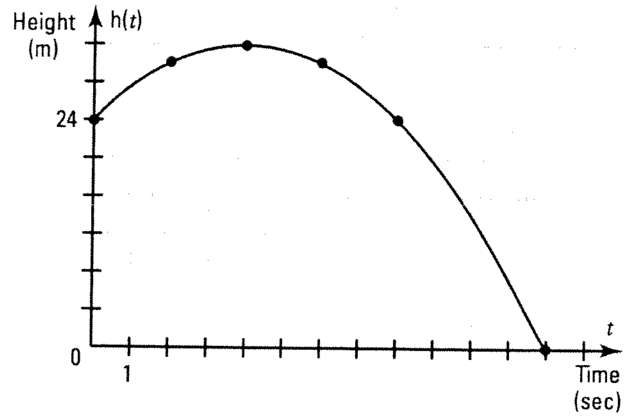
1. $f(2) =$ ***1*** 2. $f(6) =$ ***–1***
3. $f(0) =$ ***–1*** 4. $f(-2) =$ ***2***
5. $f(1) =$ ***0*** 6. $f(4) =$ ***0***

b) What are the antecedents of

1. 2. ***–2 and 7*** 2. 0. ***–1; 1; 4; 6.5 and 8***
3. –1. ***0.6 and 8.5*** 4. 3. ***None***

ACTIVITY 2 Domain and range

A projectile is launched from a height of 24 m above the ground. The function h gives the height (in metres) of the projectile as a function of time (in seconds) since it was launched. The function h with the rule: $h(t) = -\frac{t^2}{2} + 4t + 24$ is represented on the right.



a) 1. After how many seconds does the projectile hit the ground?
After 12 seconds.

2. In what interval does the variable t take its values in this situation?
In the interval [0, 12].

This interval is called the **domain** of function h.

b) 1. What is the maximum height reached by the projectile? *32 m*

2. What is the minimum height reached by the projectile? *0 m*

3. In what interval does the variable $h(t)$ take its values in this situation? *In the interval [0, 32].*

This interval is called the **range** of function h.

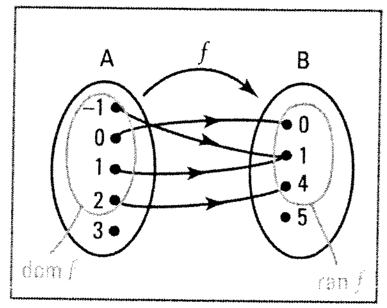
DOMAIN AND RANGE

- The domain of a function f is the subset of the elements of the source set which have an image by f. The domain of a function f is denoted dom f.

  In the mapping diagram on the right, dom $f = \{-1, 0, 1, 2\}$.

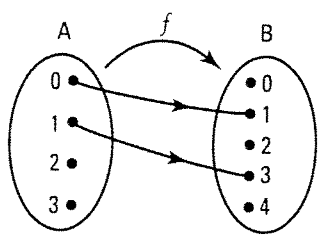
- The range of a function f is the subset of the elements of the target set which are images by f. The range of a function f is denoted ran f.

  In the mapping diagram on the right, ran $f = \{0, 1, 4\}$.



6. Determine the domain and range of the following functions.

a)



dom f = {0, 1}

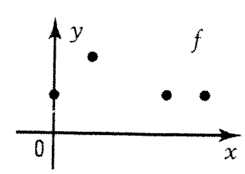
ran f = {1, 3}

b) $f = \{(1, 2), (2, 4), (3, 6), (4, 8)\}$

dom f = {1, 2, 3, 4}

ran f = {2, 4, 6, 8}

c)

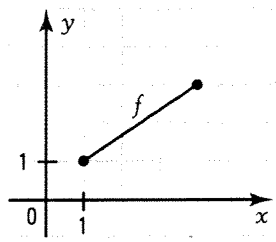


dom f = {0, 1, 3, 4}

ran f = {1, 2}

7. Determine the domain and range of the following functions.

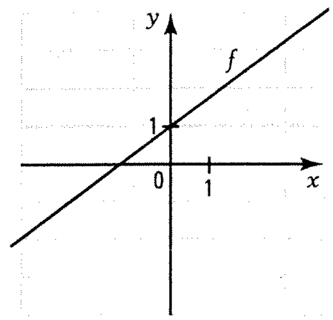
a)



dom f = [1, 4]

ran f = [1, 3]

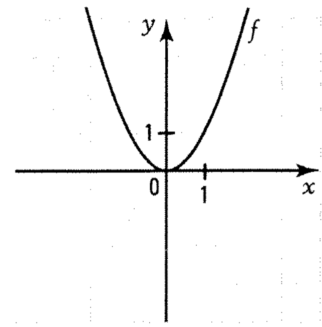
b)



dom f = ℝ

ran f = ℝ

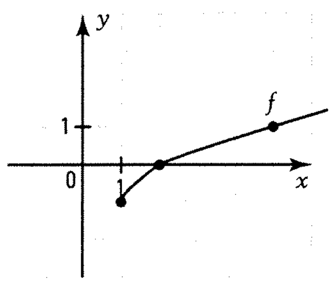
c)



dom f = ℝ

ran f = $\mathbb{R}_+$

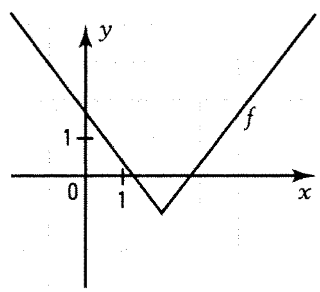
d)



dom f = [1, +∞[

ran f = [−1, +∞[

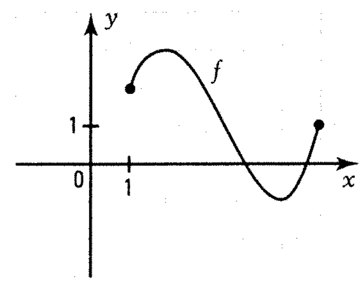
e)



dom f = ℝ

ran f = [−1, +∞[

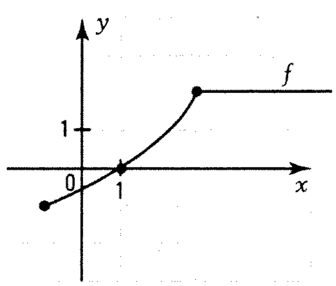
f)



dom f = [1, 6]

ran f = [−1, 3]

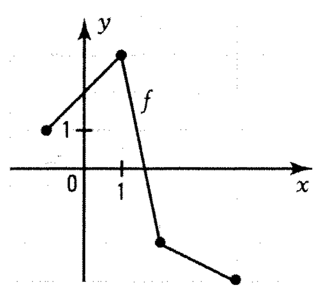
g)



dom f = [−1, +∞[

ran f = [−1, 2]

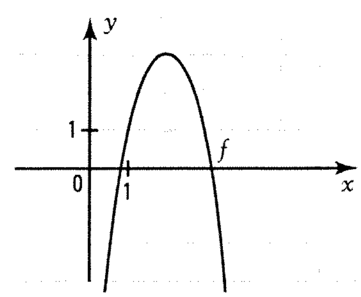
h)



dom f = [−1, 4]

ran f = [−3, 3]

i)



dom f = ℝ

ran f =]−∞, 3]

8. For each of the following functions, determine for which values of x the function f is defined and deduce the domain of the function.

a) $f(x) = x^2 + 5$ ***For any value of x, dom f =*** ℝ.

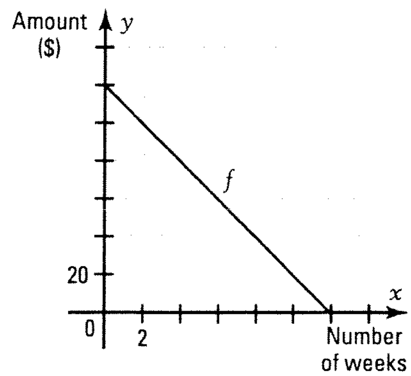
b) $f(x) = \frac{1}{x-2}$ ***For any value of x different from 2, dom f =*** ℝ \ {2}.

c) $f(x) = 3\sqrt{x-5}$ ***x − 5 ⩾ 0, x ⩾ 5, dom f = [5, +∞ [.***

d) $f(x) = -3x + 7$ ***For any value of x, dom f =*** ℝ.

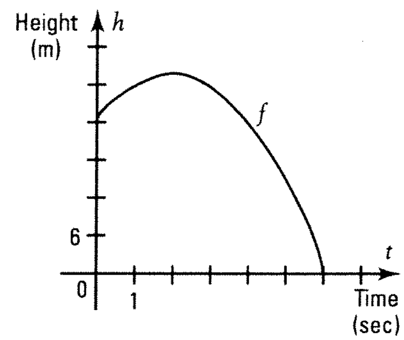
ACTIVITY 3 Initial value and zero of a function

a) Chloe takes \$10 out of her bank account each week. The function f, represented on the right, gives the amount y (in dollars) left in her account as a function of the number x of weeks. This function has the rule $y = 120 - 10x$.



1. The initial value of a function is the value of y when $x = 0$. Calculate and interpret the initial value of the function f. ***\$120. The initial value represents the initial amount of money in her bank account (t = 0).***
2. The zero(s) of a function is (are) the value(s) of x for which $y = 0$. Calculate and interpret the zero of the function. ***$120 - 10x = 0 \Rightarrow x = 12$. After 12 weeks, the bank account is empty.***

b) A stone is thrown from the top of a seaside cliff. The function f, represented on the right, gives the height h (in metres) reached by the stone as a function of the time t (in seconds) since it was thrown. This function has the rule $h = -2t^2 + 8t + 24$ ($t \geqslant 0$).

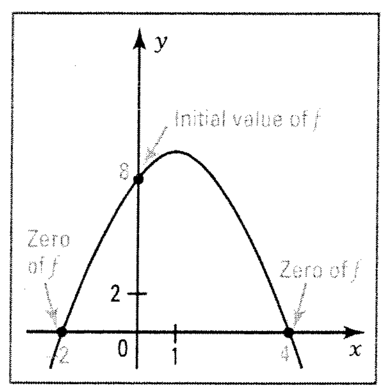


1. Calculate and interpret the initial value of this function. ***24 m. The initial value represents the height from which the stone was thrown (t = 0).***
2. Determining the zero of this function involves solving the quadratic equation $-2t^2 + 8t + 24 = 0$. Find the two solutions to this equation and indicate which of the two corresponds to the zero of this function. ***$t = -2$ and $t = 6$. 6 is the zero of the function. The stone hits the water at the instant $t = 6$ s.***

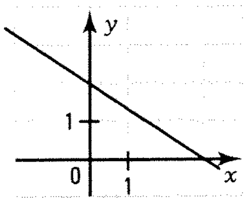
INITIAL VALUE AND ZERO OF A FUNCTION

- The initial value of a function $y = f(x)$ is the value of y when $x = 0$. To determine the initial value, we calculate $f(0)$.
- The zero(s) of a function $y = f(x)$ is (are) the value(s) of x when $y = 0$. To determine the zero(s), we solve the equation $f(x) = 0$.

Ex.: $f(x) = -x^2 + 2x + 8$
1. Initial value: $f(0) = 8$.
2. To find the zeros of f, we solve the equation: $-x^2 + 2x + 8 = 0$. The zeros are -2 and 4.

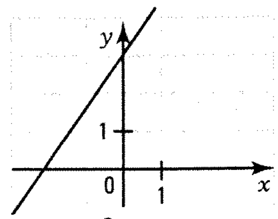


9. Determine the zero(s) and the initial value of the following functions.

a)


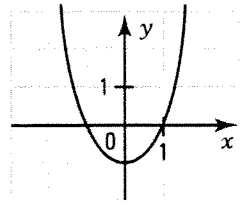
zero: 3

initial value: 2

b)


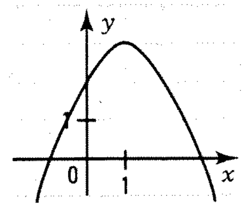
zero: −2

initial value: 3

c)


zeros: −1 and 1

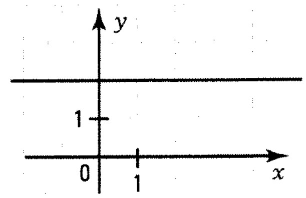
initial value: -1

d)


zeros: −1 and 3

initial value: 2

10. The function *f* on the right is represented by a horizontal line.

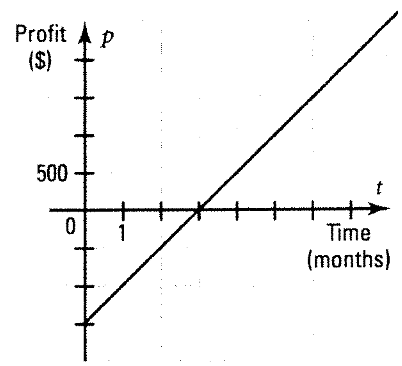


a) What can we say about the zeros of this function? Justify your answer.

The zeros do not exist, because the graph will never intersect the x-axis since the line is parallel to the x-axis.

b) What is the initial value of the function? ***2***

11. The graph on the right represents a function *f* which gives the profit *p* made by a company as a function of time *t* (in months) since its opening.



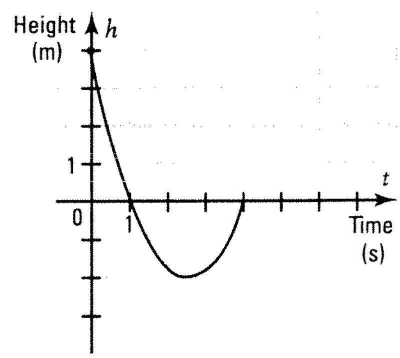
a) What is the zero of this function? What does it represent?

3; it represents the number of months since opening for the profit to be zero.

b) What is the initial value of this function? What does it represent?

−1500; it represents the company's profit at its opening.

12. The graph on the right represents a function *f* which gives the height *h* reached by a diver (in m) as a function of time *t* (in seconds).



a) What are the zeros of this function? What do they represent?

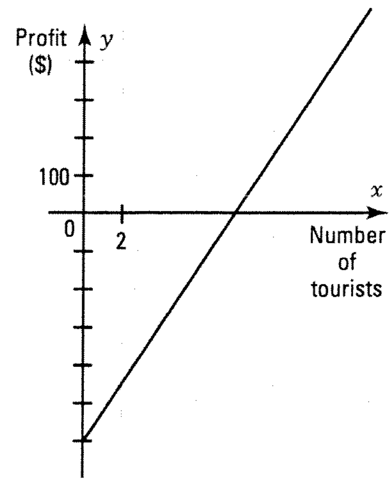
1 s and 4 s. They represent the time at which the diver was at the surface of the water.

b) What is the initial value of this function? What does it represent?

4 m. It represents the height from which the diver jumped.

ACTIVITY 4 Sign of a function

The graph on the right illustrates the function which gives the profit of a travel agency as a function of the number of tourists taking a guided tour.



a) What is the zero of the function? What does it represent?
8. The profit is zero if 8 tourists take the guided tour.

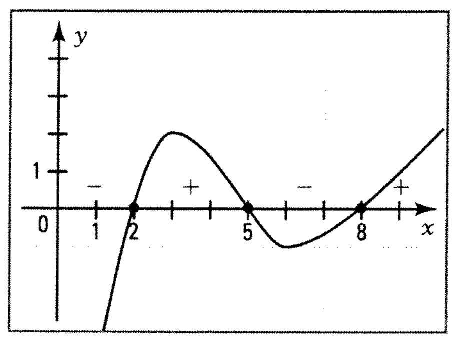
b) In what interval must the number of tourists be for the travel agency to lose money?
In the interval [0, 8[.

c) In what interval must the number of tourists be for the travel agency to make a profit?
In the interval]8, +∞[.

SIGN OF A FUNCTION

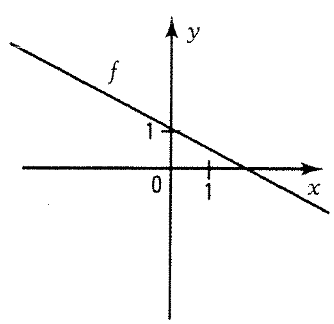
- Studying the sign of a function consists of finding the values of x for which the function is positive or those for which the function is negative.

  Ex.: Consider the function f represented on the right.
  This function has 3 zeros: 2, 5 and 8.
  $f(x) \leqslant 0$ in $]-\infty, 2] \cup [5, 8]$.
  $f(x \geqslant 0$ in $[2, 5] \cup [8, +\infty[$.



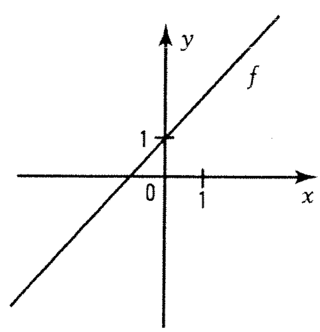
13. Determine the interval for which the functions below are
1. positive.
2. negative.

a)



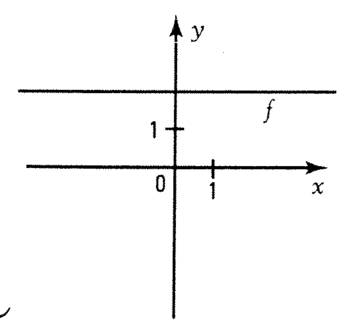
$f \geqslant 0$ in $]-\infty, 2]$
$f \leqslant 0$ in $[2, +\infty[$

b)

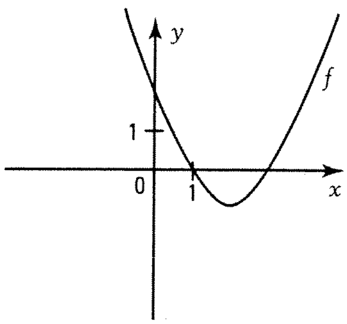


$f \geqslant 0$ in $]-\infty, -1]$
$f \leqslant 0$ in $[-1, +\infty[$
mistake

c)

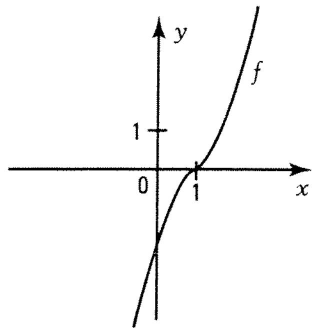


$f \geqslant 0$ in $\mathbb{R}$
$f \leqslant 0$ never

d) 

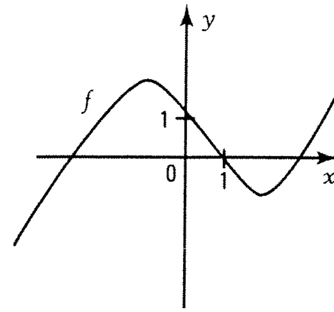
$f \geq 0$ in $]-\infty, 1] \cup [3, +\infty[$

$f \leq 0$ in $[1, 3]$

e) 

$f \geq 0$ in $[1, +\infty[$

$f \leq 0$ in $]-\infty, 1]$

f) 

$f \geq 0$ in $[-3, 1] \cup [3, +\infty[$

$f \leq 0$ in $]-\infty, -3] \cup [1, 3]$

14. Consider the function f represented on the right.

a) What is the domain of f? ***$[-1, +\infty[$***

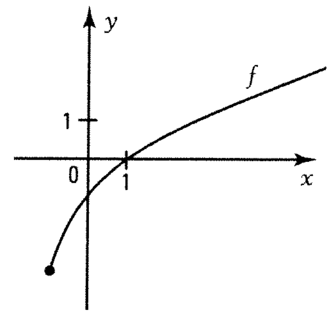
b) What is the range of f? ***$[-3, +\infty[$***

c) What is the zero of f? ***1***

d) What is the initial value of f? ***−1***

e) Find the interval for which the function f is

1. positive. ***$[1, +\infty[$***
2. strictly negative. ***$[-1, 1[$***



15. Consider the function f represented on the right.

a) What is the domain of f? $\mathbb{R}$

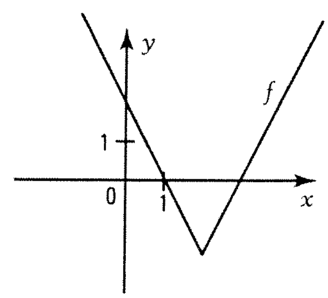
b) What is the range of f? ***$[-2, +\infty[$***

c) What is (are) the zero(s) of f? ***1 and 3***

d) What is the initial value of f? ***2***

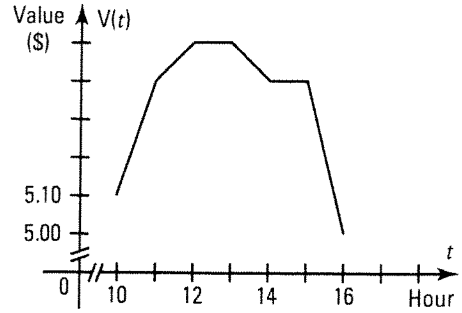
e) Find the interval for which the function f is

1. strictly positive. ***$]-\infty, 1[\cup]3, +\infty[$***
2. negative. ***$[1, 3]$***



ACTIVITY 5 Variation of a function

The graph on the right illustrates, hour by hour, the value of a share for the Kandev Company traded on the Montreal stock exchange.



a) Does the value of the share increase or decrease between the 10th and 11th hour? ***It increases.***

b) What can be said about the value of the share between the 12th and 13th hour? ***It remains constant.***

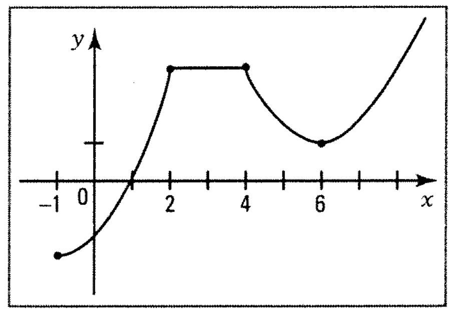
c) Does the value of the share increase or decrease between the 13th and 14th hour? ***It decreases.***

d) Indicate over which interval (or union of intervals)

1. the value of the share increases. ***[10, 12]***
2. the value of the share decreases. ***[13, 14] ∪ [15, 16]***
3. the value of the share is constant. ***[12, 13] ∪ [14, 15]***

VARIATION OF A FUNCTION

- A function f is constant over an interval if, when x increases over that interval, the image $f(x)$ remains constant.
- A function f is increasing over an interval if, when x increases over that interval, the image $f(x)$ increases or remains constant.
- A function f is decreasing over an interval if, when x increases over that interval, the image $f(x)$ decreases or remains constant.



Ex.: Consider the function f represented on the right.
f is constant over $[2, 4]$.
f is strictly increasing $f \nearrow$ over $[-1, 2] \cup [6, +\infty[$.
f is strictly decreasing $f \searrow$ over $[4, 6]$.
f is increasing over $[-1, 4] \cup [6, +\infty[$.
f is decreasing over $[2, 6]$.

Definition

- A function f is considered increasing over an interval if for all values x_1 and x_2 in this interval, $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$.

The symbol ∀ means "for all"

$$\forall x_1, x_2 \in [a, b]: x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

The symbol ⇒ means ... logically implies ...

- A function f is considered decreasing over an interval if for all values x_1 and x_2 in this interval, $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$.

$$\forall x_1, x_2 \in [a, b]: x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

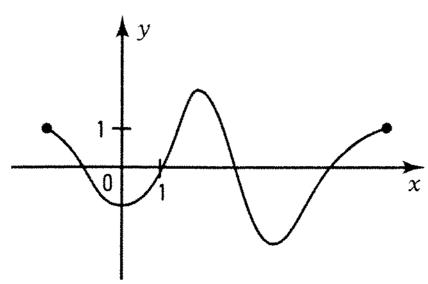
- A function f is considered constant over an interval if for all values x_1 and x_2 in this interval, $x_1 < x_2$ implies $f(x_1) = f(x_2)$.

$$\forall x_1, x_2 \in [a, b]: x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$$

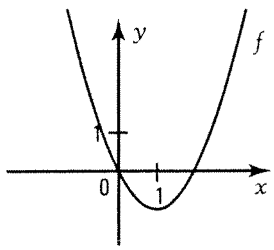
16. Consider the function represented on the right. Find the interval over which the function is

a) increasing. ***[0, 2] ∪ [4, 7]***

b) decreasing. ***[−2, 0] ∪ [2, 4]***

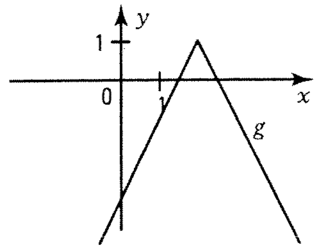


17. Give the increasing and decreasing intervals of the following functions.

a) 

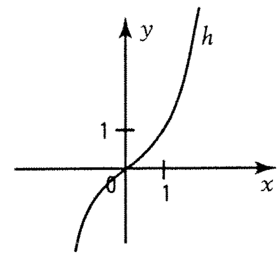
f is decreasing over]−∞, 1].

f is increasing over [1, +∞[.

b) 

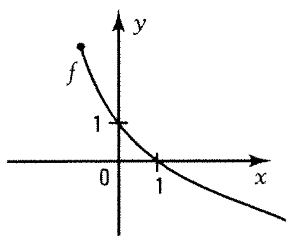
g is increasing over]−∞, 2].

g is decreasing over [2, +∞[.

c) 

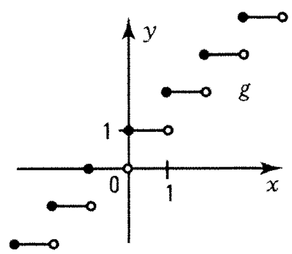
h is increasing over ℝ.

h is never decreasing.

d) 

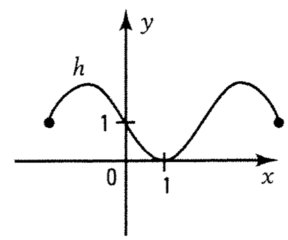
f is never increasing.

f is decreasing over [−1, +∞[.

e) 

g is increasing over ℝ.

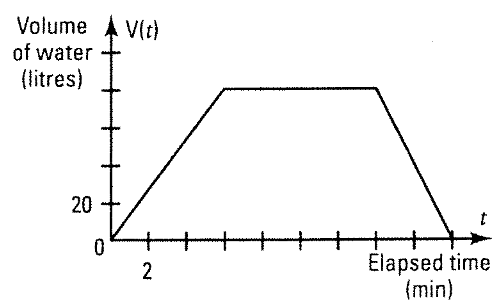
g is never decreasing.

f) 

h is increasing over [−2, −1] ∪ [1, 3].

h is decreasing over [−1, 1] ∪ [3, 4].

18. Raphael has taken a bath. The graph on the right illustrates the variation of the volume of water in the bath from the moment he turned on the faucet.



a) Indicate and interpret the interval where the function V is strictly increasing.

his, 6]. From 0 to 6 minutes, Raphael is filling his bath.

b) Indicate and interpret the interval where the function V is constant.

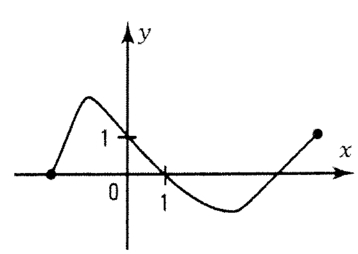
[6, 14]. It is over this interval of time, lasting 8 minutes, that Raphael bathed.

c) Indicate and interpret the interval of time where the function V is strictly decreasing.

[14, 18]. It is over this interval of time, lasting 4 minutes, that Raphael emptied his bath.

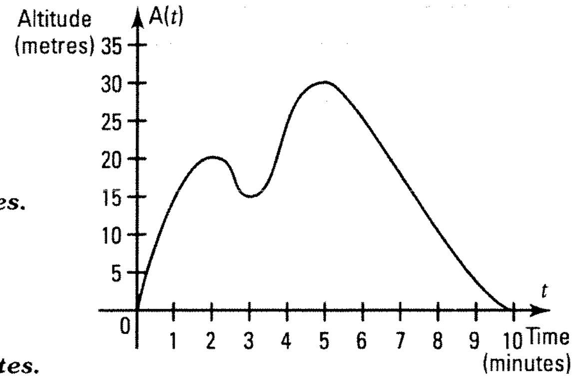
19. Draw the graph of a function f that satisfies the following conditions.

1. $f \geqslant 0$ in $[-2, 1] \cup [4, 5]$.
2. $f \leqslant 0$ in $[1, 4]$.
3. $f \nearrow$ in $[-2, -1] \cup [3, 5]$.
4. $f \searrow$ in $[-1, 3]$.



ACTIVITY 6 Extrema of a function

The graph on the right illustrates the flight of a kite, as a function of the elapsed time since the kite's release.



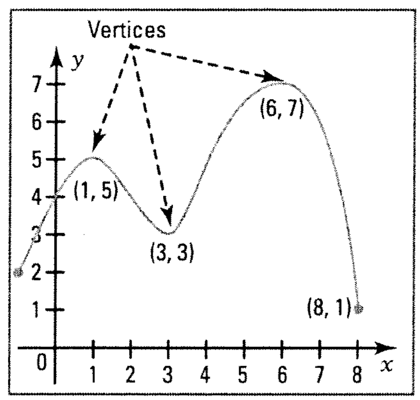
a) Determine and interpret the domain of the function.
dom A = [0, 10]. The kite was in the air for 10 minutes.

b) Determine at what instant the kite reaches its maximum altitude. What is this maximum altitude?
It reaches a maximum altitude of 30 m after 5 minutes.

c) What is the kite's minimum altitude? At what time(s) is it reached?
The minimum altitude is 0 m at the start (0 min) and when it returns 10 minutes later.

MAXIMUM AND MINIMUM OF A FUNCTION

- The absolute maximum (absolute minimum) of a function is the greatest image (lowest image) when it exists.



Thus, given M ∈ ran f and m ∈ ran f,

The symbol ⇔ means ... is logically equivalent to...

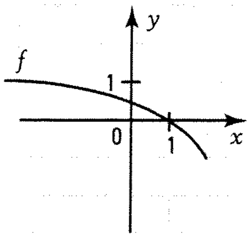
M is the absolute maximum of $f \Leftrightarrow \forall x \in \text{dom } f: f(x) \leqslant M$.
m is the absolute minimum of $f \Leftrightarrow \forall x \in \text{dom } f: f(x) \geqslant m$.

Ex.: – The absolute maximum of f is 7 and is reached when $x = 6$. We write: max $f = 7$
– The absolute minimum of f is 1 and is reached when $x = 8$. We write: min $f = 1$.
– The function f is increasing before reaching the vertex (1, 5), then decreasing. The y-coordinate of this vertex, 5, is therefore a relative maximum for f.
– The function f is decreasing before reaching the vertex (3, 3), then increasing. The y-coordinate of this vertex, 3, is therefore a relative minimum for f.

- The extrema of a function are any values that are either a maximum or a minimum.

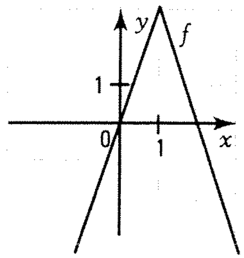
20. Determine, when they exist, the maximum and minimum of the following functions.

a)



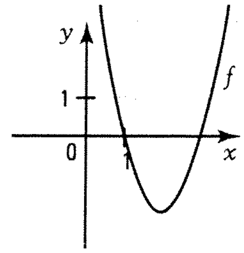
min f = −1

b)



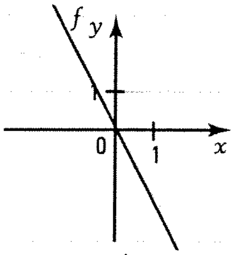
max f = 3

c)



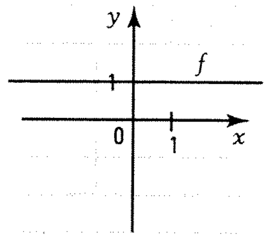
min f = −2

d)



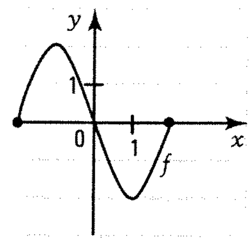
no extrema

e)



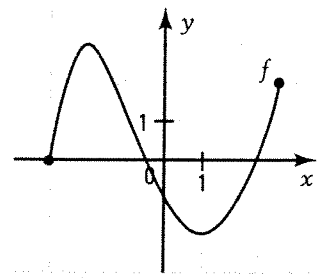
max f = 1, min f = 1

f)



max f = 2, min f = −2

21. Given the function on the right.



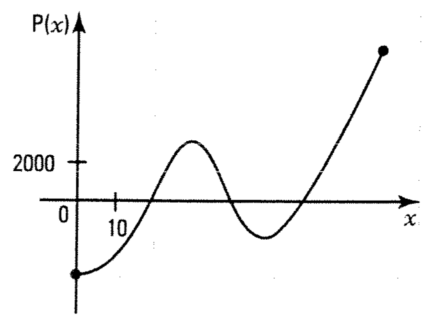
a) Determine

1. dom *f*. *[−3, 3]* 2. ran *f*. *[−2, 3]*

b) For what value of x does the function reach its absolute maximum? What is this absolute maximum?
The absolute maximum of f is 3 when x = −2.

c) For what value of x does the function reach its absolute minimum? What is this absolute minimum? *The absolute minimum of f is −2 when x = 1.*

22. The graph on the right illustrates the profit P(x) (in $) generated from selling x video cameras in one week. This company cannot produce more than 80 cameras per week.



a) Determine

1. dom P. *[0, 80]* 2. ran P. *[−4000, 8000]*

b) Determine and interpret the zeros of P.
The profit is zero when 20, 40 or 60 cameras are sold.

c) Determine the increasing and decreasing intervals for P.
The profit increases over [0, 30] ∪ [50, 80] and decreases over [30, 50].

d) Determine and interpret the absolute maximum and minimum of this profit function.
For 0 cameras sold, the company generates its absolute minimum profit, which is a loss of $4000. For 80 cameras sold, the company generates its absolute maximum profit of $8000.

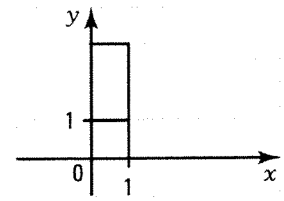
e) Determine the relative maximum and minimum of this function.
Rel. max. = 3000 Rel. min. = −1000

2.2 Transformations of graphs

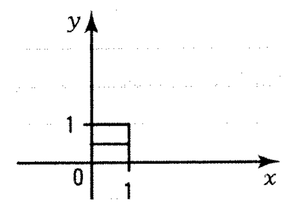
ACTIVITY 1 Impact of a change in scale on a figure

a) A vertical scale change is a transformation of the plane such that $(x, y) \rightarrow (x, ky)$.

1. Draw the image of the square on the right according to the vertical scale change $(x, y) \rightarrow (x, 3y)$ and indicate whether you observe a stretch or a reduction. ***A stretch***

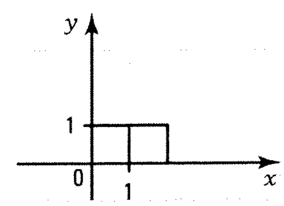


2. Draw the image of the square on the right according to the vertical scale change $(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$ and indicate whether you observe a stretch or a reduction. ***A reduction***

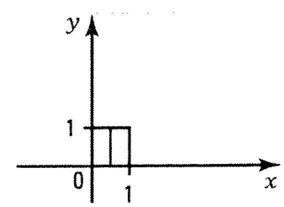


b) A horizontal scale change is a transformation of the plane such that $(x, y) \rightarrow (kx, y)$.

1. Draw the image of the square on the right according to the horizontal scale change $(x, y) \rightarrow (2x, y)$ and indicate whether you observe a stretch or a reduction. ***A stretch***

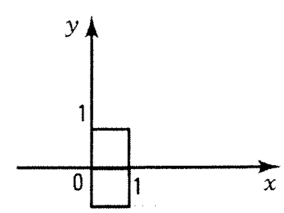


2. Draw the image of the square on the right according to the horizontal scale change $(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$ and indicate whether you observe a stretch or a reduction. ***A reduction***

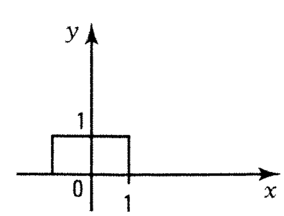


ACTIVITY 2 Impact of a reflection

a) 1. Draw the square on the right by the reflection $(x, y) \rightarrow (x, -y)$.
2. What do you notice? ***The square is reflected about the x-axis.***



b) 1. Draw the square on the right by the reflection $(x, y) \rightarrow (-x, y)$.
2. What do you notice? ***The square is reflected about the y-axis.***

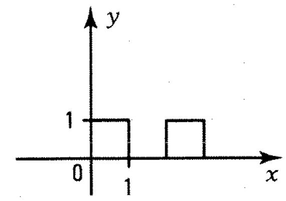


ACTIVITY 3 Impact of a translation on a figure

a) A horizontal translation is a transformation of the plane such that $(x, y) \rightarrow (x + h, y)$.

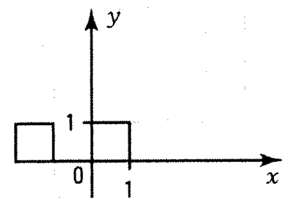
1. Draw the image of the square on the right according to the horizontal translation $(x, y) \rightarrow (x + 3, y)$ and indicate whether you observe a shift to the right or to the left.

   To the right



2. Draw the image of the square on the right according to the horizontal translation $(x, y) \rightarrow (x - 2, y)$ and indicate whether you observe a shift to the right or to the left.

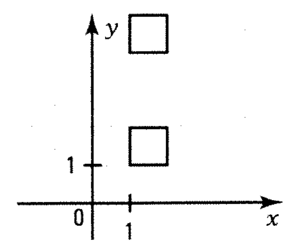
   To the left



b) A vertical translation is a transformation of the plane such that $(x, y) \rightarrow (x, y + k)$.

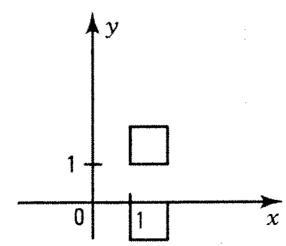
1. Draw the image of the square on the right according to the vertical translation $(x, y) \rightarrow (x, y + 3)$ and indicate whether you observe a shift upward or downward.

   Upward



2. Draw the image of the square on the right according to the vertical translation $(x, y) \rightarrow (x, y - 2)$ and indicate whether you observe a shift upward or downward.

   Downward

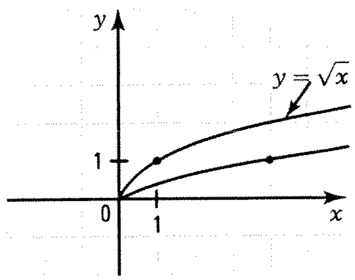


TRANSFORMATION OF THE PLANE APPLIED TO A FIGURE

- The scale change $(x, y) \mapsto (x, ky)$ causes a vertical stretch (reduction) if $k > 1$ $(0 < k < 1)$.
- The scale change $(x, y) \mapsto (kx, y)$ causes a horizontal stretch (reduction) if $k > 1$ $(0 < k < 1)$.
- The reflection $(x, y) \mapsto (x, -y)$ causes a reflection about the x-axis.
- The reflection $(x, y) \mapsto (-x, y)$ causes a reflection about the y-axis.
- The translation $(x, y) \mapsto (x + h, y)$ causes a horizontal shift to the right if h is positive and to the left if h is negative.
- The translation $(x, y) \mapsto (x, y + k)$ causes a vertical shift upward if k is positive and downward if k is negative.

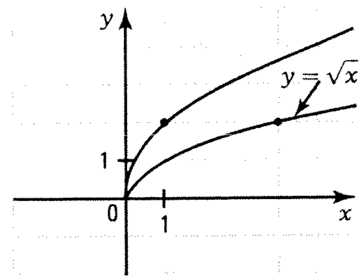
1. Draw the image of the figure according to the given transformation and indicate the resulting change.

a) $(x, y) \rightarrow (4x, y)$



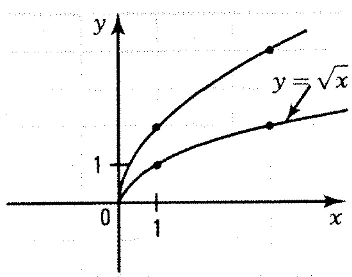
Horizontal stretch

b) $(x, y) \rightarrow \left(\frac{1}{4}x, y\right)$



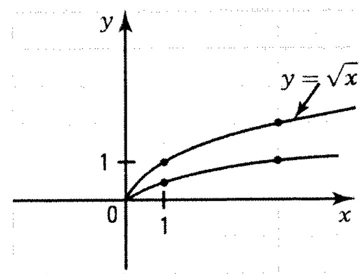
Horizontal reduction

c) $(x, y) \rightarrow (x, 2y)$



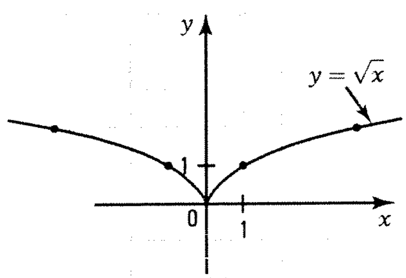
Vertical stretch

d) $(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$



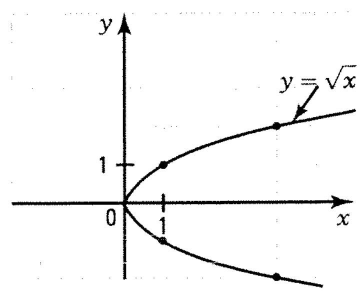
Vertical reduction

e) $(x, y) \rightarrow (-x, y)$



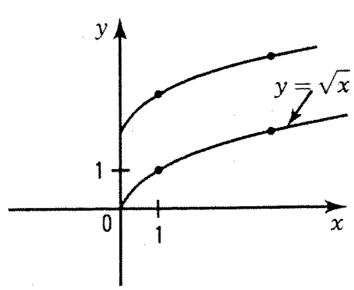
Reflection about the y-axis

f) $(x, y) \rightarrow (x, -y)$



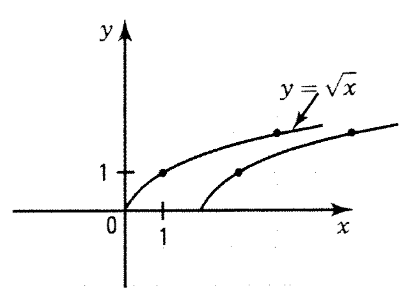
Reflection about the x-axis

g) $(x, y) \rightarrow (x, y + 2)$



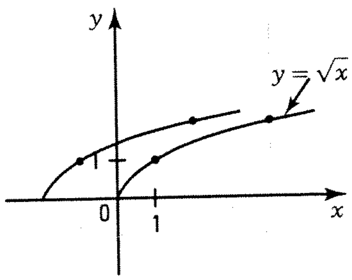
Translation of 2 units upward

h) $(x, y) \rightarrow (x + 2, y)$



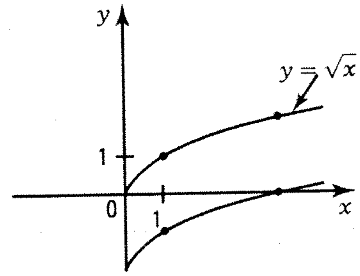
Translation of 2 units to the right

i) $(x, y) \rightarrow (x - 2, y)$



Translation of 2 units to the left

j) $(x, y) \rightarrow (x, y - 2)$



Translation of 2 units downward

2. a) Which variable must be multiplied by a constant to cause a scale change that is

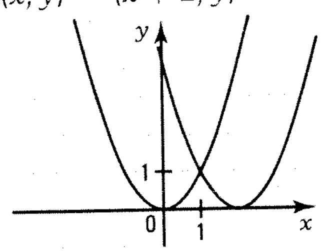
1. horizontal? ***x*** 2. vertical? ***y***

b) To which variable must we add or subtract a constant to cause a translation that is

1. horizontal? ***x*** 2. vertical? ***y***

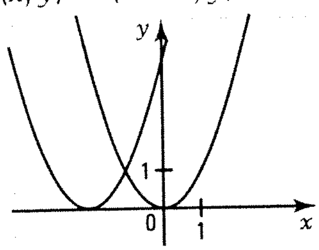
3. Draw the image of each figure according to the given transformation and indicate the resulting change.

a) $(x, y) \mapsto (x + 2, y)$



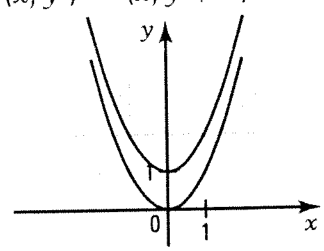
Horizontal shift of 2 units to the right

b) $(x, y) \mapsto (x - 2, y)$



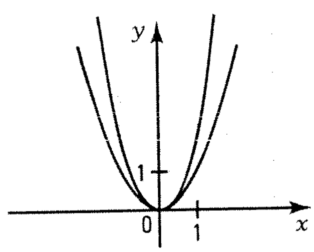
Horizontal shift of 2 units to the left

c) $(x, y) \mapsto (x, y + 1)$



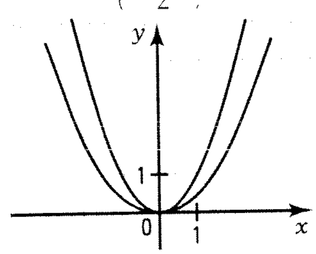
Vertical shift of 1 unit upward

d) $(x, y) \mapsto (x, 2y)$



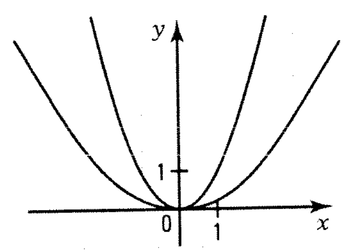
Vertical stretch

e) $(x, y) \mapsto \left(x, \frac{1}{2}y\right)$



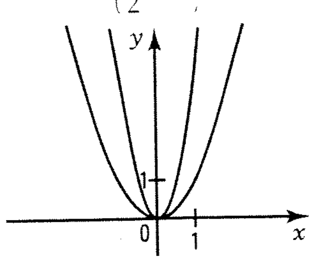
Vertical reduction

f) $(x, y) \mapsto (2x, y)$



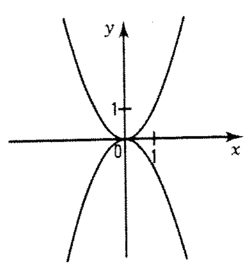
Horizontal stretch

g) $(x, y) \mapsto \left(\frac{1}{2}x, y\right)$



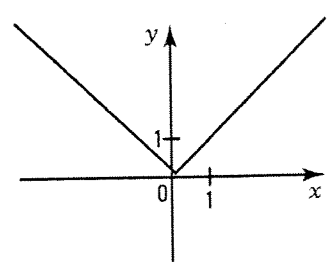
Horizontal reduction

h) $(x, y) \mapsto (x, -y)$



Reflection about the x-axis

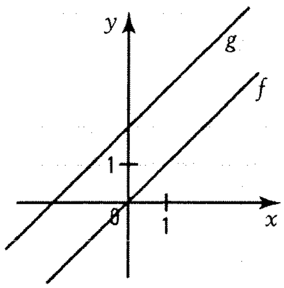
i) $(x, y) \mapsto (-x, y)$



Reflection about the y-axis

4. Identify the transformation that associates the graphs of f and g.

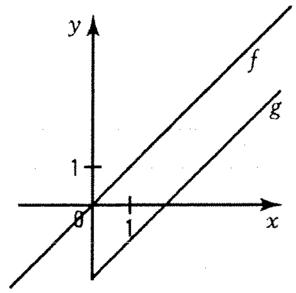
a)



(x, y) → (x, y + 2)

or (x, y) → (x − 2, y)

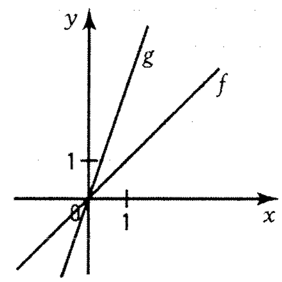
b)



(x, y) → (x, y − 2)

or (x, y) → (x + 2, y)

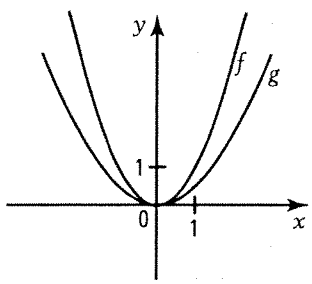
c)



(x, y) → (x, 3y)

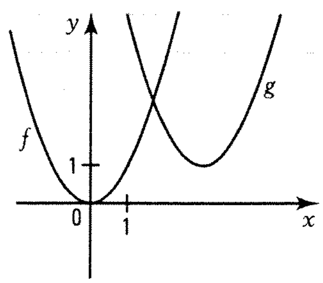
or $(x, y) \rightarrow \left(\frac{x}{3}, y\right)$

d)



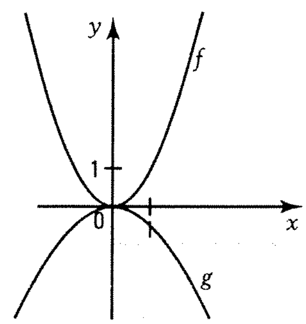
$(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$

e)



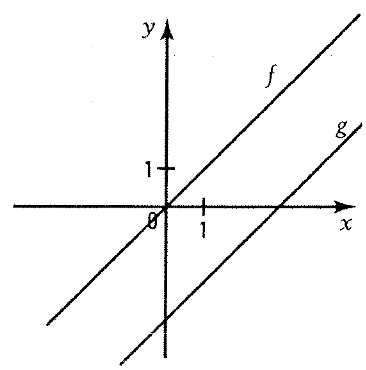
(x, y) → (x + 3, y + 1)

f)



$(x, y) \rightarrow \left(x, \frac{-1}{2}y\right)$

5. The functions f and g are represented on the right and have respectively the rule $f(x) = x$ and $g(x) = x - 3$.



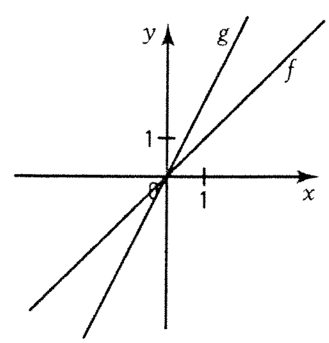
a) Find and describe two translations that map the graph of f onto g.

1. ***The horizontal translation: (x, y) → (x + 3, y).***
2. ***The vertical translation: (x, y) → (x, y − 3).***

b) Find and describe two translations that map the graph of g onto f.

1. ***The horizontal translation: (x, y) → (x − 3, y).***
2. ***The vertical translation: (x, y) → (x, y + 3).***

6. The functions f and g are represented on the right and have respectively the rule $f(x) = x$ and $g(x) = 2x$.



a) Find and describe two scale changes that map the graph of f onto g.

1. ***The vertical stretch: (x, y) → (x, 2y).***
2. ***The horizontal reduction:*** $(x, y) \rightarrow \left(\frac{x}{2}, y\right)$.

b) Find and describe two translations that map the graph of g onto f.

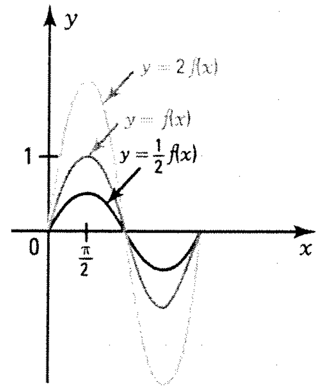
1. ***The vertical reduction:*** $(x, y) \rightarrow \left(x, \frac{y}{2}\right)$.
2. ***The horizontal stretch: (x, y) → (2x, y).***

2.3 Parameters of a function

Activity 1 Role of parameters *a* and *b*

a) Consider the function with the rule $y = a\,f(x)$.

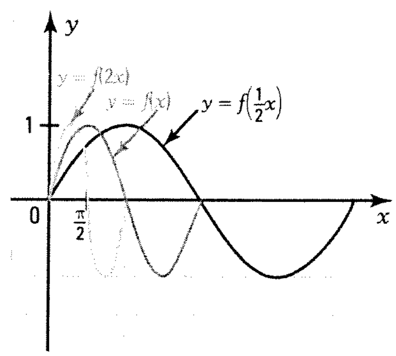
Let us analyze the graph of the function for different values of the parameter a. On the right, we have represented the graph of the function $y = f(x)$, called the basic function when $a = 1$ and the functions $y = 2f(x)$ and $y = \frac{1}{2}f(x)$ when a takes the values 2 and $\frac{1}{2}$ respectively.



1. Does a change in parameter a cause a horizontal or vertical scale change? ***Vertical***
2. Do you observe a stretch or a reduction of the graph when
   1) $a > 1$: ***Stretch*** 2) $a < 1$: ***Reduction***

b) Consider the function f with the rule $y = f(bx)$.

Let us analyze the graph of the function for different values of the parameter b. On the right, we have represented the graph of the function $y = f(x)$, called the basic function when $b = 1$ and the functions $y = f(2x)$ and $y = f(\frac{1}{2}x)$.



1. Does a change in parameter b cause a horizontal or vertical scale change? ***Horizontal***
2. Do you observe a stretch or a reduction of the graph when
   1) $b > 1$: ***Reduction*** 2) $b < 1$: ***Stretch***

ROLE OF PARAMETERS *a* AND *b*

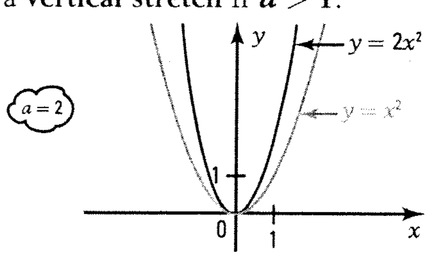
Consider the basic function $y = f(x)$ and the parameter a.

- Changing parameter a in the function $y = a\,f(x)$ causes the vertical scale change:

$$(x, y) \rightarrow (x, ay)$$

We observe

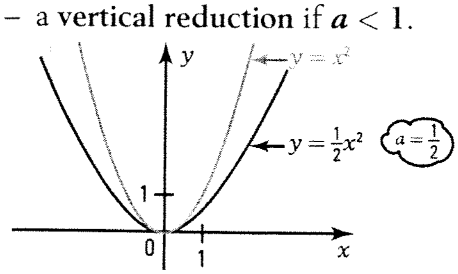
– a **vertical stretch** if $a > 1$.



$(x, y) \rightarrow (x, 2y)$

Basic function $y = x^2$

– a **vertical reduction** if $a < 1$.

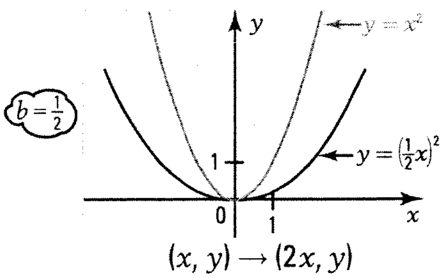


$(x, y) \rightarrow (x, \frac{1}{2}y)$

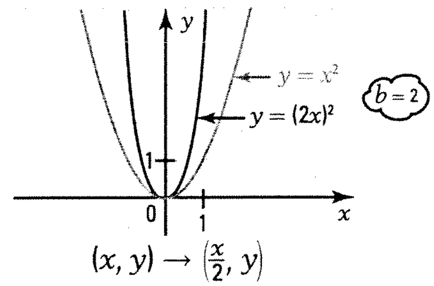
- Changing parameter b in the function $y = f(bx)$ causes the **horizontal scale change**:

$$(x, y) \rightarrow \left(\frac{x}{b}, y\right)$$

We observe
- a **horizontal stretch** if $b < 1$.
- a **horizontal reduction** if $b > 1$.



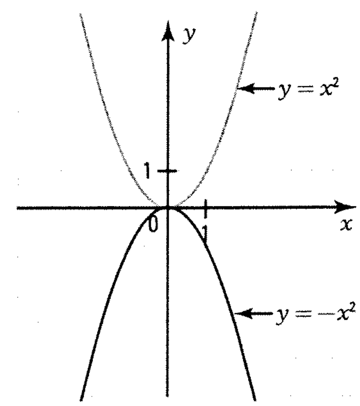
Basic function $y = x^2$



ACTIVITY 2 Reflection about the axes

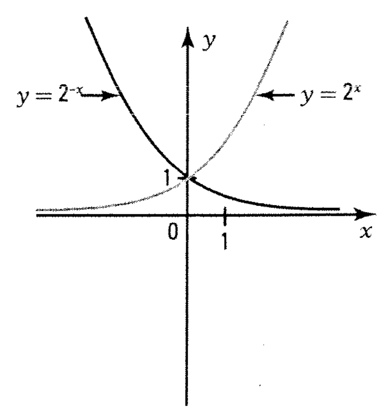
a) The function $y = x^2$ is represented on the right. Deduce the graph of the function $y = -x^2$.

b) Given the graph of a function $y = f(x)$, explain how to deduce the graph of $y = -f(x)$.
By reflecting the graph of y = f(x) about the x-axis.



c) The function $y = 2^x$ is represented on the right. Deduce the graph of the function $y = 2^{-x}$.

d) Given the graph of a function $y = f(x)$, explain how to deduce the graph of $y = f(-x)$.
By reflecting the graph of y = f(x) about the y-axis.



REFLECTION ABOUT THE AXES

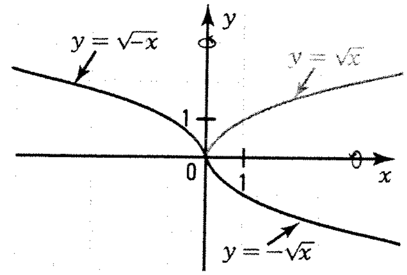
Given the graph of a function $y = f(x)$,

- the graph of $y = -f(x)$ is obtained by a reflection about the x-axis.

$$(x, y) \rightarrow (x, -y)$$

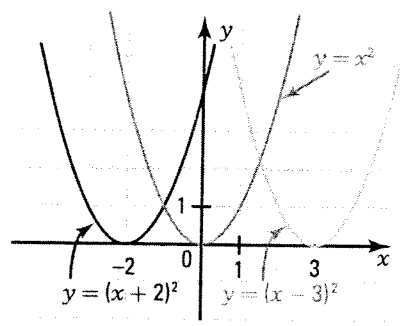
- the graph of $y = f(-x)$ is obtained by a reflection about the y-axis.

$$(x, y) \rightarrow (-x, y)$$



ACTIVITY 3 Role of parameters *h* and *k*

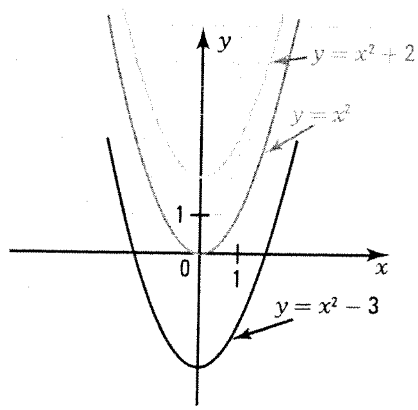
a) Consider the function f with the rule $y = (x - h)^2$.
Let us analyze the graph of function f for different values of the parameter h.
On the right, we have represented the graph of the function $y = x^2$ called the basic function when $h = 0$ and the functions $y = (x - 3)^2$ and $y = (x + 2)^2$ when h equals 3 and –2 respectively.



1. Does a change in parameter h cause a horizontal or vertical translation? ***A horizontal translation***

2. Describe the direction of the translation of the basic function's graph when
   1) $h > 0$. ***To the right*** 2) $h < 0$. ***To the left***

b) Consider the function f with the rule $y = x^2 + k$.
Let us analyze the graph of function f for different values of the parameter k.
On the right, we have represented the graph of the function $y = x^2$ called the basic function when $k = 0$ and the functions $y = x^2 + 2$ and $y = x^2 - 3$ when k equals 2 and –3 respectively.



1. Does a change in parameter k cause a horizontal or vertical translation?

   A vertical translation

2. Describe the direction of the translation when
   1) $k > 0$. ***Upward*** 2) $k < 0$. ***Downward***

ROLE OF PARAMETERS *h* AND *k*

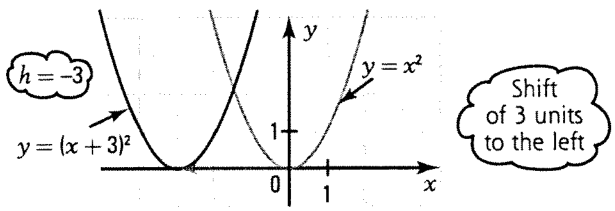
Consider the basic function $y = f(x)$.

- A change in parameter h in the function $y = f(x - h)$ causes a horizontal translation of h units:

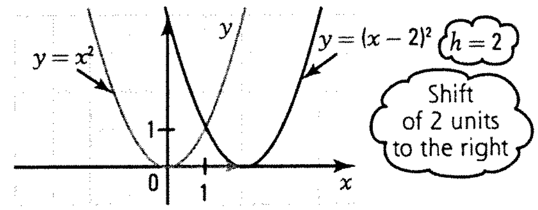
$$(x, y) \rightarrow (x + h, y)$$

We observe a

– shift to the left if $h < 0$.



– shift to the right if $h > 0$.

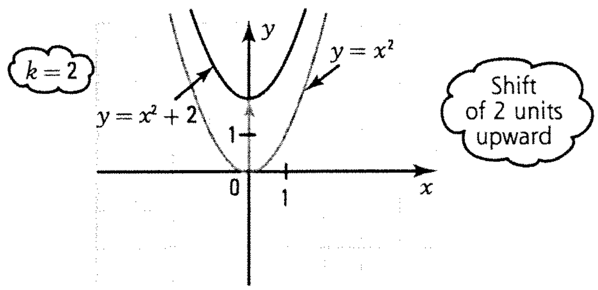


- A change in parameter k in the function $y = f(x) + k$ causes a vertical translation of k units:

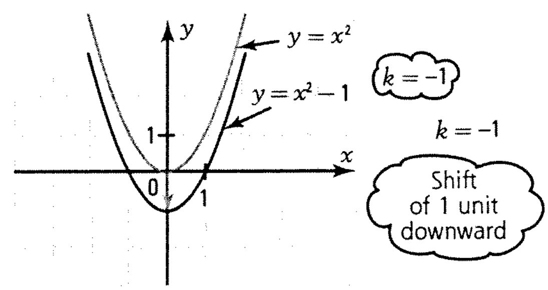
$$(x, y) \rightarrow (x, y + k)$$

We observe a

– shift upward if $k > 0$.

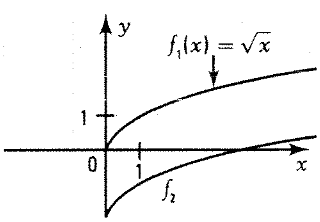


– shift downward $k < 0$.



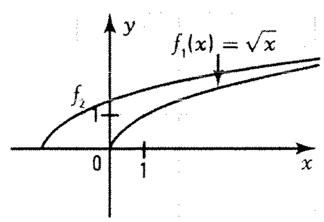
1. Given the rule of the function f_1, deduce the rule of f_2.

a)



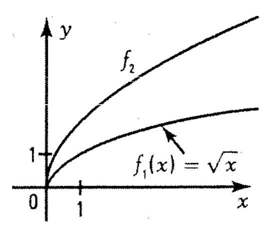
$f_2(x) = \sqrt{x} - 2$

b)



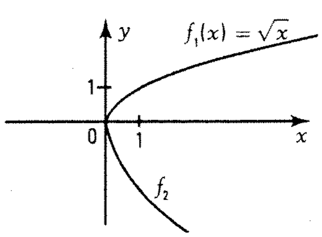
$f_2(x) = \sqrt{x + 2}$

c)



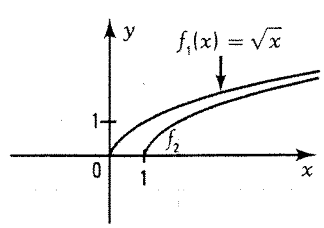
$f_2(x) = 2\sqrt{x}$

d)



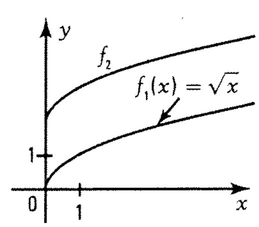
$f_2(x) = -2\sqrt{x}$

e)



$f_2(x) = \sqrt{x - 1}$

f)

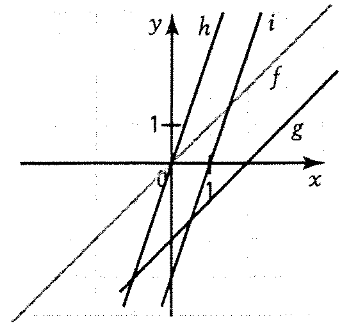


$f_2(x) = \sqrt{x} + 2$

2. The graph of the function $f(x) = x$ is drawn on the right.

a) Complete the following table of values.

x	x	$x - 2$	$3x$	$3(x - 1)$
−1	*−1*	*−3*	*−3*	*−6*
0	*0*	*−2*	*0*	*−3*
1	*1*	*−1*	*3*	*0*



b) From the table of values, graph the following functions.

1. $g(x) = x - 2$ 2. $h(x) = 3x$ 3. $i(x) = 3(x - 1)$

c) Explain how, from the graph of f, you would obtain

1. the graph of g. ***Horizontal translation: (x, y) → (x + 2, y)***
2. the graph of h. ***Vertical stretch: (x, y) → (x, 3y)***
3. the graph of i. ***Horizontal translation: (x, y) → (x + 1, y) followed by a vertical stretch (x, y) → (x, 3y).***

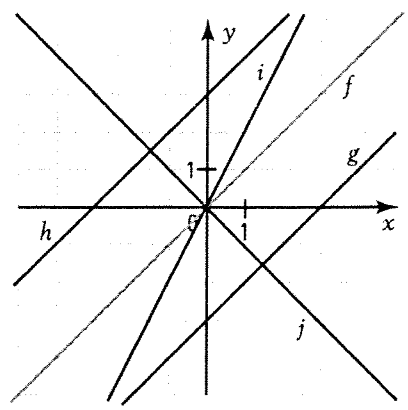
3. The graph of the function $f(x) = x$ is drawn on the right.

a) Deduce the graph of the function

1. $g(x) = x - 3$. 2. $h(x) = x + 3$. 3. $i(x) = 2x$. 4. $j(x) = -x$.

b) Explain how, from the graph of f, you would obtain

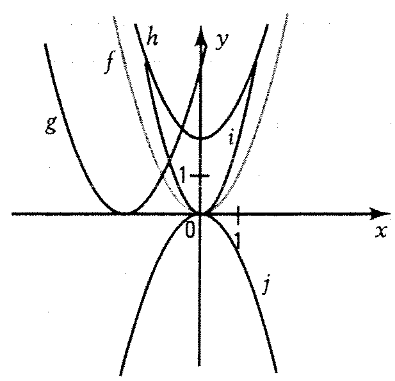
1. the graph of g. ***Horizontal translation: (x, y) → (x + 3, y)***
2. the graph of h. ***Horizontal translation: (x, y) → (x − 3, y)***
3. the graph of i. ***Vertical stretch: (x, y) → (x, 2y)***
4. the graph of j. ***Reflection about the y-axis: (x, y) → (−x, y)***



4. The graph of the function $f(x) = x^2$ is drawn on the right.

a) Complete the following table of values.

x	x^2	$(x+2)^2$	x^2+2	$2x^2$	$-x^2$
−2	*4*	*0*	*6*	*8*	*−4*
−1	*1*	*1*	*3*	*2*	*−1*
0	*0*	*4*	*2*	*0*	*0*
1	*1*	*9*	*3*	*2*	*−1*
2	*4*	*16*	*6*	*8*	*−4*



b) From the table of values, graph the following functions.

1. $g(x) = (x + 2)^2$ 2. $h(x) = x^2 + 2$ 3. $i(x) = 2x^2$ 4. $j(x) = -x^2$

c) Explain how, from the graph of f, you would obtain

1. the graph of g. ***Horizontal translation of 2 units to the left: (x, y) → (x − 2, y)***
2. the graph of h. ***Vertical translation of 2 units upward: (x, y) → (x, y + 2)***
3. the graph of i. ***Vertical stretch: (x, y) → (x, 2y)***
4. the graph of j. ***Reflection about the x-axis: (x, y) → (x, −y)***

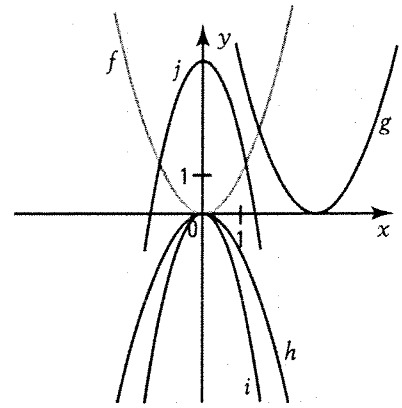
5. The graph of the function $f(x) = x^2$ is drawn on the right. Deduce the graph of

a) $g(x) = (x - 3)^2$.

b) $h(x) = -x^2$.

c) $i(x) = -2x^2$.

d) $j(x) = -2x^2 + 4$.



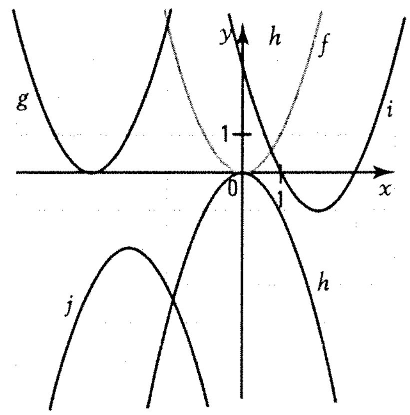
6. On the right, we have drawn the graph of the function $f(x) = x^2$ as well as the functions g, h, i and j. Determine the rule of the functions g, h, i and j and verify your answers using a graphing calculator.

$g(x) = (x + 4)^2$

$h(x) = -x^2$

$i(x) = (x - 2)^2 - 1$

$j(x) = -(x + 3)^2 - 2$



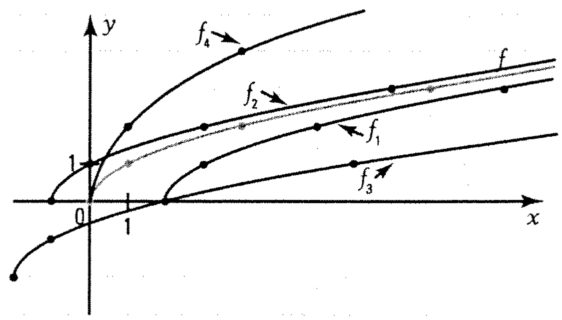
7. The function f has the rule $y = \sqrt{x}$.
Draw the graph of

a) the function $f_1(x) = \sqrt{x - 2}$.

b) the function $f_2(x) = \sqrt{x + 1}$.

c) the function $f_3(x) = \sqrt{x + 2} - 2$.

d) the function $f_4(x) = 2\sqrt{x}$.

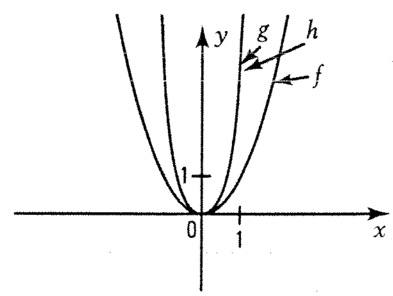


8. Consider the function $f(x) = x^2$, the function $g(x) = 4f(x)$ and the function $h(x) = f(2x)$ represented on the right.

a) Verify that g and h have the same rule, which explains why the functions g and h have the same graph.

$g(x) = 4\ f(x) = 4x^2$

$h(x) = f(2x) = (2x)^2 = 4x^2$



b) Complete

1. The graph of g is obtained from the graph of f by

a vertical stretch: $(x, y) \rightarrow (x, 4y)$

2. The graph of h is obtained from the graph of f by

a horizontal reduction: $(x, y) \rightarrow \left(\frac{x}{2}, y\right)$

Evaluation 2

1. For each of the following relations, determine

1. the domain
2. the range
3. if it is a function or not.

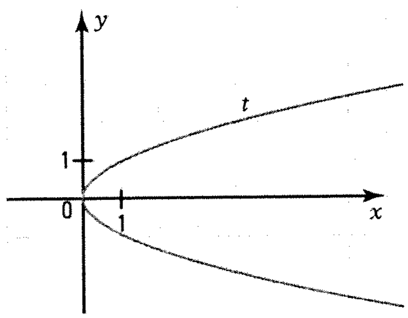
a) $r = \{(1, 2), (2, 5), (3, 4), (1, 3)\}$

1. ***Dom r = {1, 2, 3}***
2. ***Ran r = {2, 3, 4, 5}***
3. ***No***

b) $s = \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$

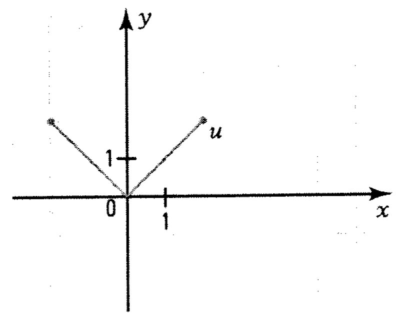
1. ***Dom s = {−2, −1, 0, 1, 2}***
2. ***Ran s = {0, 1, 4}***
3. ***Yes***

c)



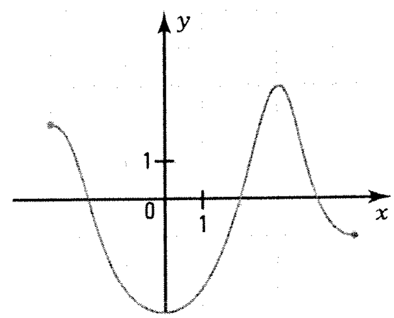
1. ***Dom t = [0, +∞[***
2. ***Ran t = ℝ***
3. ***No***

d)



1. ***Dom u = [−2, 2]***
2. ***Ran u = [0, 2]***
3. ***Yes***

2. Consider the function f represented on the right. Determine



a) 1. dom f: ***[−3, 5]*** 2. ran f. ***[−3, 3]***

b) 1. the zeros of f. ***−2, 2 and 4***
2. the y-intercept. ***−3***

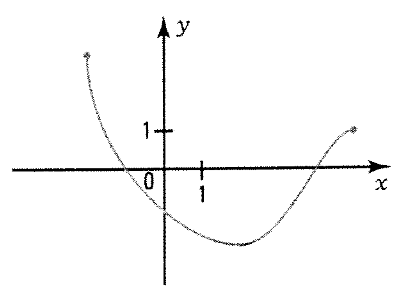
c) The values of x for which the function f is
1. positive. ***[−3, −2] ∪ [2, 4]*** 2. negative. ***[−2, 2] ∪ [4, 5]***

d) The values of x for which the function f is
1. increasing. ***[0, 3]*** 2. decreasing. ***[−3, 0] ∪ [3, 5]***

e) 1. the maximum of f. ***3*** 2. the minimum of f. ***−3***

3. Draw the graph of a function f that satisfies the following conditions.

1. dom $f = [-2, 5]$.
2. ran $f = [-2, 3]$.
3. the zeros of f are -1 and 4.
4. the y-intercept is -1.
5. f is positive over $[-2, -1] \cup [4, 5]$.
6. f is negative over $[-1, 4]$.
7. max $f = 3$ and min $f = -2$.

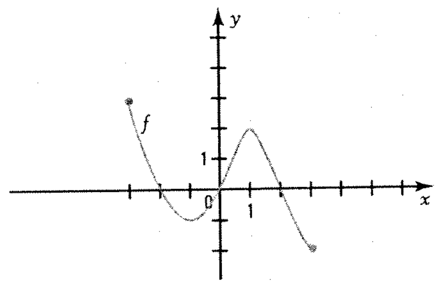


4. The study of a function consists of determining:

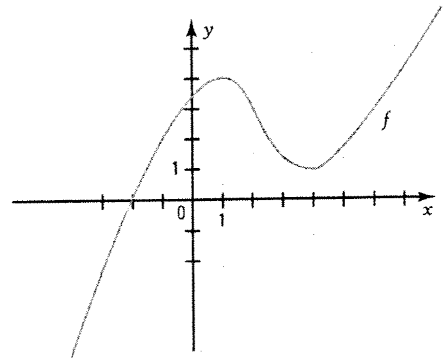
1°	the domain and range of the function.
2°	the zeros and y-intercept if they exist.
3°	the sign of the function.
4°	the increasing and decreasing intervals.
5°	the extrema of the function, if they exist.

Do a study of the following functions.

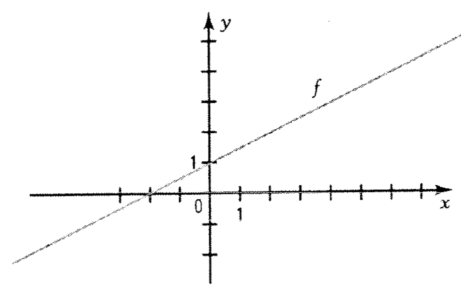
a)
1. ***Dom** $f = [-3, 3]$; **ran** $f = [-2, 3]$.*
2. ***Zeros:** -2, 0 **and** 2; y**-intercept:** 0*
3. ***f is negative over** $[-2, 0] \cup [2, 3]$.*
   ***f is positive over** $[-3, -2] \cup [0, 2]$.*
4. ***f is decreasing over** $[-3, -1] \cup [1, 3]$.*
   ***f is increasing over** $[-1, 1]$.*
5. ***max** $f = 3$; **min** $f = -2$.*



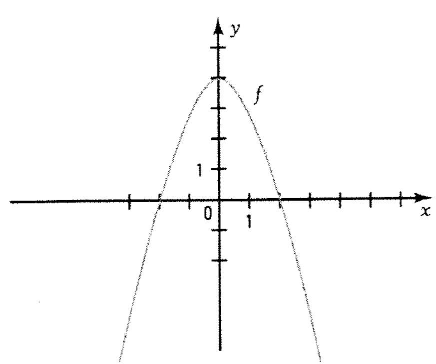
b)
1. ***Dom** $f = \mathbb{R}$; **ran** $f = \mathbb{R}$.*
2. ***Zero:** -2; y**-intercept:** 3.5*
3. ***f is negative over** $]-\infty, -2]$.*
   ***f is positive over** $[-2, +\infty[$.*
4. ***f is increasing over** $]\infty, 1] \cup [4, +\infty[$.*
   ***f is decreasing over** $[1, 4]$.*
5. ***There are no extremum.***



c)
1. ***Dom** $f = \mathbb{R}$; **ran** $f = \mathbb{R}$.*
2. ***Zeros:** -2; y**-intercept:** 1*
3. ***f is negative over** $]-\infty, -2]$.*
   ***f is positive over** $[-2, +\infty[$.*
4. ***f is increasing over** $\mathbb{R}$.*
   f is never decreasing
5. ***There are no extremum.***

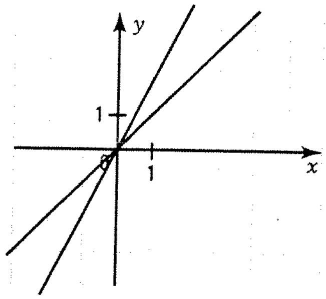


d)
1. ***Dom** $f = \mathbb{R}$; **ran** $f =]-\infty, 4]$.*
2. ***Zeros:** -2 **and** 2; y**-intercept:** 4*
3. ***f is negative over** $]-\infty, -2] \cup [2, +\infty[$.*
   ***f is positive over** $[-2, 2]$.*
4. ***f is increasing over** $]-\infty, 0]$.*
   ***f is decreasing over** $[0, +\infty[$.*
5. ***max** $f = 4$; **no minimum.***

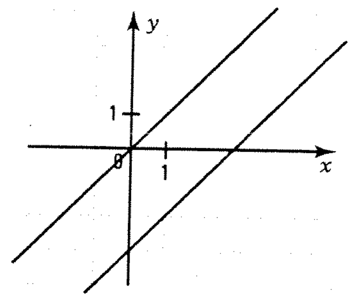


5. Draw the image of the figure according to the given transformation.

a) $(x, y) \to (x, 2y)$

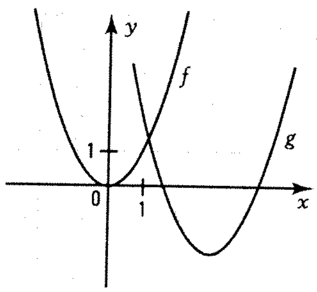


b) $(x, y) \to (x + 2, y - 1)$



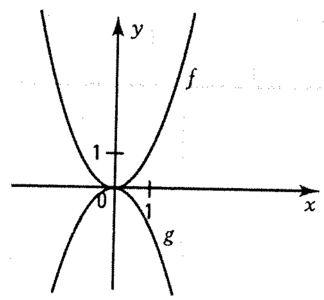
6. Identify the transformation which maps the graph of f onto the graph of g.

a)



$(x, y) \to (x + 3, y - 2)$

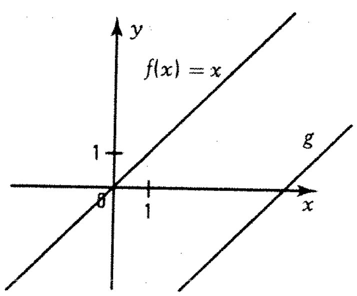
b)



$(x, y) \to (x, -y)$

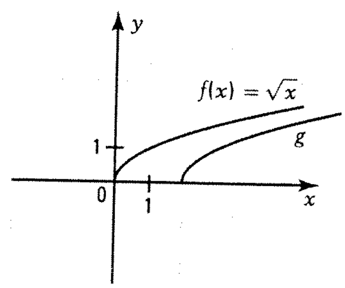
7. Determine the rule of g from the rule of f.

a)



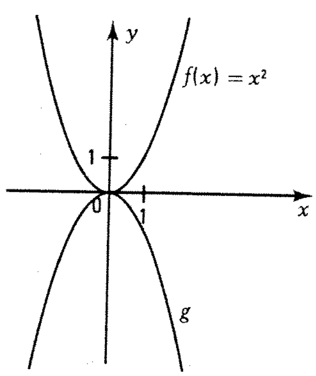
$g(x) = x - 5$

b)



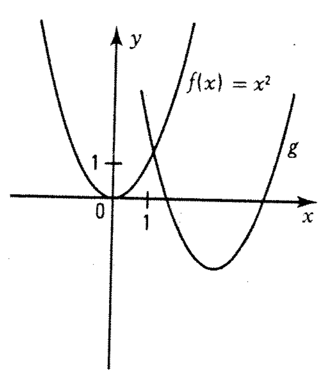
$g(x) = \sqrt{x - 2}$

c)



$g(x) = -x^2$

d)



$g(x) = (x - 3)^2 - 2$