Chapter 3

Polynomial functions

CHALLENGE 3

1. The weekly profit $p(x)$ made by a company selling x computers is given by the rule $p(x) = -20x^2 + 800x - 6000$.

This company cannot sell more than 25 computers per week. What must be the number of computers sold in a week for this company to register a loss?

The number of computers sold is less than 10.

2. The trajectory of a ball is described by the rule $h(t) = -t^2 + 4t + 5$ where $h(t)$ represents the height (in metres) of the ball as a function of t, the elapsed time (in seconds) since the ball was thrown. At what time does the ball reach its maximum height?

At $t = 2$s.

3. What are the coordinates of the vertex of the parabola that intercepts the x-axis at the points A(1, 0) and B(5, 0) and crosses the y-axis at the point C(0, 10)?

V(3, –8)

4. What are the coordinates of the points where the parabola, with the vertex V(–1, 4) and passing through the point P(0, 3), intersects the x-axis?

A(–3, 0) and B(1, 0)

5. The manager of a concert hall notices that when the admission price is set at \$20, the average attendance is 300. For each \$2 reduction in admission price, there are on average 50 spectators more. What admission price maximizes the revenue?

x: # of \$2 reductions; revenue = (20 – 2x)(300 + 50x)

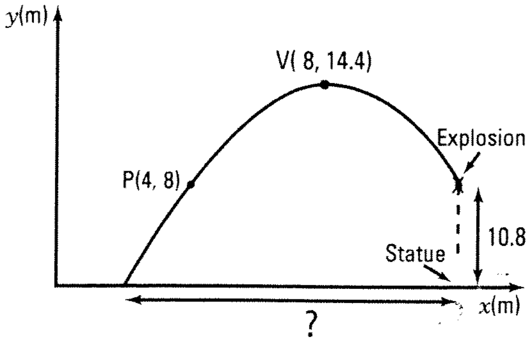
An admission price of \$16 will maximize revenue.

6. Determine the set of all real numbers that are greater than their square.

Any real number in the open interval]0, 1[.

7. During a fireworks display, a pyrotechnical device is launched from ground level as illustrated in the figure on the right. It explodes above a statue at a height of 10.8 m.

What is the distance separating the launching point and the statue, knowing that the device's trajectory is a parabola with the given vertex and point?



Rule of f: $f(x) = -0.4(x - 8)^2 + 14.4$;

Zeros of f: 2 m or 14 m

Values of x when y = 10.8 m: 5 m or 11 m. Distance = 9 m

3.1 Polynomial functions

ACTIVITY 1 Polynomial functions

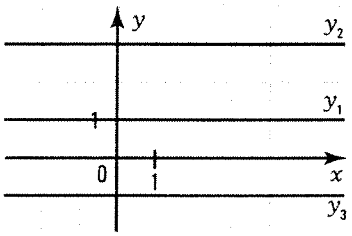
A **polynomial function** is any function with a polynomial as a rule.

a) Among the following functions, indicate which ones are polynomial functions.
If it is a polynomial function, indicate its degree.

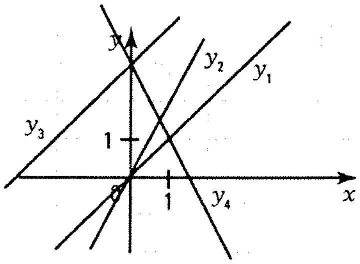
1. $P(x) = 3x^2 - 2x + 1$ ***Yes, 2nd degree***
2. $P(x) = -2x + 5$ ***Yes, 1st degree***
3. $P(x) = 5$ ***Yes, degree 0***
4. $P(x) = 2x + \frac{3}{x}$ ***No***
5. $P(x) = 3x^{-2} + x - 1$ ***No***
6. $P(x) = 2x^2 + 2\sqrt{x}$ ***No***

b) Using a graphing calculator, graph

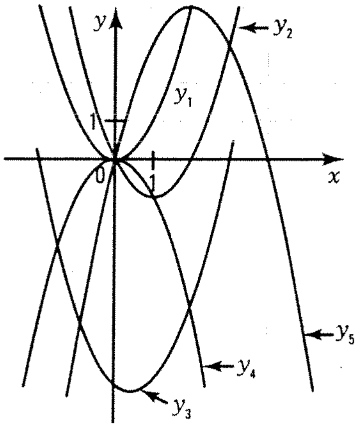
1. the zero degree polynomial functions.
   1) $y_1 = 1$ (zero degree basic function)
   2) $y_2 = 3$
   3) $y_3 = -1$



2. the 1st degree polynomial functions.
   1) $y_1 = x$ (1st degree basic function)
   2) $y_2 = 2x$
   3) $y_3 = x + 3$
   4) $y_4 = -2x + 3$



3. the 2nd degree polynomial functions.
   1) $y_1 = x^2$ (2nd degree basic function)
   2) $y_2 = x^2 - 2x$
   3) $y_3 = x^2 - x - 6$
   4) $y_4 = -x^2$
   5) $y_5 = -x^2 + 4x$



c) Complete.
The graphic representation of a polynomial function is
1. a horizontal line when the degree is ***0***.
2. an oblique line when the degree is ***1***.
3. a parabola when the degree is ***2***.

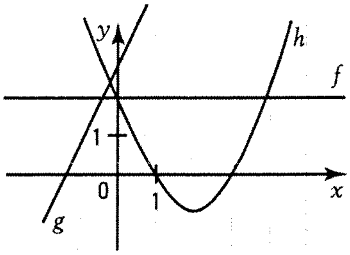
POLYNOMIAL FUNCTIONS

- A polynomial function is any function with a polynomial as a rule.

  Ex.: $f(x) = 2$ is a zero degree polynomial function.

  $g(x) = 2x + 3$ is a 1st degree polynomial function.

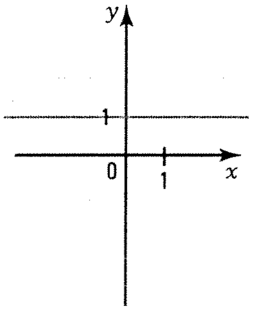
  $h(x) = x^2 - 4x + 3$ is a 2nd degree polynomial function.



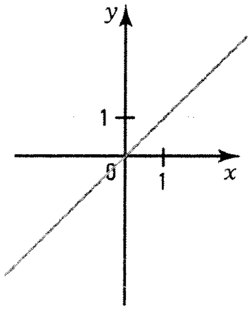
- The following table classifies polynomial functions according to their degree.

Degree	Basic polynomial function	Transformed polynomial function	Name
0	$f(x) = 1$	$f(x) = b$ where $b \in \mathbb{R}$	constant function
1	$f(x) = x$	$f(x) = ax$ where $a \in \mathbb{R}^*$	direct variation linear function
		$f(x) = ax + b$ where $a, b \in \mathbb{R}^*$	partial variation linear function
2	$f(x) = x^2$	$f(x) = ax^2 + bx + c$ where $a \in \mathbb{R}^*$	quadratic function
3	$f(x) = x^3$	$f(x) = ax^3 + bx^2 + cx + d$ where $a \in \mathbb{R}^*$	cubic function

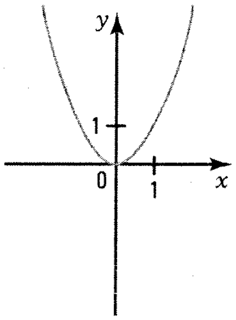
- Cartesian graphs of basic polynomial functions.



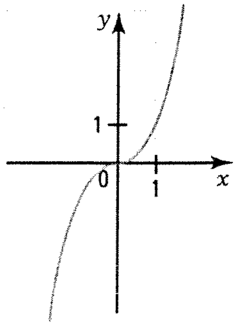
$f(x) = 1$



$f(x) = x$



$f(x) = x^2$

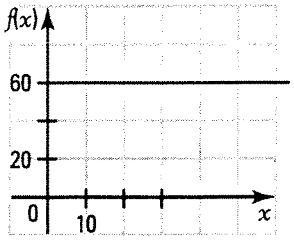


$f(x) = x^3$

3.2 Constant functions

ACTIVITY 1 Cost of a ticket

To attend a concert, a spectator must pay $60 regardless of his age.



a) If x represents the person's age and $f(x)$ the amount they must pay, what rule describes this situation? ***$f(x) = 60$***

b) Do all values of x have the same image? ***Yes***

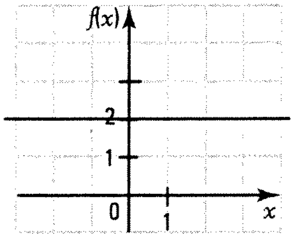
c) 1. Draw the Cartesian graph of the function f.

2. Describe the graph of function f. ***Horizontal line***

The function f is called constant since the image $f(x)$ remains constant for all values of x.

ACTIVITY 2 Study of a constant function

The rule of a function f is given by the equation $f(x) = 2$.



a) Do all the numbers x have the same image? ***Yes***

b) 1. Draw the Cartesian graph of the function f after completing the following table of values.

x	−3	−1	0	2
$f(x)$	***2***	***2***	***2***	***2***

2. Describe the graph of function f. ***Horizontal line***

c) Determine

1. the domain of this function. $\mathbb{R}$ 2. the range of this function. ***{2}***
3. the zeros of this function if they exist. ***No zeros***
4. the y-intercept of f. ***2***
5. the sign of this function. ***$f(x) > 0$ for all $x \in \mathbb{R}$***
6. the increasing and decreasing intervals of this function. ***It is constant over $\mathbb{R}$.***
7. the maximum or minimum of this function. ***max f = min f = 2***

d) By choosing two random points on the graph of f, verify that the rate of change is constant. What is it? ***0***

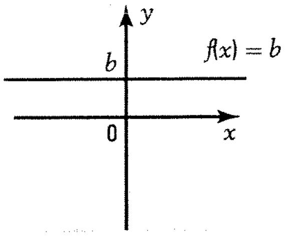
CONSTANT FUNCTIONS

- A constant function is a zero degree polynomial function. It is described by a rule of the form:

$$\boxed{f(x) = b}, b \in \mathbb{R}$$

- The Cartesian graph of a constant function is a horizontal line with the equation $y = b$.

Study of a constant function



- dom $f = \mathbb{R}$.
- ran $f = \{b\}$.
- The constant function has no zero unless $b = 0$.
- $f(x) > 0$ over $\mathbb{R}$ if $b > 0$.
  $f(x) < 0$ over $\mathbb{R}$ if $b < 0$.
- f is both increasing and decreasing over $\mathbb{R}$. We therefore say it is constant.
- $\max f = \min f = b$.

- The rate of change of any constant function is zero.
- A zero function is a constant function described by the rule $f(x) = 0$. Its Cartesian graph is represented by the x-axis.

1. The rental cost $c(x)$ of a reception hall as a function of the number x of guests is given by the rule $c(x) = 600$.

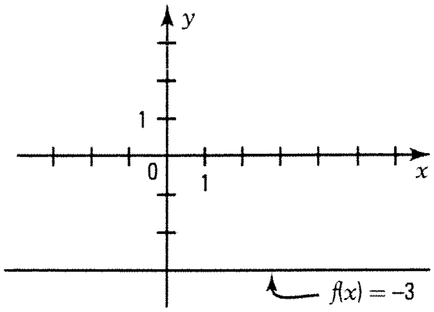
a) What is the rental cost if there are 30 guests? ***$600***

b) What is the rate of change of function c? ***0***

c) Describe the Cartesian graph of function c. ***Its Cartesian graph is a horizontal line starting at (0, 600). (A ray with end-points (0, 600) and (p, 600) depending on the capacity p of the hall).***

d) What is the domain and range of function c if the maximum capacity of the hall is 240 guests? ***dom c = [0, 240] and ran c = {600}***

2. Given the real function defined by $f(x) = -3$.



a) Draw the Cartesian graph of f.

b) Determine

1. the domain of this function. $\mathbb{R}$
2. the range of this function. ***{-3}***
3. the zeros of this function if they exist. ***No zeros***
4. the y-intercept. ***-3***
5. the sign of this function. ***f(x) < 0 over*** $\mathbb{R}$
6. the increasing and decreasing intervals of this function. ***f is constant***
7. the maximum or minimum of this function. ***max f = min f = -3***

c) What is the rate of change of this function? ***0***

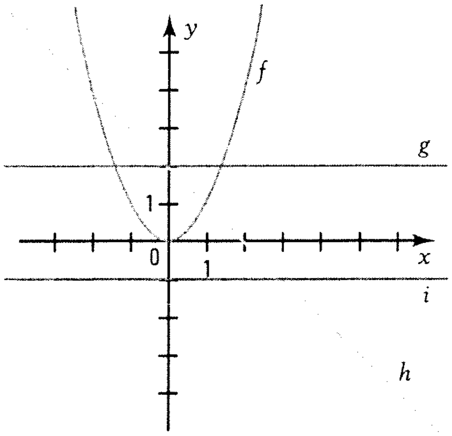
d) Is there a real number x for which its image in f is 2? ***No***

3. a) Which functions represented in the figure on the right are graphs of constant functions?

g and i

b) Determine the rule of these constant functions.

$g(x) = 2$ and $i(x) = -1$



4. Among the following situations, indicate which ones can be described by a constant function.

a) The price of a lottery ticket as a function of the buyer's income. ***Yes***

b) The price of piece of fabric as a function of the number of metres purchased. ***No***

c) The velocity of a free falling object as a function of time. ***No***

d) The volume of a classroom as a function of the number of students in the class. ***Yes***

e) The basic monthly cost of a telephone as a function of the number of people using it. ***Yes***

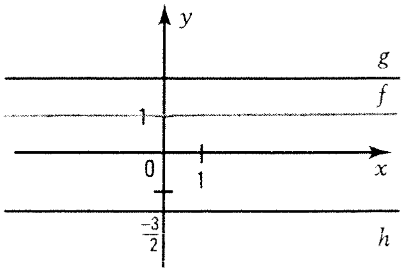
f) The weekly salary of an employee as a function of the amount of his sales, if he receives a base salary and

1. a commission on his sales. ***No***
2. no commission on his sales. ***Yes***

5. The given Cartesian graph of a relation is a vertical line (parallel to the y-axis). Is this the graph of a constant function? Justify your answer.

No, it is not a function.

6. The basic constant function $f(x) = 1$ and the constant function $g(x) = 2$ are represented on the right.



a) Which scale change will enable you to get the graph of g from the graph of f?

$(x, y) \to (x, 2y)$

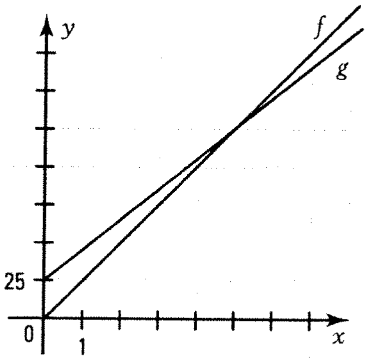
b) 1. Draw the image of the basic function according to the scale change $(x, y) \to \left(x, -\frac{3}{2}y\right)$ to get the graph of a function h.

2. What is the rule of function h? ***$h(x) = -\frac{3}{2}$***

3.3 Linear functions

ACTIVITY 1 Linear functions

Frank and Gina work on weekends in a boutique selling cell phones. Each day, Frank receives \$25 per cell phone sold whereas Gina receives a daily base salary of \$25 plus \$20 per cell phone sold.



a) Determine the rule of the function f which gives Frank's salary $f(x)$ as a function of the number x of cell phones he sold that day.

$f(x) = 25x$

b) Determine the rule of the function g which gives Gina's salary $g(x)$ as a function of the number x of cell phones she sold that day.

$g(x) = 20x + 25$

c) Represent functions f and g in the same Cartesian plane.

Function f is called a direct variation linear function since it is represented by an oblique line passing through the origin.

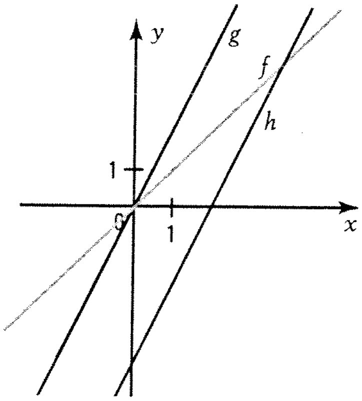
Function g is called a partial variation linear function since it is represented by an oblique line not passing through the origin.

d) How many cell phones must Frank and Gina sell to earn the same salary?

5 cell phones

ACTIVITY 2 Transformation of a basic linear function

The basic 1st degree linear function $f(x) = x$ is represented on the right.



a) 1. Draw the image of the graph of f according to the vertical scale change $(x, y) \rightarrow (x, 2y)$ in order to obtain the Cartesian graph of a function g.

2. What is the rule of function g? ***$g(x) = 2x$***

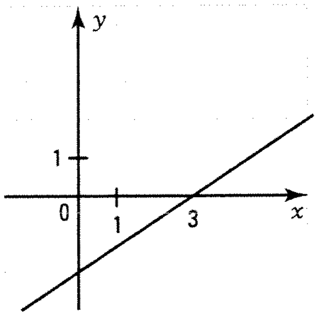
b) 1. Draw the image of the graph of g according to the vertical translation $(x, y) \rightarrow (x, y - 4)$ in order to obtain the Cartesian graph of a function h.

2. What is the rule of function h? ***$h(x) = 2x - 4$***

c) Which of the two linear functions g and h is

1. direct variation? ***g*** 2. partial variation? ***h***

ACTIVITY 3 Study of a linear function



Consider the linear function $f(x) = \frac{2}{3}x - 2$.

a) Represent function f in the Cartesian plane.

b) Determine

1. dom f. **$\mathbb{R}$** 2. ran f. **$\mathbb{R}$**

c) What is the zero of function f? ***3***

d) What is the y-intercept of function f? ***-2***

e) What is the sign of function f?

$f(x) \geqslant 0$ over $[3, +\infty[$; $f(x) \leqslant 0$ over $]-\infty, 3]$

f) What is the variation of function f? ***It is increasing over $\mathbb{R}$.***

g) Does this function have any extrema? ***No***

h) By choosing two random points on the graph of f, verify that the rate of change is constant. What is it?

$\frac{2}{3}$

LINEAR FUNCTIONS

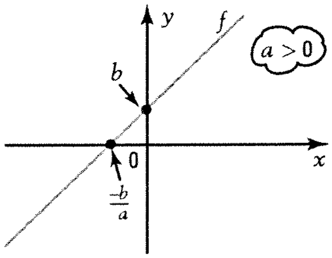
- A linear function is a 1st degree polynomial function. It is described by a rule of the form:

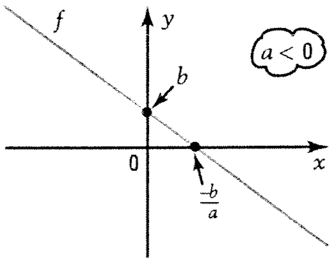
$$\boxed{f(x) = ax + b}, a \neq 0$$

- The Cartesian graph of a linear function is an oblique line.
  - dom $f = \mathbb{R}$
  - ran $f = \mathbb{R}$
  - The linear function has one real zero equal to $\frac{-b}{a}$.
  - The y-intercept of the function (or initial value) is b.
  - If $a > 0$; $f(x) \leqslant 0$ if $x \in \left]-\infty, \frac{-b}{a}\right]$

    $f(x) \geqslant 0$ if $x \in \left[\frac{-b}{a}, +\infty\right[$
  - If $a < 0$; $f(x) \geqslant 0$ if $x \in \left]-\infty, \frac{-b}{a}\right]$

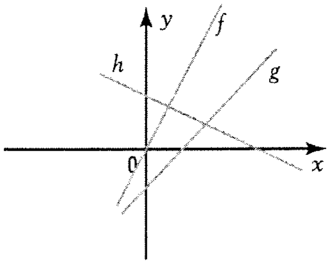
    $f(x) \leqslant 0$ if $x \in \left[\frac{-b}{a}, +\infty\right[$
  - If $a > 0$, f is increasing over $\mathbb{R}$.
    If $a < 0$, f is decreasing over $\mathbb{R}$.
  - There is no maximum or minimum.





- The rate of change of a linear function is constant and is equal to parameter a.
- A linear function is said to be of
  - direct variation when it is represented by an oblique line passing through the origin ($b = 0$).
  - partial variation when it is represented by an oblique line not passing through the origin ($b \neq 0$).

  Ex.: f is a direct variation linear function, g and h are partial variation linear functions.

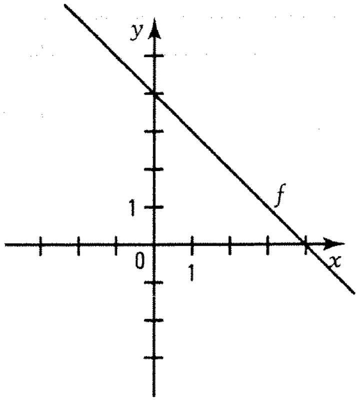


1. Given the real function f defined by $f(x) = -x + 4$.

a) Draw the Cartesian graph of f.

b) Determine

1. the domain of this function. $\mathbb{R}$
2. the range of this function. $\mathbb{R}$
3. the zero of this function. ***4***
4. the y-intercept. ***4***
5. the sign of this function.
   $f(x) \geqslant 0$ if $x \in]-\infty, 4]$; $f(x) \leqslant 0$ if $x \in [4, +\infty[$
6. the maximum or minimum of this function if they exist.
   None
7. the increasing and decreasing intervals of this function. ***f is decreasing over $\mathbb{R}$.***



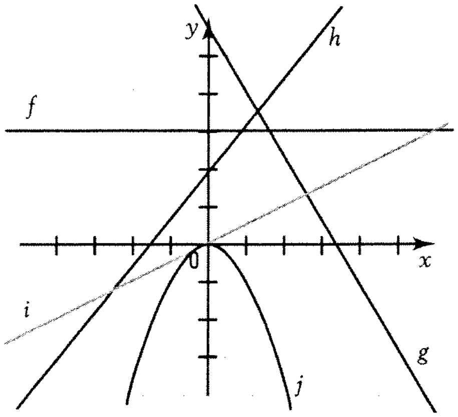
2. Given the graphs represented in the figure on the right, which functions are

a) constant? ***f***

b) linear? ***g, h and i***

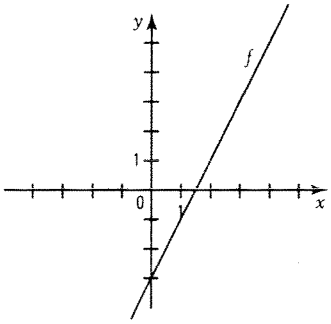
c) linear of direct variation? ***i***

d) linear of partial variation? ***g and h***

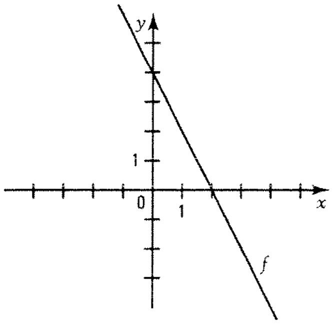


3. Draw the Cartesian graphs of the linear functions f defined by the following equations.

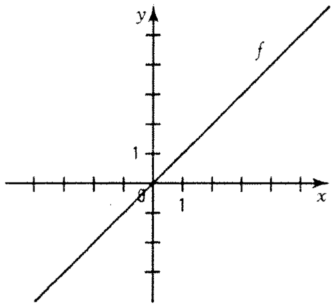
a) $f(x) = 2x - 3$



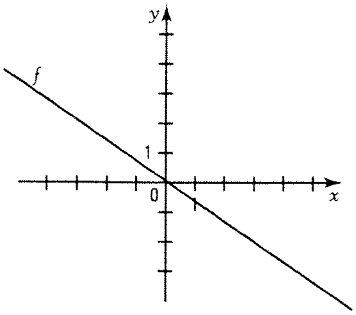
b) $f(x) = -2x + 4$



c) $f(x) = x$



d) $f(x) = -\frac{2}{3}x$



4. A real function f is defined by $f(x) = ax + b$. What constraints must be placed on parameters a and b for the function f to be a function that is

a) constant? $a = 0; b \neq 0$

b) linear? $a \neq 0$

c) linear of direct variation? $a \neq 0; b = 0$

d) linear of partial variation? $a \neq 0; b \neq 0$

e) strictly increasing? $a > 0$

f) strictly decreasing? $a < 0$

g) one with a rate of change of 3? $a = 3$

h) one with a y-intercept of -2? $b = -2$

i) one with a zero of 0? $b = 0$

j) one with a zero of -0.5? $a = 2b$

5. Complete the following table.

	$f(x) = 3$	$f(x) = x$	$f(x) = -2x + 3$	$f(x) = \frac{2}{5}x - \frac{1}{2}$
Dom f	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
Ran f	$\{3\}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
Rate of change	0	1	-2	$\frac{2}{5}$
Zero (if it exists)	***none***	0	$\frac{3}{2}$	$\frac{5}{4}$
Initial value	3	0	3	$-\frac{1}{2}$
Variation	***constant***	***strictly increasing***	***strictly decreasing***	***strictly increasing***
Sign	$f(x) > 0$ ***over*** $\mathbb{R}$	$f(x) < 0$ if $x \in]-\infty, 0[$ $f(x) > 0$ if $x \in]0, +\infty[$	$f(x) > 0$ if $x \in]-\infty, \frac{3}{2}[$ $f(x) < 0$ if $x \in]\frac{3}{2}, +\infty[$	$f(x) < 0$ if $x \in]-\infty, \frac{5}{4}[$ $f(x) > 0$ if $x \in]\frac{5}{4}, +\infty[$

6. The graph of a linear function passes through the points A(1, 8) and B(−2, 32).

a) Find the rule of this function. $y = -8x + 16$

b) What is the zero of this function? $x = 2$

c) Indicate over which interval this function is negative? $[2, +\infty[$

d) What is the variation of this function? ***f is decreasing over*** $\mathbb{R}$.

7. Consider the function $f(x) = -5x + 70$. Over what interval is this function positive?

$]-\infty, 14]$

8. Before going on a trip, Eric filled his gas tank. The quantity $f(x)$ of gas remaining in the tank as a function of the number x of kilometers traveled is given by $f(x) = 72 - 0.12x$.

a) What is the rate of change of function f? ***−0.12 l/km***

Interpret this rate. ***The car consumes 0.12 l/km traveled.***

b) Does the function f represent an increasing or a decreasing situation? Justify your answer.

A decreasing situation since the rate of change −0.12 is negative.

c) What is the initial value of function f? What does it represent? ***The initial value is 72.***

It represents the initial quantity of gas in the tank before going on the trip.

d) How many litres of gas are remaining after traveling 350 km? ***30 litres***

e) 1. After how many kilometers is the tank empty? ***600 km***

2. What does this value represent for the function *f*? ***The zero of function f***

f) What is the distance traveled if there remains 15 *l* of gas in the tank? ***475 km***

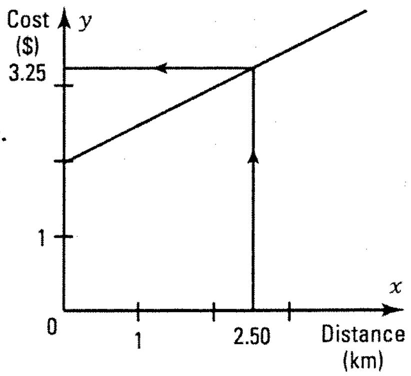
9. The table of values on the right describes the function which gives the cost y of a taxi ride as a function of the distance traveled x.

Distance (km)	0	1	2	3	4
Cost ($)	2.00	2.50	3.00	3.50	4.00

a) What is the rule of this function? ***$y = 0.50x + 2$***

b) Is this function increasing or decreasing? Justify your answer.
The function is increasing since the rate of change is positive.

c) Represent this function in the Cartesian plane.



d) Calculate the cost of a 2.5 km trip using:

1. the rule. ***$3.25***
2. the graph. ***$3.25***

10. Let C represent the temperature (in degrees Celsius) and F represent the temperature (in degrees Fahrenheit).

a) The rule of the linear function which converts a temperature in degrees Celsius into degrees Fahrenheit is: $F = \frac{9}{5}C + 32$.

1. Indicate and interpret the rate of change for this function. ***$\frac{9}{5} = 1.8$. Each increase of 1° Celsius corresponds to an increase of 1.8° Fahrenheit.***
2. Indicate and interpret the initial value of this function. ***32. A temperature of 0 °C corresponds to 32 °F.***

b) 1. What is the rule of the inverse function which converts a temperature in degrees Fahrenheit into degrees Celsius? ***$C = \frac{5}{9}(F - 32)$***

2. Indicate and interpret the rate of change for this function. ***$\frac{5}{9}$. Each increase of 1° Fahrenheit corresponds to an increase of $\frac{5}{9}$° Celsius.***
3. Indicate and interpret the initial value of this function. ***−18°. A temperature of 0 °F corresponds to a temperature of −18°C.***

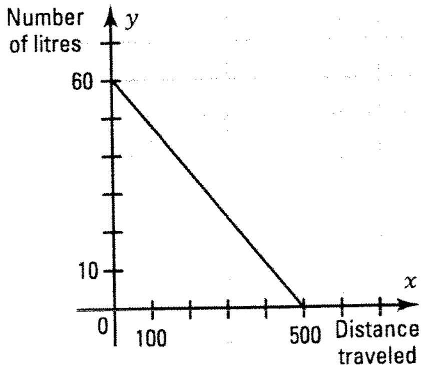
11. Alan and Muriel train at a health club for an upcoming equestrian competition. They must pay a club membership fee plus an amount for each training session. Alan pays $360 for 4 sessions whereas Muriel pays $600 for 10 sessions.

a) What is the rule of the function which gives the total amount y paid to the club as a function of the number x of training sessions. ***$y = 40x + 200$***

b) Determine and interpret

1. the rate of change of this function. ***$40/session. The cost per training session is $40.***
2. the initial value of this function. ***$200. The club membership fee is $200.***

12. At the beginning of a car trip, Julie fills her gas tank. She has 36 litres left in the tank after traveling 200 km and 30 litres left after 250 km.



a) What is the rule of the function which gives the number of litres y remaining in the tank as a function of the number x of kilometers traveled? ***$y = 60 - 0.12x$***

b) Determine and interpret

1. the rate of change of the function. ***0.12 l/km***
The car consumes 12 l/100 km.
2. the initial value of the function. ***60 l. The capacity of the gas tank is 60 litres.***
3. the zero of the function. ***After 500 km, the tank is empty.***

c) Represent this function in the Cartesian plane.

d) Determine for this function

1. the domain. ***[0, 500]***
2. the range. ***[0, 60]***

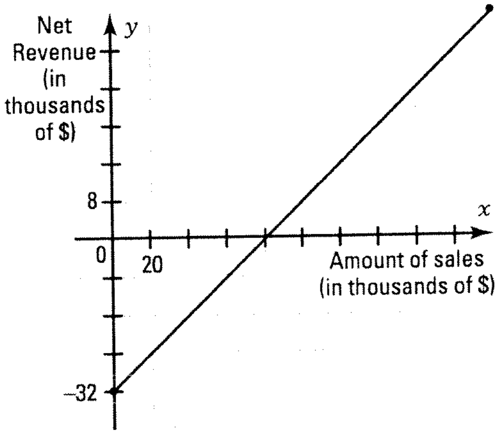
e) Is the function increasing or decreasing? Justify your answer.
The function is decreasing since the rate of change is negative.

13. The manager of a clothing store establishes that the monthly revenue corresponds to 40% of the amount x of sales during that month. The store's fixed monthly costs are \$32 000 and the store cannot sell more than \$200 000 in one month.

a) What is the rule which gives the net revenue y as a function of the amount x of sales? ***$y = 0.4x - 32\ 000$***

b) Determine and interpret

1. the zero of the function. ***80 000. The store must sell \$80 000 to have a net revenue of zero.***
2. the initial value of the function. ***−32 000. The initial value corresponds to the fixed monthly costs of \$32 000.***



c) Represent this function in the Cartesian plane.

d) Is the function increasing or decreasing? Justify your answer.
The function is increasing since the rate of change is positive.

e) Determine for this function

1. the domain. ***[0, 200 000]***
2. the range. ***[-32 000, 48 000]***

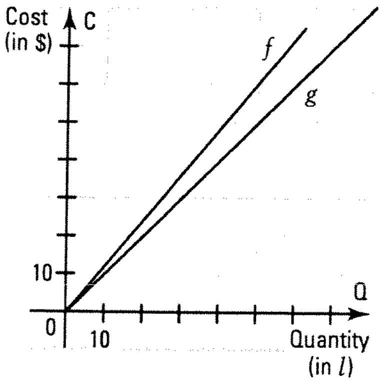
f) Study the sign of this function.
$f(x) \leq 0$ if $x \in [0, 80\ 000]$; $f(x) \geq 0$ if $x \in [80\ 000, 200\ 000]$.

14. The graph on the right represents the function f which gives the cost C (in dollars) of filling a gas tank with Q litres of gas.

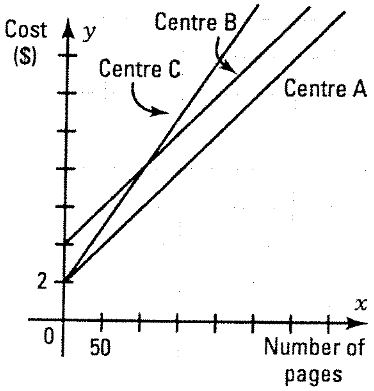
a) What is the rule of this function? ***C = 1,2 Q***

b) A few months later, the price per litre of gas is reduced by 20 cents.

1. Explain the consequence on the graph of function f. ***A vertical scale reduction of the graph.***
2. What is the rule of the function g which gives this new cost for x litres. ***C = Q***
3. Draw the graph of function g.



15. Photocopy centre A charges an amount for binding a document plus a cost per page. The graph on the right illustrates the function which gives the total cost charged by centre A for a document of x pages.



a) For centre A, what is

1. the cost per page? ***$0.04***
2. the cost of the binding? ***$2***
3. the rule which gives the total cost y charged by centre A as a function of the number x of pages? ***y = 0.04x + 2***

b) Photocopy centre B charges the same amount per page as centre A but $4 for the binding.

1. Explain how to modify the graph representing centre A to get the graph representing the centre B. ***The graph of centre A is translated vertically 1 unit ($2) upward.***
2. What is the rule which gives the total cost charged by centre B as a function of the number of pages? ***y = 0.04x + 4***
3. Graph the function representing centre B.

c) Photocopy centre C charges the same amount for the binding as centre A but 2 cents more per page.

1. What is the rule which gives the total cost charged by centre C as a function of the number of pages? ***y = 0.06x + 2***
2. Graph the function representing centre C.

d) Which one of the three centres is a better deal for the consumer? ***Centre A***

e) Which of the two remaining centres is a better deal for the consumer if centre A is closed?

Centre C if the document has less than 100 pages.

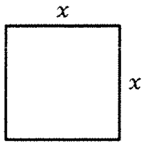
Centre B if the document has more than 100 pages.

Centre B or C if the document has exactly 100 pages.

3.4 Quadratic functions – Standard form

ACTIVITY 1 Area of a square

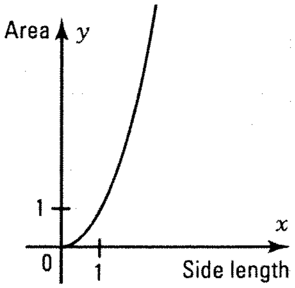
Consider the square with sides x given on the right.



a) What is the rule of the function which gives the area y of the square as a function of the side length x? ***$y = x^2$***

b) Complete the table of values giving the area y as a function of the side length x.

x	0	0.5	1	1.5	2	3
y	***0***	***0.25***	***1***	***2.25***	***4***	***9***



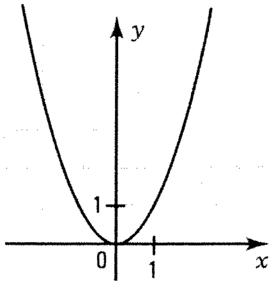
c) Represent this function in the Cartesian plane.

d) Explain why the domain of the function is $\mathbb{R}_+$.
The side length x of a square cannot be negative.

e) Is the rate of change between any two points on the graph constant? ***No***

ACTIVITY 2 Basic quadratic function

Consider the function $f(x) = x^2$.



a) Complete the following table of values.

x	−2	−1	0	1	2
y	***4***	***1***	***0***	***1***	***4***

b) Represent the function in the Cartesian plane.

c) 1. Explain why $f(-x) = f(x)$, for any x.
Two opposite numbers have the same square. Thus, $(-x)^2 = x^2$.

2. Therefore, what does the y-axis represent for the drawn curve? ***An axis of symmetry.***

d) Determine

1. dom f. $\mathbb{R}$
2. ran f. $\mathbb{R}_+$
3. the zero of f. ***0***
4. the y-intercept of f. ***0***

e) What is the sign of function f? ***$f(x) \geqslant 0$ over $\mathbb{R}$***

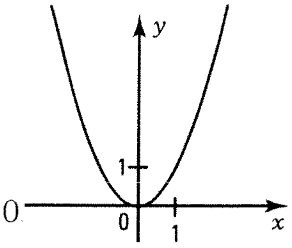
f) Over what interval is the function f

1. decreasing? ***$]-\infty, 0]$***
2. increasing? ***$[0, +\infty[$***

g) What is the minimum of function f? ***0***

BASIC QUADRATIC FUNCTION

- The function $f(x) = x^2$ is called the basic quadratic function.
- The Cartesian graph is a parabola with vertex V (0, 0).
  - dom $f = \mathbb{R}$.
  - ran $f = \mathbb{R}_+$.
  - The y-intercept of f is 0.
  - The function has only one zero which is equal to 0. $\forall x \in \mathbb{R}: f(x) \geqslant 0$
  - The function is decreasing over $]-\infty, 0]$, increasing over $[0, +\infty[$.
  - The minimum of the function is 0.
  - The y-axis with equation $x = 0$ is an axis of symmetry for the parabola. $\forall x \in \mathbb{R}: f(-x) = f(x)$.



1. Consider the basic quadratic function $f(x) = x^2$.

a) Explain how to deduce the graph of the function $g(x) = -x^2$.
By a reflection over the x-axis.

b) Draw the graph of function g.

c) Determine

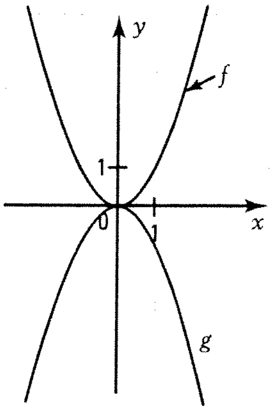
1. dom g. $\mathbb{R}$
2. ran g. $\mathbb{R}_-$
3. the zero of g. ***0***
4. the y-intercept of g. ***0***

d) What is the sign of function g? $g(x) \leqslant 0, \ \forall x \in \mathbb{R}$

e) Over what interval is the function g

1. increasing? $]-\infty, 0]$
2. decreasing? $[0, +\infty[$

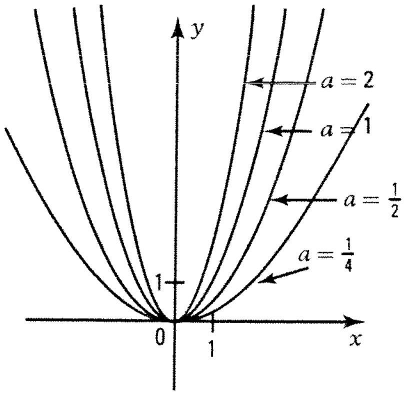
f) What is the maximum of function g? ***0***



ACTIVITY 3 Role of parameter *a*

a) Consider the basic quadratic function $y = x^2$ and the function $f(x) = ax^2$ $(a > 0)$.

1. Represent the function f when
   1) $a = \frac{1}{4}$ 2) $a = \frac{1}{2}$ 3) $a = 2$
2. As the parameter a increases, do we observe a vertical stretch or reduction of the parabola?
A vertical stretch.



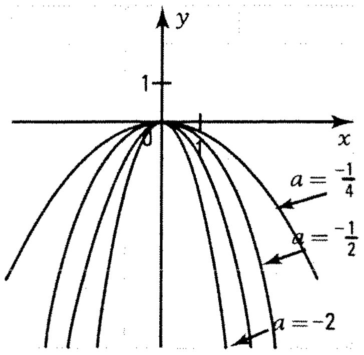
b) Consider the quadratic function $y = -x^2$ and the function $f(x) = ax^2$ $(a < 0)$.

1. Represent the function f when

   1) $a = -\frac{1}{4}$ 2) $a = \frac{1}{2}$ 3) $a = -2$

2. As the absolute value of parameter a increases, do we observe a vertical stretch or reduction of the parabola?

   A vertical stretch.



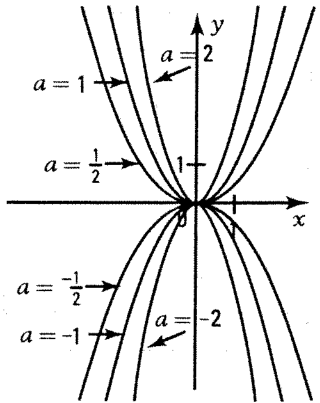
c) Consider the parabola with equation $y = ax^2$.
Is the parabola open upward or downward when

1. $a > 0$? ***Upward*** 2. $a < 0$? ***Downward***

ROLE OF PARAMETER a

Consider the parabola with equation $y = ax^2$.

- The sign of a determines whether the parabola is open upward or downward.
  - $a > 0$: The parabola is open upward.
  - $a < 0$: The parabola is open downward.
- The absolute value of a influences the opening of the parabola.
  As the absolute value of parameter a increases, we observe a vertical stretch of the parabola.



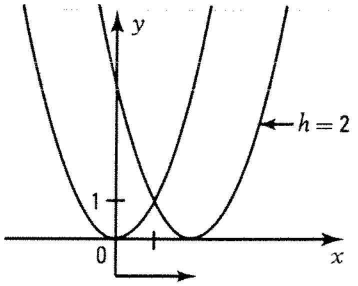
ACTIVITY 4 Role of parameter h

Consider the basic quadratic function $y = x^2$ and the function $y = (x - h)^2$.

a) 1. Represent the function when $h = 2$.

2. Explain how to deduce its graph from the graph of the basic function.

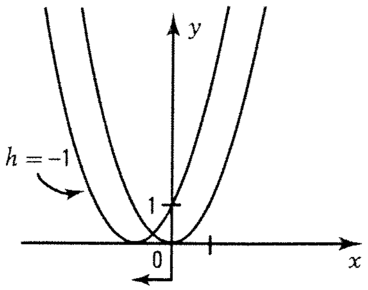
   A horizontal translation of 2 units to the right.



b) 1. Represent the function when $h = -1$.

2. Explain how to deduce its graph from the graph of the basic function.

   A horizontal translation of 1 unit to the left.



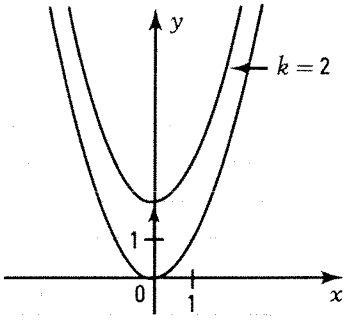
ACTIVITY 5 Role of parameter k

Consider the basic quadratic function $y = x^2$ and the function $y = x^2 + k$.

a) 1. Represent the function when $k = 2$.

2. Explain how to deduce its graph from the graph of the basic function.

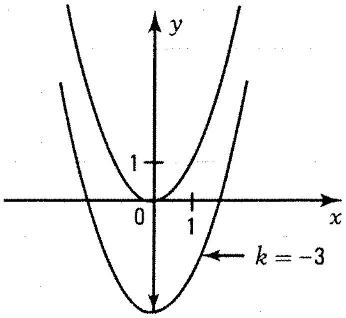
A vertical translation of 2 units upward.



b) 1. Represent the function when $k = -3$.

2. Explain how to deduce its graph from the graph of the basic function.

A vertical translation of 3 units downward.



ACTIVITY 6 Role of parameters h and k

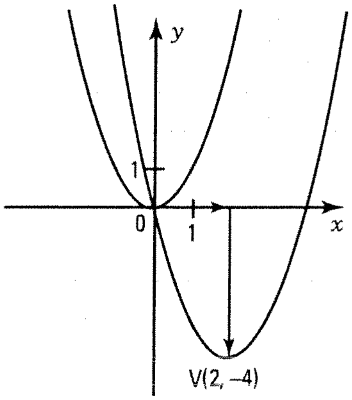
Consider the basic quadratic function $y = x^2$ and the function $y = (x - h)^2 + k$.

a) Represent the function $y = (x - 2)^2 - 4$.

b) Explain how to deduce its graph from the graph of the basic function.

A horizontal translation of 2 units to the right followed by a vertical translation of 4 units downward.

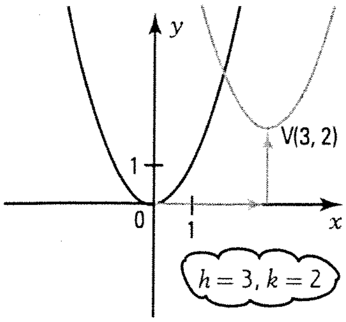
c) What are the coordinates of the vertex V of the resulting parabola? ***V(2, −4)***



ROLE OF PARAMETERS h AND k

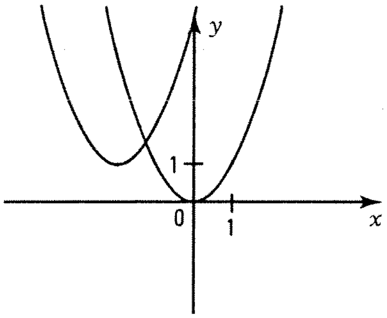
- The graph of the quadratic function with equation $y = (x - h)^2 + k$ can be deduced from the basic quadratic function with equation $y = x^2$ by
  - a horizontal translation of $|h|$ units to the right if $h > 0$ or to the left if $h < 0$.
  - a vertical translation of $|k|$ units upward if $k > 0$ or downward if $k < 0$.
- The vertex V of the parabola has the coordinates (h, k).

Ex.: The graph of $y = (x - 3)^2 + 2$ is deduced from the graph of $y = x^2$ by a horizontal translation of 3 units to the right followed by a vertical translation of 2 units upward. The vertex of the parabola is V(3, 2).



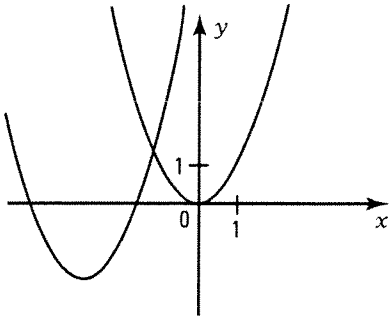
2. For each of the following functions,
1. determine parameters h and k.
2. deduce the graph from the graph of the basic quadratic function.

a) $y = (x + 2)^2 + 1$



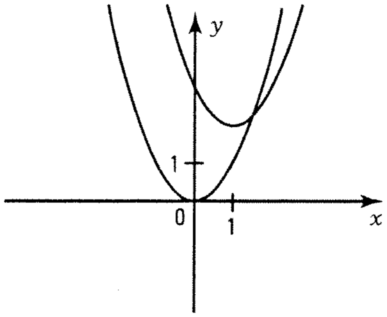
h = -2, k = 1

b) $y = (x + 3)^2 - 2$



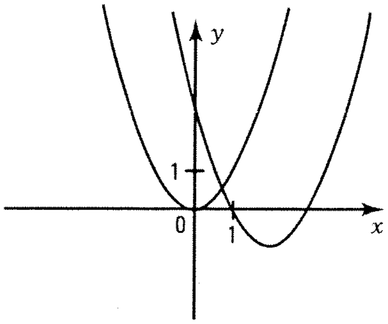
h = -3, k = -2

c) $y = (x - 1)^2 + 2$



h = 1, k = 2

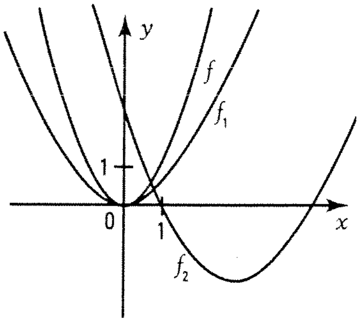
d) $y = (x - 2)^2 - 1$



h = 2, k = -1

ACTIVITY 7 Quadratic function – Standard form

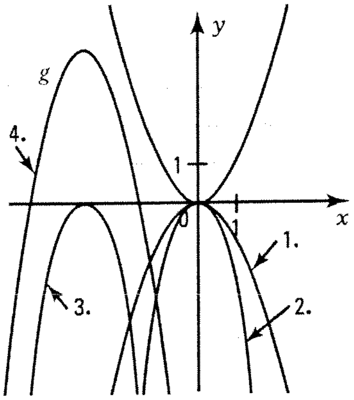
a) Consider the basic quadratic function $f(x) = x^2$, the function $f_1(x) = \frac{1}{2}x^2$ and the function $f_2(x) = \frac{1}{2}(x - 3)^2 - 2$.



1. Explain how to deduce the graph of f_1 from the graph of the basic function and draw the graph of f_1.
   By a vertical reduction of factor $\frac{1}{2}$.
2. Explain how to deduce the graph of f_2 from the graph of f_1 and draw the graph of f_2.
   By a horizontal translation of 3 units to the right followed by a vertical translation of 2 units downward.
3. Complete the table of values to verify the graph of f_2.
4. What are the coordinates of the parabola's vertex representing f_2? ***V (3, -2)***

x	1	2	3	4	5
y	***0***	***-1.5***	***-2***	***-1.5***	***0***

b) Consider the function $g(x) = -2(x + 3)^2 + 4$.



1. Draw the graph of this function from the graph of the basic function. Explain your procedure.

   1. I draw the curve $y = -x^2$ by symmetry about the x-axis.

   2. I draw the curve $y = -2x^2$ by a vertical stretch of factor 2.

   3. I translate horizontally 3 units to the left.

   4. I translate vertically 4 units upward.

2. Complete the table of values on the right to verify the graph of g.

3. What are the coordinates of the vertex?

   V(–3, 4)

x	–5	–4	–3	–2	–1
y	**–4**	**2**	**4**	**2**	**–4**

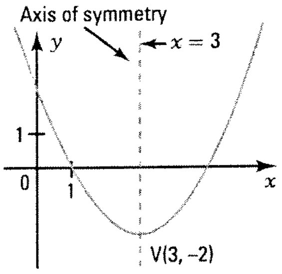
QUADRATIC FUNCTION – STANDARD FORM

- The standard form of the rule of a quadratic function is:

$$\boxed{y = a(x - h)^2 + k}$$

- If $a > 0$, the parabola is open upward. ∪

  If $a < 0$, the parabola is open downward. ∩
- The vertex of the parabola is: $V(h, k)$.
- The parabola's axis of symmetry is the vertical line passing through the parabola's vertex. Its equation is: $x = h$.



Ex.: $y = \frac{1}{2}(x - 3)^2 - 2$

x	1	2	3	4	5
y	0	–1,5	–2	–1,5	0

ACTIVITY 8 Finding the zeros – Standard form

a) Justify the steps of finding the zeros of the quadratic function $f(x) = a(x - h)^2 + k$.

$a(x - h)^2 + k = 0$

$\Leftrightarrow a(x - h)^2 = -k$ ***Subtract k from each side.***

$\Leftrightarrow (x - h)^2 = -\frac{k}{a}$ ***Divide each side by a***

$\Leftrightarrow x - h = -\sqrt{\frac{-k}{a}}$ or $x - h = \sqrt{\frac{-k}{a}}$ if $-\frac{k}{a} > 0$ ***Two opposite numbers have the same square.***

$\Leftrightarrow x = h - \sqrt{\frac{-k}{a}}$ or $x = h + \sqrt{\frac{-k}{a}}$ ***Isolate x in each equation.***

b) Indicate the number of zeros when

1. $-\frac{k}{a} > 0$. ***2 zeros*** 2. $-\frac{k}{a} = 0$. ***1 zero*** 3. $-\frac{k}{a} < 0$. ***no zeros***

c) What are the zeros when

1. $-\frac{k}{a} > 0$. ***$h - \sqrt{-\frac{k}{a}}$ and $h + \sqrt{\frac{-k}{a}}$*** 2. $-\frac{k}{a} = 0$. ***h***

d) Find the zeros of the following quadratic functions.

1. $f(x) = 2\left(x - \frac{1}{2}\right)^2 - 8$ $-\frac{3}{2}$ ***and*** $\frac{5}{2}$
2. $f(x) = 2(x - 1)^2 - 10$ ***$1 - \sqrt{5}$ and $1 + \sqrt{5}$***
3. $f(x) = -2(x + 3)^2$ ***-3***
4. $f(x) = 3(x - 1)^2 + 6$ ***no zeros***

FINDING THE ZEROS – STANDARD FORM

- The number of zeros of the quadratic function $f(x) = a(x - h)^2 + k$ depends on the sign of $-\frac{k}{a}$.
- $-\frac{k}{a} > 0$: There are two zeros x_1 and x_2.

$$\boxed{x_1 = h - \sqrt{-\frac{k}{a}} \text{ and } x_2 = h + \sqrt{-\frac{k}{a}}}$$

- $-\frac{k}{a} = 0$: There is only one zero or the two zeros x_1 and x_2 are equal

$$\boxed{x_1 = x_2 = h}$$

- $-\frac{k}{a} < 0$: There are no zeros.
- Note that the function has no zeros when a and k have the same sign.

Ex.: $f(x) = 2(x - 1)^2 - 8$
$a = 2;\ h = 1;\ k = -8;\ -\frac{k}{a} = 4$
$x_1 = 1 - \sqrt{4} = -1$ and $x_2 = 1 + \sqrt{4} = 3$
The zeros are -1 and 3.

Ex.: $f(x) = 2(x - 1)^2$
$a = 2;\ h = 1;\ k = 0;\ \frac{-k}{a} = 0$
$x_1 = x_2 = 1$
The only zero is 1.

Ex.: $f(x) = 2(x - 1)^2 + 8$
$a = 2;\ h = 1;\ k = 8;\ -\frac{k}{a} = -4$
There is no zero since $-\frac{k}{a} < 0$.

3. Find the zeros of the following functions.

a) $f(x) = -4(x + 2)^2 + 16$ ***-4 and 0***

b) $f(x) = \frac{1}{2}(x + 3)^2 - 2$ ***-5 and -1***

c) $f(x) = 2(x + 1)^2 - 10$ ***$-1 - \sqrt{5}$ and $-1 + \sqrt{5}$***

d) $f(x) = (x - 1)^2 - 7$ ***$1 - \sqrt{7}$ and $1 + \sqrt{7}$***

e) $f(x) = -2(x + 3)^2$ ***-3***

f) $f(x) = 3(x - 2)^2 - 27$ ***-1 and 5***

g) $f(x) = 3(x - 1)^2 + 6$ ***none***

h) $f(x) = -(x + 1)^2$ ***-1***

4. Consider the quadratic function $f(x) = a(x - h)^2 + k$.

a) If $a > 0$, indicate the number of zeros when

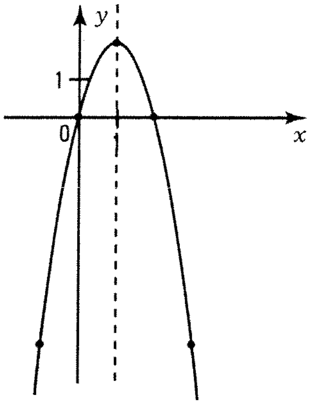
1. $k > 0$. ***none*** 2. $k = 0$. ***only one*** 3. $k < 0$. ***two zeros***

b) If $a < 0$, indicate the number of zeros when

1. $k > 0$. ***two zeros*** 2. $k = 0$. ***only one*** 3. $k < 0$. ***none***

ACTIVITY 9 Graphing a parabola – Standard form

Consider the function $f(x) = -2(x - 1)^2 + 2$.



a) Use the following procedure to graph the quadratic function.

1. Identify the parameters a, h and k. ***$a = -2$; $h = 1$; $k = 2$***
2. Is the parabola open upward or downward?
   Justify your answer. ***Downward since $a < 0$.***
3. What are the coordinates of the parabola's vertex? ***V(1, 2)***
4. Find, if they exist, the points where the parabola intersects the x-axis, in other words the zeros of the function.
   $-\frac{k}{a} > 0$. There are two zeros: $x_1 = 0$ and $x_2 = 2$.
5. Complete the following table of values.
6. Graph the parabola.

x	−1	0	1	2	3
y	***−6***	***0***	***2***	***0***	***−6***

b) Find the y-intercept, in other words the initial value of the function.
$f(0) = 0$. The y-intercept is equal to 0.

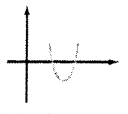
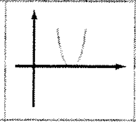
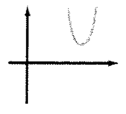
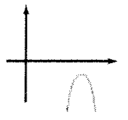
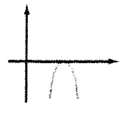
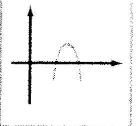
c) 1. Draw the parabola's axis of symmetry.
2. What is its equation? ***$x = 1$***

d) Use a graphing calculator to verify your answers.

GRAPHING A PARABOLA – STANDARD FORM

Procedure

1. Identify the parameters a, h and k.
2. Determine the opening according to the sign of a.
3. Determine the coordinates of the vertex. $V(h, k)$.
4. Find, if they exist, the zeros.
   $x_1 = h - \sqrt{\frac{-k}{a}}$, $x_2 = h + \sqrt{\frac{-k}{a}}$
5. Find the y-intercept.
6. Complete a table of values.
7. Graph the parabola. We observe 6 possible situations according to the signs of a and k.

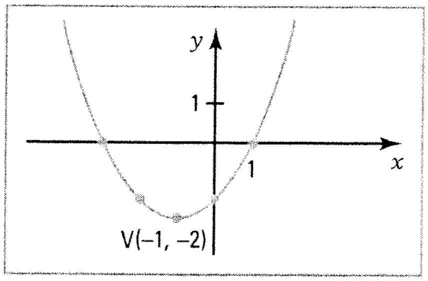
	$k < 0$	$k = 0$	$k > 0$
$a > 0$			
$a < 0$			

Ex.: $f(x) = \frac{1}{2}(x + 1)^2 - 2$

1. $a = \frac{1}{2}$; $h = -1$; $k = -2$.
2. Open upward since $a > 0$.
3. V(−1, −2)
4. $x_1 = -3$ and $x_2 = 1$ are the zeros of f.
5. $f(0) = -1.5$. The y-intercept is −1.5.
6.

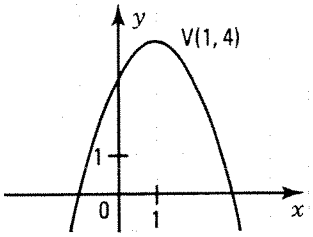
x	−3	−2	−1	0	1
y	0	−1.5	−2	−1.5	0

7.

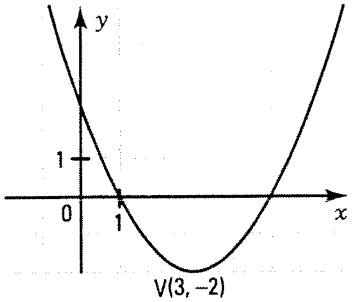


5. Graph the following parabolas.

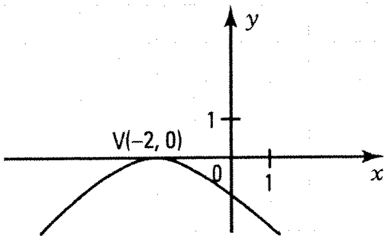
a) $y = -(x-1)^2 + 4$



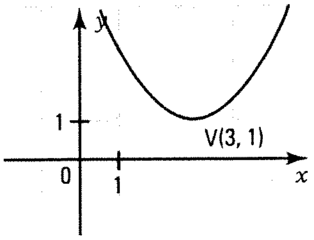
b) $y = \frac{1}{2}(x-3)^2 - 2$



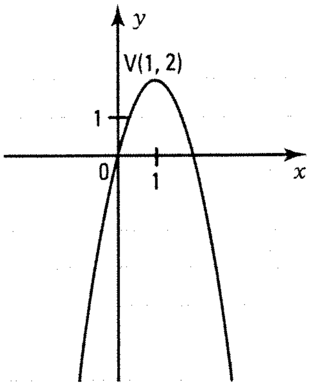
c) $y = -\frac{1}{4}(x+2)^2$



d) $y = \frac{1}{2}(x-3)^2 + 1$



6. Consider the function $f(x) = -2(x-1)^2 + 2$ represented on the right.



a) What is the domain of *f*? ***dom f = ℝ***

b) What is the range of *f*? ***ran f =]−∞, 2]***

c) What are the zeros of *f*? ***0 and 2***

d) What is the *y*-intercept of *f*? ***0***

e) What is the sign of *f*? $f(x) \leqslant 0$ ***if*** $x \in \,]-\infty, 0] \cup [2, +\infty[$

$f(x) \geqslant 0$ ***if*** $x \in [0, 2]$

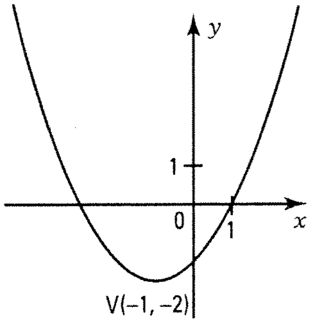
f) Complete the study of the variation of *f*.

1. *f* is increasing over ***]−∞, 1]***. 2. *f* is decreasing over ***[1, +∞[***.

g) 1. Does function *f* reach a maximum? If yes, what is it? ***Yes; max f = 2***

2. Does function *f* reach a minimum? ***no***

7. Consider the function $f(x) = \frac{1}{2}(x+1)^2 - 2$ represented on the right.



Find

a) 1. dom *f*. ℝ 2. ran *f*. ***[−2, +∞[***

b) 1. the zeros of *f*. ***−3 and 1*** 2. the *y*-intercept of *f*. ***−1.5***

c) the sign of *f*. $f(x) \geqslant 0$ ***over*** $]-\infty, -3] \cup [1, +\infty[$; $f(x) \leqslant 0$ ***over*** $[-3, 1]$

d) the variation of *f*. ***f is increasing over [−1, +∞[.***

f is decreasing over]−∞, −1].

e) the minimum of *f*. ***min f = −2***

8. Find the domain and range of the following functions.

a) $f(x) = -2(x + 1)^2 + 5$

dom f = ℝ; ran f =]–∞, 5]

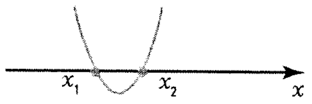
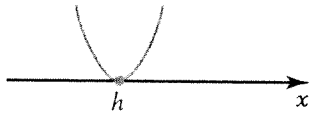
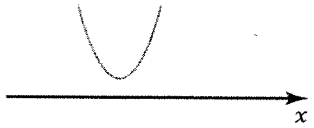
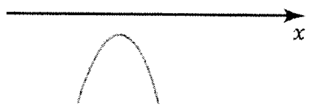
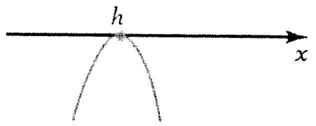
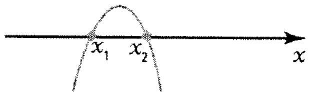
b) $f(x) = \frac{3}{2}(x - 1)^2 - 2$

dom f = ℝ; ran f = [–2, +∞[

9. Determine the zeros of the function $y = -2(x - 3)^2 + 18$. ***0 and 6***

10. What is the y-intercept of the function $y = -2(x - 3)^2 + 7$? ***-11***

11. The sign of the quadratic function $f(x) = a(x - h)^2 + k$ depends on the signs of a and k. Indicate, in each of the 6 following cases, the intervals where $f(x) > 0$ and $f(x) < 0$.

	$k < 0$	$k = 0$	$k > 0$
$a > 0$	 $f(x) > 0$ if $x \in]-\infty, x_1 [\cup] x_2, +\infty[$ $f(x) < 0$ if $x \in]x_1, x_2[$	 $f(x) > 0$ if $x \in \mathbb{R} \backslash \{h\}$	 $f(x) > 0, \forall x \in \mathbb{R}$
$a < 0$	 $f(x) < 0, \forall x \in \mathbb{R}$	 $f(x) < 0$ if $x \in \mathbb{R} \backslash \{h\}$	 $f(x) > 0$ if $x \in]x_1, x_2[$ $f(x) < 0$ if $x \in]-\infty, x_1 [\cup] x_2, +\infty[$

12. Determine, in each case, the values of x for which

1. $f(x) > 0$.
2. $f(x) \geqslant 0$.
3. $f(x) < 0$.
4. $f(x) \leqslant 0$.

a) $f(x) = 2(x - 1)^2 - 2$

1. ***$f(x) > 0$ if $x \in]-\infty, 0 [\cup] 2, +\infty[$***
2. ***$f(x) \geqslant 0$ if $x \in]-\infty, 0]\cup[2, +\infty[$***
3. ***$f(x) < 0$ if $x \in]0, 2[$***
4. ***$f(x) \leqslant 0$ if $x \in [0, 2]$***

b) $f(x) = -4(x - 3)^2 + 16$

1. ***$f(x) > 0$ if $x \in]1, 5[$***
2. ***$f(x) \geqslant 0$ if $x \in [1, 5]$***
3. ***$f(x) < 0$ if $x \in]-\infty, 1 [\cup] 5, +\infty[$***
4. ***$f(x) \leqslant 0$ if $x \in]-\infty, 1] \cup] [5, +\infty[$***

13. Determine the values of x for which $y = 3(x - 1)^2 - 27$ is positive.

$x \in]-\infty, -2] \cup [4, +\infty[$

14. Study the variation of the following functions.

a) $f(x) = 3(x - 1)^2 - 2$

f is decreasing over]–∞, 1].

f is increasing over [1, +∞[.

b) $f(x) = -2(x + 1)^2 + 1$

f is increasing over]–∞, –1].

f is decreasing over [–1, +∞[.

15. Determine the interval over which the function $f(x) = 2(x + 4)^2 + 2$ is increasing. ***[-4, +∞[***

16. Determine the values of x for which the function $y = -3(x + 1)^2 + 12$ is increasing.

$x \in]-\infty, -1]$

17. In each of the following cases, indicate whether the function reaches a maximum or a minimum and determine it.

a) $f(x) = -2(x - 3)^2 - 1$ ***A maximum: max $f = -1$***

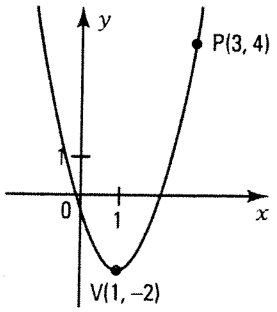
b) $f(x) = \frac{3}{4}(x + 1)^2 - 2$ ***A minimum: min $f = -2$***

18. Find the extrema and its nature (maximum or minimum) of $f(x) = 3(x - 1)^2 - 4$.
A minimum; −4

19. What is the axis of symmetry of the parabola with equation $y = (x - 1)^2$?
The line $x = 1$

20. Find the values of x for which the function $f(x) = -2(x - 1)^2 - 4$ is equal to −36.
−3 and 5

ACTIVITY 10 Finding the rule – Given the vertex and a point



The parabola on the right has the vertex V(1, −2) and passes through the point P(3, 4). The quadratic function represented by this parabola has the rule: $y = a(x - h)^2 + k$ (standard form).

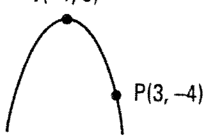
Use the following procedure to determine the rule.

1. Identify h and k. ***$h = 1$; $k = -2$***
2. Determine a knowing that the point P(3, 4) verifies the function's rule.
   We have: $y = a(x - 1)^2 - 2$
   $4 = a(3 - 1)^2 - 2$
   $4 = 4a - 2$
   $a = \frac{3}{2}$
3. What is the rule of this function? $y = \frac{3}{2}(x - 1)^2 - 2$

FINDING THE RULE – GIVEN THE VERTEX AND A POINT

$$\boxed{y = a(x - h)^2 + k}$$

1. Identify h and k.	1. $h = -1$, $k = 8$ $y = a(x + 1)^2 + 8$
2. Find a after replacing x and y in the rule by the coordinates of the given point P.	2. $-4 = a(3 + 1)^2 + 8$ $-4 = 16a + 8$ $a = \frac{-3}{4}$
3. Deduce the rule.	3. $y = \frac{-3}{4}(x + 1)^2 + 8$



21. Determine the equation of the parabola with vertex V and passing through the given point P.

a) V(−1, 4) and P(2, −2) $y = -\frac{2}{3}(x + 1)^2 + 4$

b) V(0, 0) and P(−1, 2) $y = 2x^2$

c) V(2, 0) and P(1, 4) $y = 4(x - 2)^2$

d) V(0, −1) and P(2, 1) $y = \frac{1}{2}x^2 - 1$

22. A parabola with vertex V(3, 16) has a y-intercept equal to 7. What is the y-coordinate of the point A on the parabola whose x-coordinate is 5?

$y = -(x - 3)^2 + 16$; A(5, 12). The y-coordinate of point A is 12.

23. A parabola with vertex V(3, 8) passes through the point A(6, –10). What are the points on this parabola whose y-coordinates are equal to 6?

$y = -2(x - 3)^2 + 8$; $P_1(2, 6)$ and $P_2(4, 6)$

24. What are the zeros of the parabola whose vertex is V(–1, 12) and passes through the point A(2, –15)?

$y = -3(x + 1)^2 + 12$. The zeros are –3 and 1.

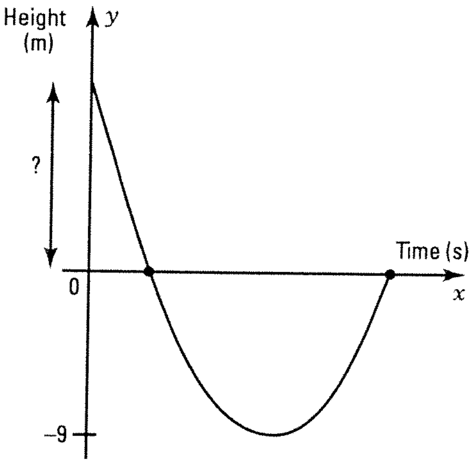
25. A parabola with vertex V(6, 10) passes through the point P(10, 6). What is the initial value of this function?

$f(x) = -\frac{1}{4}(x - 6)^2 + 10$; $f(0) = 1$. The initial value is equal to 1.

26. During a competition, a diver enters the water 2 seconds after jumping from the diving board and reaches a maximum depth of 9 m. The portion of the parabola on the right represents the diver's trajectory. If the diver remains underwater for 6 seconds, determine the height of the diving board.

$f(x) = (x - 5)^2 - 9$; $f(0) = 16$

The diving board is at a height of 16 m.



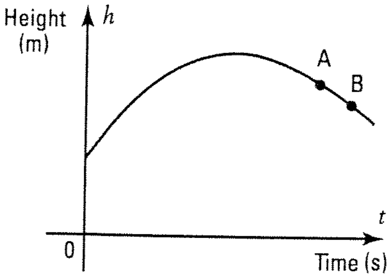
27. At its purchase, a share is worth \$6. We observe that the function f, which gives the value y of the share as a function of the time x in months since its purchase, is a quadratic function. The share reaches a maximum value of \$8 six months after its purchase. What is the value of this share 9 months after its purchase?

$f(x) = \frac{-1}{18}(x - 6)^2 + 8$; $f(9) = 7.5$. The share is worth \$7.50.

28. We have represented on the right the trajectory of two fireworks launched at the same time.
The rule $h = -2(t - 4)^2 + 100$ gives the height h, in metres, as a function of the elapsed time, in seconds, since they were launched. Knowing that firework A explodes at a height of 92 m and that firework B explodes 1 second later, determine at what height firework B explodes at.

Firework A explodes 6 seconds after its launch.

Firework B explodes 7 seconds after its launch at a height of 82 m.



3.5 Quadratic functions – General form

ACTIVITY 1 Quadratic function – General form

a) Consider the quadratic function with the rule (standard form) $f(x) = 2(x - 1)^2 - 3$.

1. What are the coordinates of the parabola's vertex? ***V(1, -3)***
2. Expand the rule to obtain the form $f(x) = ax^2 + bx + c$ called **general form**.
   $f(x) = 2x^2 - 4x - 1$
3. Identify the coefficients a, b and c. ***$a = 2$, $b = -4$, $c = -1$***
4. Verify that the x-coordinate h of the vertex is equal to $-\frac{b}{2a}$ and that the y-coordinate k of the vertex is equal to $\frac{4ac - b^2}{4a}$.
   $h = 1$ and $-\frac{b}{2a} = 1$; $k = -3$ and $\frac{4ac - b^2}{4a} = -3$

b) Given $f(x) = a(x - h)^2 + k$ (standard form). When this rule is expanded we get:
$f(x) = ax^2 - 2ahx + ah^2 + k$.
By letting $b = -2ah$ and $c = ah^2 + k$, we get the general form: $f(x) = ax^2 + bx + c$.

1. Show that $h = -\frac{b}{2a}$. ***Since $b = -2ah$ then $h = -\frac{b}{2a}$***
2. Show that $k = \frac{4ac - b^2}{4a}$. ***Since $c = ah^2 + k$ then $k = c - ah^2 = c - a\left(\frac{-b}{2a}\right)^2 = c - \frac{ab^2}{4a^2}$***
   $= \frac{4a^2c - ab^2}{4a^2} = \frac{4ac - b^2}{4a} (a \neq 0)$

c) Consider the quadratic function $f(x) = -2x^2 + 4x + 6$ (general form).

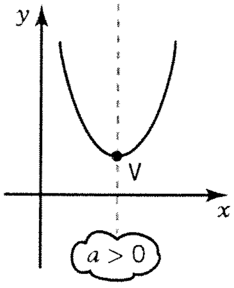
1. Identify the coefficients a, b and c. ***$a = -2$; $b = 4$; $c = 6$***
2. What are the coordinates of the parabola's vertex? ***$h = -\frac{b}{2a} = 1$; $k = \frac{4ac - b^2}{4a} = 8$; V(1, 8)***

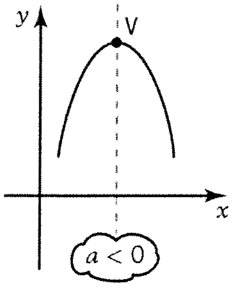
QUADRATIC FUNCTION – GENERAL FORM

- The general form of a quadratic function's rule is:

$$\boxed{y = ax^2 + bx + c}$$

- The vertex of the parabola is $V\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ or $V\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$ where $\Delta = b^2 - 4ac$.
- The parabola's axis of symmetry has the equation: $x = -\frac{b}{2a}$.





1. For each of the following parabolas, find the coordinates of the vertex and the equation of the axis of symmetry.

a) $y = 2x^2 + 8x + 2$

$V(-2, -6); x = -2$

b) $y = -3x^2 + 6x - 6$

$V(1, -3); x = 1$

c) $y = 2x^2 - 3x$

$V\left(\frac{3}{4}, \frac{-9}{8}\right); x = \frac{3}{4}$

d) $y = -2x^2 + 6$

$V(0, 6); x = 0$

ACTIVITY 2 Finding the zeros – General form

Consider the quadratic function $f(x) = 2x^2 - 7x + 3$ (general form).

a) What equation must we solve to determine the zeros of the function? **$2x^2 - 7x + 3 = 0$**

b) Solve this equation to determine the zeros. **$a = 2; b = -7; c = 3; \Delta = 25; x_1 = \frac{1}{2}$ and $x_2 = 3$**

FINDING THE ZEROS – GENERAL FORM

Consider the function $f(x) = ax^2 + bx + c$.
To determine the zeros of function f, we solve the quadratic equation $ax^2 + bx + c = 0$.

(See solving quadratic equations page 33.)

Ex.: To find the zeros of $f(x) = 2x^2 - 5x - 12$, we solve $2x^2 - 5x - 12 = 0$. We get the zeros $\frac{-3}{2}$ and 4.

ACTIVITY 3 Graphing a parabola – General form

Consider the function $f(x) = 2x^2 - 4x - 6$.

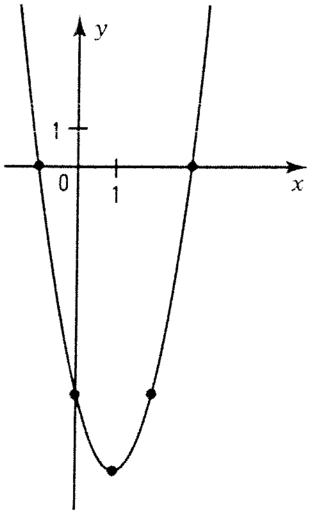
Use the following procedure to graph the function.

1. Identify the parameters a, b and c. **$a = 2; b = -4; c = -6$**
2. Is the parabola open upward or downward? **Upward since $a > 0$**
3. What are the coordinates of the vertex? **$V(1, -8)$**
4. Find, if they exist, the zeros of the function.
   $\Delta = 64$. There are two zeros: $x_1 = -1$ and $x_2 = 3$.
5. What is the y-intercept? **-6**
6. Complete the following table of values.

x	-1	0	1	2	3
y	**0**	**-6**	**-8**	**-6**	**0**

7. Graph the parabola.

The axis of symmetry is useful in graphing the parabola.

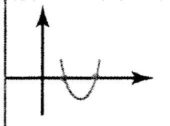
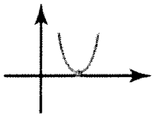
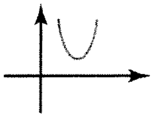
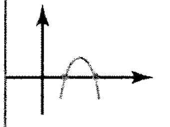
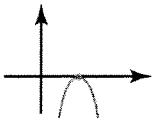
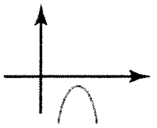


GRAPHING A PARABOLA – GENERAL FORM

Procedure

1. Identify the parameters a, b and c.
2. Determine the opening according to the sign of a.
3. Determine the coordinates of the vertex V.
   $V = \left(-\frac{b}{2a}, -\frac{\Delta}{2a}\right)$ where $\Delta = b^2 - 4ac$.
4. Find the zeros.
   $x_1 = \frac{-b - \sqrt{\Delta}}{2a}$; $x_2 = \frac{-b + \sqrt{\Delta}}{2a}$
5. Find the y-intercept.
6. Complete a table of values.
7. Graph the parabola.

We observe 6 possible situations according to the signs of a and Δ.

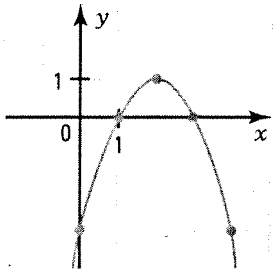
	$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
$a > 0$			
$a < 0$			

Ex.: $f(x) = -x^2 + 4x - 3$

1. $a = -1$, $b = 4$, $c = -3$
2. Open downward since $a < 0$
3. V(2, 1).
4. $\Delta = 4$. There are 2 zeros: $x_1 = 1$ and $x_2 = 3$.
5. $f(0) = -3$.
6.

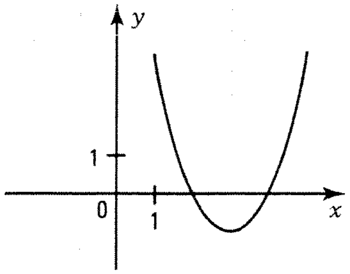
x	0	1	2	3	4
y	–3	0	1	0	–3

7.

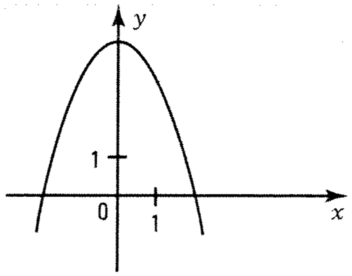


2. Graph the following parabolas.

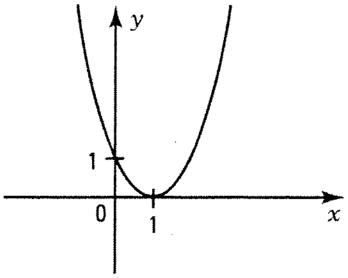
a) $y = x^2 - 6x + 8$



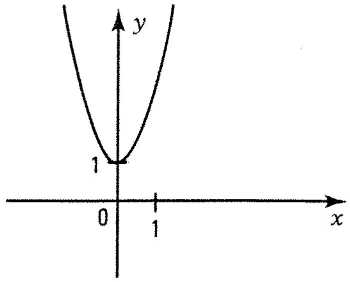
b) $y = -x^2 + 4$



c) $y = x^2 - 2x + 1$



d) $y = 2x^2 + 1$

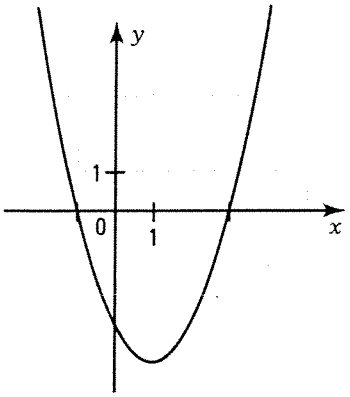


3. Consider the function $f(x) = x^2 - 2x - 3$.

a) Graph this function in the Cartesian plane.

b) Determine

1. dom f. $\mathbb{R}$
2. ran f. ***$[-4, +\infty[$***
3. the zeros of f. ***-1 and 3***
4. the y-intercept of f. ***-3***
5. the sign of f.
   $f(x) \geq 0$, if $x \in]-\infty, -1] \cup [3, +\infty[$; $f(x) \leq 0$, if $x \in [-1, 3]$
6. the increasing and decreasing intervals of the function.
   $f \nearrow$ if $x \in [1, +\infty[$; $f \searrow$ if $x \in]-\infty, 1]$
7. the extrema of f. ***min $f = -4$***



ACTIVITY 4 Sign of a quadratic function – Graphical method

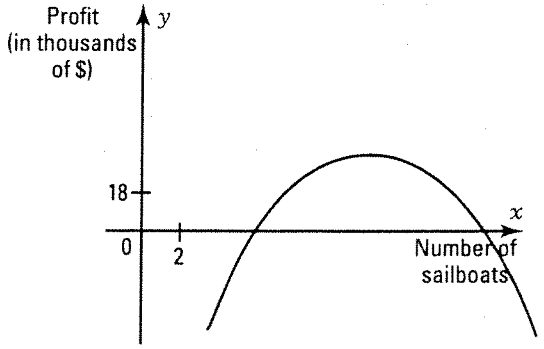
a) The sales manager of a company making sailboats has established that the profit $f(x)$, in thousands of dollars, resulting from the selling of x sailboats in a month is represented by a quadratic function with the rule $f(x) = -x^2 + 24x - 108$. Determine the interval in which the number of sailboats sold must be for the profit to be positive using two methods.

1. Number line method (see page 39).

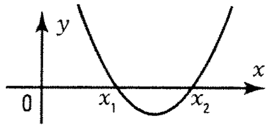
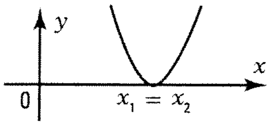
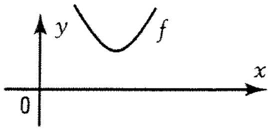
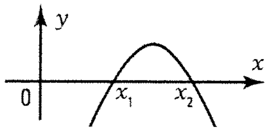
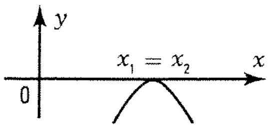
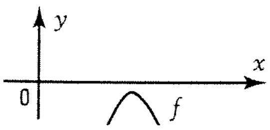
The profit $f(x)$ is positive if $x \in [6, 18]$.

2. Graphical method. This method consists of using the function's graph to determine the values of x for which $f(x) \geqslant 0$.

The profit $f(x)$ is positive if $x \in [6, 18]$.



b) The sign of the quadratic function $f(x) = ax^2 + bx + c$ depends on the sign of a and the sign of the discriminant Δ. Indicate the sign of the function in each of the following 6 cases.

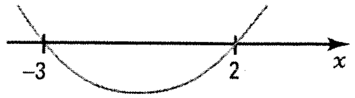
	$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
$a > 0$	 ***$f(x) > 0$ if $x \in]-\infty, x_1[\cup]x_2, +\infty]$*** ***$f(x) < 0$ if $x \in]x_1, x_2[$***	 ***$f(x) > 0$, $\forall x \in \mathbb{R} \setminus \{x_1\}$***	 ***$f(x) > 0$, $\forall x \in \mathbb{R}$***
$a < 0$	 ***$f(x) > 0$ if $x \in]x_1, x_2[$*** ***$f(x) < 0$ if $x \in]-\infty, x_1 [\cup] x_2, +\infty[$***	 ***$f(x) < 0$, $\forall x \in \mathbb{R} \setminus \{x_1\}$***	 ***$f(x) < 0$, $\forall x \in \mathbb{R}$***

SIGN OF A QUADRATIC FUNCTION – GRAPHICAL METHOD

To determine the sign of the quadratic function $f(x) = x^2 + x - 6$,

1. we determine the zeros of the function which are then placed on a number line.
2. we draw a sketch of the parabola taking into account its opening which depends on the sign of a.
3. we deduce the sign of the function.

   $f(x) \geqslant 0$ if $x \in]-\infty, -3] \cup [2, +\infty[$. $f(x) \leqslant 0$ if $x \in [-3, 2]$.



4. Determine the sign of the following quadratic functions.

a) $f(x) = x^2 + 2x - 15$ ***$f(x) \geqslant 0$ if $x \in]-\infty, -5] \cup [3, +\infty[$; $f(x) \leqslant 0$, if $x \in [-5, 3]$***

b) $f(x) = -2x^2 + 7x - 6$ ***$f(x) \geqslant 0$ if $x \in \left[\frac{3}{2}, 2\right]$; $f(x) \leqslant 0$ if $x \in \left]-\infty, \frac{3}{2}\right] \cup [2, +\infty[$***

c) $f(x) = x^2 - 2x + 1$ ***$f(x) \geqslant 0, \forall x \in \mathbb{R}$***

d) $f(x) = -4x^2 + 4x - 1$ ***$f(x) \leqslant 0, \forall x \in \mathbb{R}$***

5. Determine the domain and range of the following functions.

a) $f(x) = -x^2 + 4x + 5$ ***Dom $f = \mathbb{R}$; ran $f =]-\infty, 9]$***

b) $f(x) = x^2 + 2x - 15$ ***Dom $f = \mathbb{R}$; ran $f = [-16, +\infty[$***

6. Study the variation of the following functions.

a) $f(x) = x^2 - x - 6$ ***$f \searrow$ if $x \in \left]-\infty, \frac{1}{2}\right]$ and $f \nearrow$ if $x \in \left[\frac{1}{2}, +\infty\right[$***

b) $f(x) = -2x^2 + 3x - 1$ ***$f \nearrow$ if $x \in \left]-\infty, \frac{3}{4}\right]$ and $f \searrow$ if $x \in \left[\frac{3}{4}, +\infty\right[$***

7. What are the zeros of the function $y = -3x^2 + 11x - 6$? ***$\frac{2}{3}$ and 3***

8. Find the values of x for which $f(x) = x^2 + 5x - 14$ is positive. ***$]-\infty, -7] \cup [2, +\infty[$***

9. What is the range of the function $f(x) = -x^2 + 2x + 15$? ***Ran $f =]-\infty, 16]$***

10. What is the y-intercept of $y = 3x^2 - 2x + 5$? ***5***

11. Find the extrema and its nature (maximum or minimum) for $y = -x^2 - 2x + 3$.

A maximum; 4

12. What is the equation of the axis of symmetry for the parabola $y = -2x^2 + 5x - 3$?

The line with equation $x = \frac{5}{4}$

13. For what values of x is the function $f(x) = 2x^2 - x - 6$ decreasing?

$x \in \left]-\infty, \frac{1}{4}\right]$

3.6 Quadratic functions – Factored form

ACTIVITY 1 Quadratic function – Factored form

Consider the quadratic function $f(x) = 2x^2 - 7x + 3$.

a) What is the value of parameter a? $a = 2$

b) Determine the zeros x_1 and x_2 of the function. $x_1 = \frac{1}{2}$ and $x_2 = 3$

c) The factored form of the quadratic function is $f(x) = a(x - x_1)(x - x_2)$.

Determine the factored form of $f(x) = 2x^2 - 7x + 3$. $f(x) = 2\left(x - \frac{1}{2}\right)(x - 3)$

d) Expand the factored form to get back to the general form.

$2\left(x - \frac{1}{2}\right)(x - 3) = (2x - 1)(x - 3) = 2x^2 - 7x + 3$

QUADRATIC FUNCTION – FACTORED FORM

- Given the general form of a quadratic function $f(x) = ax^2 + bx + c$ with zeros x_1 and x_2. The factored form of the quadratic function is:

$$\boxed{f(x) = a(x - x_1)(x - x_2)}$$

Ex.: – $f(x) = -2x^2 + 5x - 3$ yields the zeros: $x_1 = \frac{3}{2}$ and $x_2 = 1$.

The factored form of f is $f(x) = -2\left(x - \frac{3}{2}\right)(x - 1)$.

– $f(x) = 4x^2 - 12x + 9$ yields only one zero or two equal zeros: $x_1 = x_2 = \frac{3}{2}$.

The factored form of f is $f(x) = 4\left(x - \frac{3}{2}\right)\left(x - \frac{3}{2}\right) = 4\left(x - \frac{3}{2}\right)^2$.

1. In each of the following cases, determine the factored form of the function.

a) $f(x) = 3x^2 - 5x - 2$ $f(x) = 3\left(x + \frac{1}{3}\right)(x - 2)$

b) $f(x) = 2x^2 + 7x + 6$ $f(x) = 2\left(x + \frac{3}{2}\right)(x + 2)$

c) $f(x) = x^2 - 8x + 15$ $f(x) = (x - 3)(x - 5)$

d) $f(x) = -2x^2 + x + 3$ $f(x) = -2\left(x - \frac{3}{2}\right)(x + 1)$

e) $f(x) = 4x^2 - 4x + 1$ $f(x) = 4\left(x - \frac{1}{2}\right)^2$

Converting forms

2. Consider the three forms of a quadratic function: $f(x) = a(x - h)^2 + k$ (standard form). $f(x) = ax^2 + bx + c$, (general form) and $f(x) = a(x - x_1)(x - x_2)$ (factored form).

For each given form, determine the two other forms.

a) $f(x) = 2(x - 1)^2 - 8$

$f(x) = 2x^2 - 4x - 6$ (general form)

$f(x) = 2(x + 1)(x - 3)$ (factored form)

b) $f(x) = x^2 - 10x + 16$

$f(x) = (x - 5)^2 - 9$ (standard form)

$f(x) = (x - 2)(x - 8)$ (factored form)

c) $f(x) = 4x^2 - 8x + 3$

$f(x) = 4(x - 1)^2 - 1$ (standard form)

$f(x) = 4\left(x - \frac{3}{2}\right)\left(x - \frac{1}{2}\right)$ (factored form)

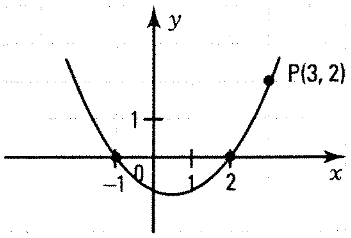
d) $f(x) = 2(x - 1)(x - 5)$

$f(x) = 2x^2 - 12x + 10$ (general form)

$f(x) = 2(x - 3)^2 - 8$ (standard form)

ACTIVITY 2 Finding the rule – Given the zeros and a point

The parabola on the right has two zeros: −1 and 2 and passes through the point P(3, 2).



The quadratic function represented by this parabola has the rule:
$y = a(x - x_1)(x - x_2)$ (factored form).

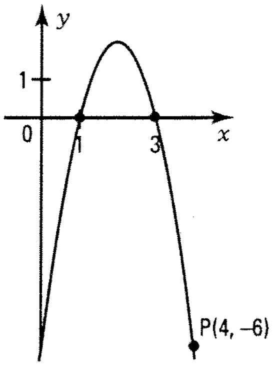
Use the following procedure to determine the factored form of the rule.

1. Identify x_1 and x_2. ***$x_1 = -1$ and $x_2 = 2$***
2. Determine a knowing the coordinates of the point (3, 2) verify the rule.
   We have: ***$y = a(x + 1)(x - 2)$***
   $2 = a(3 + 1)(3 - 2)$
   $2 = 4a$
   $a = \frac{1}{2}$
3. What is therefore the factored form of the rule? ***$y = \frac{1}{2}(x + 1)(x - 2)$***
4. What is the general form? ***$y = \frac{1}{2}x^2 - \frac{1}{2}x - 1$***

FINDING THE RULE – GIVEN THE ZEROS AND A POINT

$$\boxed{y = a(x - x_1)(x - x_2)}$$

1. Identify the zeros x_1 and x_2.	1. $x_1 = 1$; $x_2 = 3$ $y = a(x - 1)(x - 3)$
2. Determine a after replacing x and y in the rule by the coordinates of the point P.	2. $-6 = a(4 - 1)(4 - 3)$ $-6 = 3a$ $a = -2$
3. Deduce the rule of the function.	3. $y = -2(x - 1)(x - 3)$ (factored form) $y = -2x^2 + 8x - 6$ (general form)



Converting forms

3. Find the rule, in general form, of each of the following quadratic functions.

a) A function with –5 and 2 as zeros and passing through the point P(3, 16).
$y = 2x^2 + 6x - 20$

b) A function with –3 and –1 as zeros and an initial value of –6.
$y = -2x^2 - 8x - 6$

c) A function with the unique zero –2 and passing through the point P(–1, 3).
$y = 3x^2 + 12x + 12$

d) A function with the vertex V(–1, 4) and passing through the point P(2, –5).
$y = -x^2 - 2x + 3$

e) A function with the vertex (1, –8) and one of the zeros equal to 3.
$y = 2x^2 - 4x - 6$

4. What is the vertex of the parabola that has –2 and 4 for zeros and passes through the point A(5, 21)?
$y = 3(x + 2)(x - 4)$; V(1, –27)

5. A parabola with zeros –1 and 3 passes through the point A(2, 6). What is the y-coordinate of the point B on the parabola that has an x-coordinate of 4?
$y = -2(x + 1)(x - 3)$; B(4, –10). The y-coordinate of point B is –10.

6. A parabola with zeros –3 and 4 passes through the point A(2, –20). What are the points on this parabola that have a y-coordinate equal to 16?
$y = 2(x + 3)(x - 4)$; $P_1(-4, 16)$ and $P_2(5, 16)$

7. What is the y-intercept of the parabola with zeros –3 and –1 and passing through the point A(–2, 2)?
The y-intercept is equal to –6.

8. What is the equation of the axis of symmetry of a parabola with zeros –3 and 4?
$x = \frac{1}{2}$

9. The table of values on the right gives the coordinates of different points on a parabola. What is the equation of this parabola?
Axis of symmetry: $x = 2$; The zeros are –1 and 5.
$y = -(x + 1)(x - 5)$; $y = -x^2 + 4x + 5$

x	y
0	5
1	8
3	8
5	0

10. Determine the range of the quadratic function f with zeros 3 and 5 which verifies $f(2) = -6$.
$f(x) = -2x^2 + 16x - 30$; V(4, 2); ran f = $]-\infty, 2]$

11. What is the rule of the function f that has a range of $]-\infty, 4]$ and is positive over the interval $[-1, 3]$? ***$f(x) = -x^2 + 2x + 3$***

12. The value of a share, in dollars, x weeks after its purchase is given by the rule $y = -0.1x^2 + x + 4.5$. Do you make a profit or a loss if the share is sold two weeks after reaching its maximum value?

Value at purchase: \$4.50; V(5, 7); f(7) = \$6.60. A profit of \$2.10 per share is made.

13. The position $f(t)$, in metres, of a diver relative to the surface is described by the rule $f(t) = 0.5t^2 - 6t + 10$ where t represents elapsed time, in seconds. How long was the diver under water?

$f(t) \leqslant 0 \Leftrightarrow 2 \leqslant t \leqslant 10$. The diver was under water during 8 seconds.

14. The trajectory of a stone thrown from a seaside cliff is a partial parabola. The position $f(t)$, in metres, of the stone relative to sea level is given by $f(t) = -t^2 + 8t + 20$ where t represents elapsed time in seconds since it was thrown. How many seconds after reaching its maximum height will the stone hit the water?

After 6 seconds.

15. The manager of a movie theatre has calculated the following results. When the cost of admission is set at \$10, he observes on average 100 spectators per showing and for each \$0.50 rebate on the admission price, he notices an average of 10 more spectators.

a) Find the rule which gives the total revenue per showing as a function of the number x of \$0.50 rebates.

R(x): Total revenue per showing.

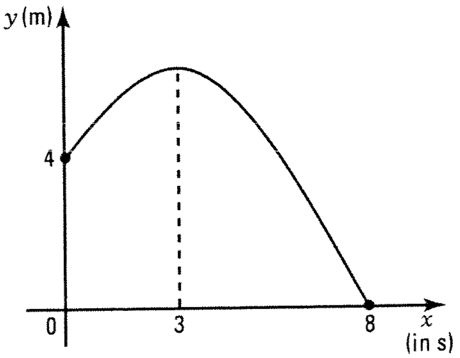
$R(x) = (10 - 0.5x)(100 + 10x)$; $R(x) = -5x^2 + 50x + 1000$

b) 1. At what amount should the manager set the cost of admission in order to maximize the revenue per showing?

The function R reaches its maximum when x = 5. The price of admission should be set at \$7.50.

2. What is the total maximum revenue per showing? ***\$1125***

16. A stone is thrown upward from a height of 4 m. After 3 s, it reaches its maximum height and after 8 s, it hits the ground. Its trajectory is parabolic.



1. What is the maximum height reached by the stone?

6.25 m

2. Determine the elapsed time from the moment the stone was at a height of 2.25 m during its descent to the moment it hit the ground.

1 second

Evaluation 3

1. Find the zeros of the following polynomial functions.

a) $f(x) = -2x + 7$ $\frac{7}{2}$

b) $f(x) = -2x^2 + 7x - 3$ $\frac{1}{2}$ ***and 3***

c) $f(x) = -2(x - 3)^2 + 18$ ***0 and 6***

d) $f(x) = -4x^2 + 12x - 9$ $\frac{3}{2}$

2. Study the sign of the following functions.

a) $f(x) = -3x + 6$ $f(x) \geqslant 0$ ***if*** $x \in]-\infty, 2]$; $f(x) \leqslant 0$ ***if*** $x \in [2, +\infty[$

b) $f(x) = 2x^2 - 15x + 7$ $f(x) \geqslant 0$ ***if*** $x \in \left]-\infty, \frac{1}{2}\right] \cup [7, +\infty[$; $f(x) \leqslant 0$ ***if*** $x \in \left[\frac{1}{2}, 7\right]$

3. What is the range of the following functions?

a) $f(x) = 5(x - 1)^2 - 20$

ran $f = [-20, +\infty[$

b) $f(x) = -2x^2 + 18x - 21$

ran $f = \left]-\infty, \frac{39}{2}\right]$

4. Study the variation of the following functions.

a) $f(x) = -2x + 1$

$f \searrow, \forall x \in \mathbb{R}$

b) $f(x) = 3x - 1$

$f \nearrow, \forall x \in \mathbb{R}$

c) $f(x) = -2(x + 1)^2 + 9$

$f \nearrow$ ***if*** $x \in]-\infty, -1]$

$f \searrow$ ***if*** $x \in [-1, +\infty[$

d) $f(x) = x^2 - 6x + 8$

$f \searrow$ ***if*** $x \in]-\infty, 3]$

$f \nearrow$ ***if*** $x \in [3, +\infty[$

5. Find the rule of the following functions.

a) f is a constant function such that $f(1) = 2$. $f(x) = 2$

b) f is a linear function such that $f(1) = 2$ and $f(3) = 5$. $f(x) = \frac{3}{2}x + \frac{1}{2}$

c) f is a quadratic function with zeros -3 and 2 and an initial value of -12.

$f(x) = 2x^2 + 2x - 12$

d) f is represented by a parabola with the vertex V(2, 1) and passing through the point P(4, −7).

$f(x) = -2(x - 2)^2 + 1$

6. At the start of a car ride, the gas tank contains 66 litres. After traveling 50 km, the gas tank contains 60 litres. What is the rule of the linear function which gives the remaining quantity y of gas in the tank as a function of the distance traveled x in km?

$y = 66 - 0.12x$

7. A parabola with the vertex V(3, 16) passes through the point A(5, 12). What is its y-intercept? ***7***

8. A parabola intersects the x-axis at −2 and 4 and passes through the point A(2, −24). Find the coordinates of its vertex. ***V(1, −27)***

9. The trajectory of a ball thrown by David is a partial parabola. The height $h(t)$, in metres, reached by the ball is described by the rule $h(t) = -(t - 3)^2 + 9$.

Determine over what interval of time the ball is at a height greater than or equal to 8 m above ground.

[2, 4]

10. A share purchased for $4 reaches its maximum value of $4.50 five weeks after its purchase. The function associating the value $v(t)$, in dollars, of the share as a function of time t, in weeks, has been shown to be a quadratic function. What is the value of the share 8 weeks after its purchase?

$v(t) = -0.02(t - 5)^2 + 4.50;\ v(8) = 4.32$

The share is worth $4.32 eight weeks after its purchase.

11. The number of units $q(x)$ produced per day by x employees is given by the rule $q(x) = -0.25x^2 + 10x\ (x \leq 25)$.

a) What is the maximum number of units produced in one day? How many employees are required to produce this maximum?

100 units produced by 20 employees.

b) How many employees are required to produce 75 units?

10 employees

12. A truck with a height of 190 cm enters a tunnel with a parabolic ceiling. The width of the tunnel is 20 m and the maximum height of the tunnel is 10 m. At what minimal distance from the edge at ground level can this truck pass through the tunnel? ***1 m*** Good

13. The rule $p = 1000 - 2q$ enables you to calculate the selling price p of a parasol as a function of the number q of parasols ordered. What must the number of parasols ordered be to maximize the revenue generated by the sale of the parasols? ***250 parasols***

14. A stone is thrown vertically upward. The function which gives the height $h(t)$, in metres, as a function of elapsed time t, in seconds, since the stone was thrown is a quadratic function with the rule: $h(t) = -2t^2 + 12t$.

a) 1. What is the maximum height reached by the stone? ***18 m***

2. At what time does the stone reach its maximum height? ***3 seconds***

b) At what time, during its descent, does the stone reach a height of 16 m?

4 seconds after it was thrown.

~~15.~~ A rectangular yard is to be fenced in with 80 m of fence. What must the dimensions of the yard be in order to maximize the area of the field?

x: width of the yard; 40 – x: length of the yard

A(x): area of the yard. $A(x) = -x^2 + 40x$

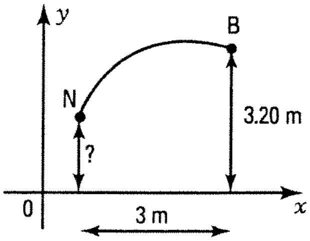
The vertex of the parabola representing A(x) has the coordinates V(20, 400).

The yard must be in the shape of a square with 20 m sides.

16. Nancy throws a ball toward a basket located 3.2 m off the ground. The ball's trajectory is represented on the right.

The rule associated with this trajectory is: $y = -0.4(x - 6)^2 + 3.6$.

Nancy throws the ball at a distance of 3 m from the basket. From what height did Nancy throw the ball?

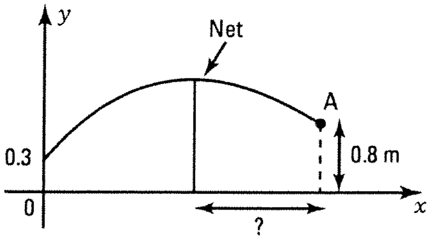


$y_B = 3.20$; $x_B = 7$; $x_N = 4$; $y_N = 2$

The ball is thrown from a height of 2 m.

17. During a tennis match, Karen hits the ball to Alex. The trajectory of the ball is represented in the Cartesian plane by a parabola with its vertex over the net. The equation of the trajectory is: $f(x) = -0.1(x - 3)^2 + 1.2$.

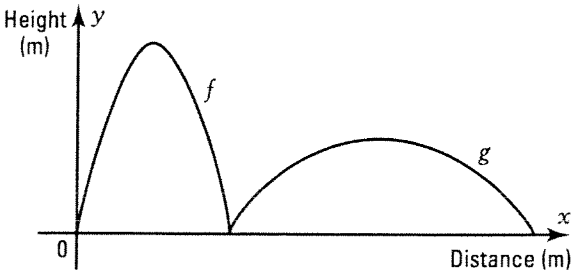
The ball is hit by Karen at a height of 0.3 m and reaches Alex at a height of 0.8 m on its descent. How far is Alex from the net if the vertex of the ball's trajectory is directly over the net?



$x_A = 5$; $x_V = 3$; $x_A - x_V = 2$. Alex is located 2 m from the net.

18. A kangaroo makes two consecutive jumps. The trajectory is represented by two portions of parabolas associated with the functions f and g.

The rule associated with the second jump is $g(x) = -0.25(x - 6.4)^2 + 2.56$. What is the rule associated with the first jump if the kangaroo jumped twice as high on the first jump as the second jump? (The variables x and y are expressed in metres.)



The zeros of g are 3.2 and 9.6.

Vertex of the 1st parabola: V(1.6, 5.12); $f(x) = -2(x - 1.6)^2 + 5.12$.