Chapter 4

Greatest integer function

CHALLENGE 4

- **4.1** Step function
- **4.2** Greatest integer of a real number
- **4.3** Basic greatest integer function
- **4.4** Transformed greatest integer function

EVALUATION 4

CHALLENGE 4

1. A salesman receives a weekly base salary of \$100 and a \$25 bonus for every \$1000 in weekly sales.

a) What will his salary be if he sells for \$8425 during the week?

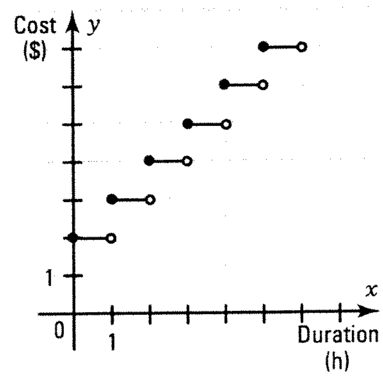
\$300

b) In what interval is the amount of his weekly sales if he receives a salary of \$525?

[17 000, 18 000[

2. At a parking lot, the cost y for parking is calculated the following way: a minimum cost of \$2 for a parking time of less than an hour and an extra \$1 for every complete 60 minute interval.

Represent, in the Cartesian plane on the right, the function which gives the total cost y (in dollars) of parking as a function of the parking time x (in hours).

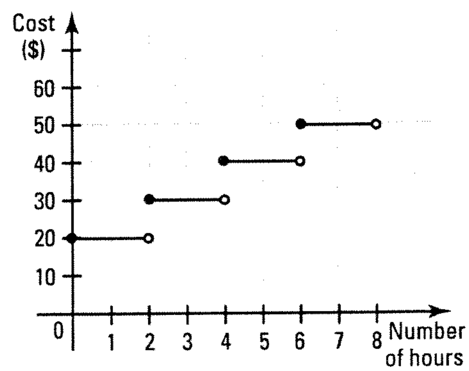


3. The rental cost $C(x)$ of a specialized tool is given by an equation of the form $C(x) = a[b(x - h)] + k$ where x represents the number of rental hours.

The graph on the right represents this situation.

What could the possible values of the parameters a, b, h and k be?

$a = 10$, $b = \frac{1}{2}$, $h = 0$, $k = 20$



4. The cost C, in dollars, of mailing a package is given by the function $C(x) = [2.75x] + 1.25$ where x represents the mass (in kg).
Peter wants to mail a 4.4 kg package. How much will it cost to mail this package?

\$13.35

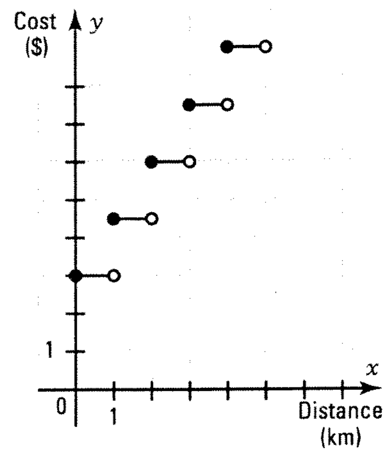
5. The cost, in dollars, of mailing a package varies according to the rule $y = 2\left[\frac{1}{3}(x + 5)\right] + 8$ where x represents the mass of the package (in kg). For what mass will the mailing cost be \$24?

$19 \leq x < 22$

4.1 Step function

ACTIVITY 1 Cost of a taxi ride

The graph on the right indicates the cost of a taxi ride as a function of the distance traveled.



a) What will it cost to travel
   1. 2 km? ***$6*** 2. 2.9 km? ***$6*** 3. 3 km? ***$7.50***

b) What will it cost if the distance traveled x verifies $4 \leqslant x < 5$?
   $ 9

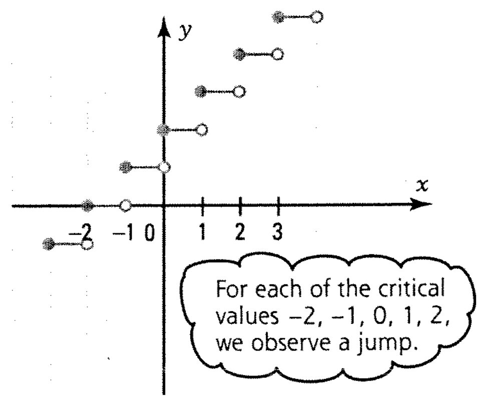
c) If the cost of the ride is $7.50, in what interval is the distance traveled x? ***$x \in [3, 4[$***

d) Explain how the customer is being charged for a taxi ride. ***The minimal cost is $3 for a ride of less than 1 km. There is then an additional $1.50 charge for every complete 1 km traveled.***

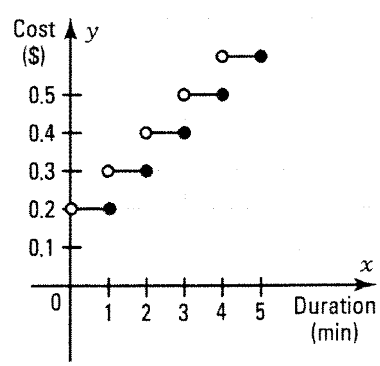
STEP FUNCTION

The graph of a step function contains horizontal segments (steps), generally closed on one end and open on the other.

- step closed on the left and open on the right: ●—o
- step open on the left and closed on the right: o—●



1. A telephone company charges long distance calls in North America the following way: $0.20 for the first minute or part thereof plus $0.10 per minute or fraction of additional minute.



a) What is the cost of a call lasting
   1. 2 min? ***$0.30*** 2. 2.1 min? ***$0.40*** 3. 3 min? ***$0.40***

b) Draw the Cartesian graph of the function which gives the cost $f(x)$ of a call, in dollars, as a function of the duration x, in minutes, of the call.

c) What is the duration of a call costing $0.50? ***$x \in]3, 4]$***
   Complete using the term "open" or "closed" that applies.
   Each step is ***open*** on the left and ***closed*** on the right.

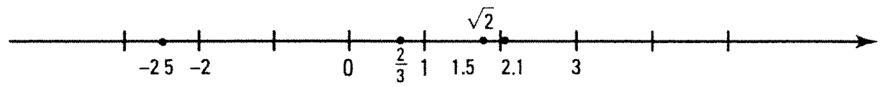
4.2 Greatest integer of a real number

ACTIVITY 1 Greatest integer of a real number

The greatest integer of a real number x, noted $[x]$, is the greatest integer less than or equal to x.

a) Locate the following real numbers on the real number line and find their greatest integer.

1.5; 2.1; $\sqrt{2}$; $\frac{2}{3}$; -2.5; 3; -2



$[1.5] =$ ***1*** $[2.1] =$ ***2*** $[\sqrt{2}] =$ ***1*** $\left[\frac{2}{3}\right] =$ ***0*** $[-2.5] =$ ***−3*** $[3] =$ ***3*** $[-2] =$ ***−2***

b) Determine the interval in which the real number x is located if

1. $[x] = 2$. ***$2 \leqslant x < 3$***
2. $[x] = -2$. ***$-2 \leqslant x < -1$***

GREATEST INTEGER OF A REAL NUMBER

- The greatest integer of a real number x, noted $[x]$, is the greatest integer less than or equal to x.

  Ex.: $[2] = 2$; $[2.99] = 2$; $[-1.01] = -2$; $\left[\frac{-2}{3}\right] = -1$
- Note that: $[x] = a \Leftrightarrow a \leqslant x < a + 1$ $(a \in \mathbb{Z})$.

---

1. If $f(x) = [x]$, calculate

a) $f(-2)$. ***−2*** b) $f\left(\frac{2}{3}\right)$. ***0*** c) $f(\pi)$. ***3*** d) $f(-\pi)$. ***−4***

2. If $f(x) = -2[x]$, calculate

a) $f(-1.5)$. ***4*** b) $f(0.\overline{6})$. ***0*** c) $f(-1.01)$. ***4*** d) $f(-0.01)$. ***2***

3. If $f(x) = [-2x]$, calculate

a) $f(-1)$. ***2*** b) $f(\pi)$. ***−7*** c) $f(-\sqrt{2})$. ***2*** d) $f(0.\overline{3})$. ***−1***

4. If $f(x) = -2[3(x - 1)] + 1$, calculate

a) $f(1)$. ***1*** b) $f(2.5)$. ***−7*** c) $f(-2.5)$. ***23*** d) $f\left(\frac{1}{8}\right)$. ***7***

5. Determine the interval in which the real number x is located if

a) $[x] = 3$. ***[3, 4[*** b) $[x] = -1$. ***[−1, 0[***

c) $[2x] = 1$. ***$\left[\frac{1}{2}, 1\right[$*** d) $2[x] = 1$. ***∅***

e) $[x - 2] = 1$. ***[3, 4[*** f) $[2x + 1] = 2$. ***$\left[\frac{1}{2}, 1\right[$***

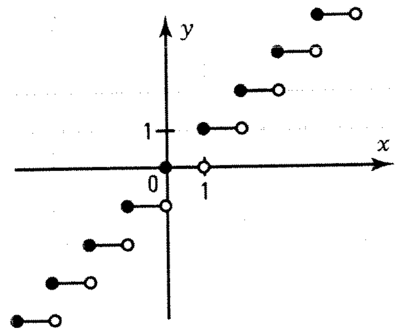
4.3 Basic greatest integer function

ACTIVITY 1 Basic greatest integer function

Consider the function $f(x) = [x]$ called the basic greatest integer function.

a) Complete the table of values and draw the graph of f.

Interval	$f(x)$
$-3 \leqslant x < -2$	*-3*
$-2 \leqslant x < -1$	*-2*
$-1 \leqslant x < 0$	*-1*
$0 \leqslant x < 1$	*0*
$1 \leqslant x < 2$	*1*
$2 \leqslant x < 3$	*2*



b) Determine
1. dom f. $\mathbb{R}$
2. ran f. $\mathbb{Z}$

c) Determine
1. the set of all zeros of f. *[0, 1[*
2. the y-intercept of f. *0*

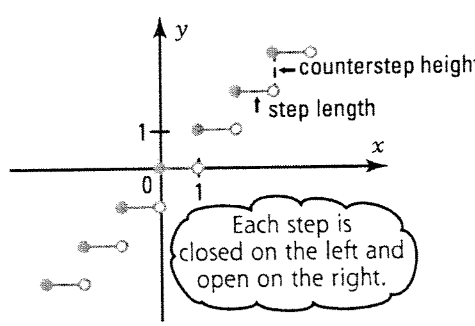
d) Complete
1. $f(x) = 0$ if *$x \in [0, 1[$*.
2. $f(x) > 0$ if *$x \in [1, +\infty[$*.
3. $f(x) < 0$ if *$x \in]-\infty, 0[$*.

e) Is the function f increasing or decreasing over $\mathbb{R}$? *Increasing*

f) Does the function f have any extrema? *No*

BASIC GREATEST INTEGER FUNCTION

- The basic greatest integer function is the function $f(x) = [x]$.
  The function $f(x) = [x]$ is a step function.
  - The length of a step is 1.
  - The height of the counterstep is 1.
- – dom $f = \mathbb{R}$ – ran $f = \mathbb{Z}$
  - zeros of f: $[0, 1[$ – y-intercept of f: 0
  - sign of f: $f(x) \geqslant 0$ if $x \in \mathbb{R}_+$ and $f(x) \leqslant 0$ if $x \in \mathbb{R}_-^*$
  - the function f is increasing over $\mathbb{R}$.
  - the function f has no extrema.



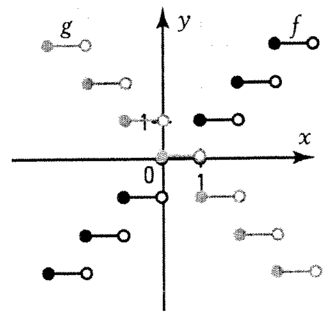
1. The function $f(x) = [x]$ is represented on the right. Let $g(x) = -[x]$.

a) Represent, in the same Cartesian plane, the function g.

b) Explain how to obtain the graph of function g from the graph of function f.

By a reflection about the x-axis.

c) Is the function f increasing or decreasing? ***Decreasing***



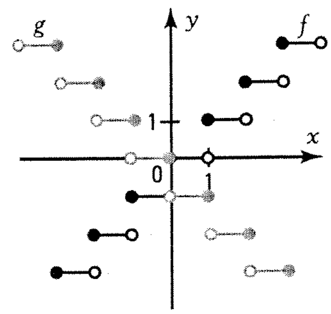
2. The function $f(x) = [x]$ is represented here on the right. Let $g(x) = [-x]$.

a) Represent, in the same Cartesian plane, the function g.

b) Explain how to obtain the graph of function g from the graph of function f.

By a reflection about the y-axis.

c) Is the function g increasing or decreasing? ***Decreasing***



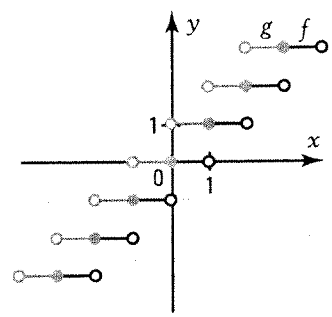
3. The function $f(x) = [x]$ is represented on the right. Let $g(x) = -[-x]$.

a) Represent, in the same Cartesian plane, the function g.

b) Explain how to obtain the graph of function g from the graph of function f.

By a reflection about the y-axis followed by a reflection about the x-axis.

c) Is the function g increasing or decreasing? ***Increasing***

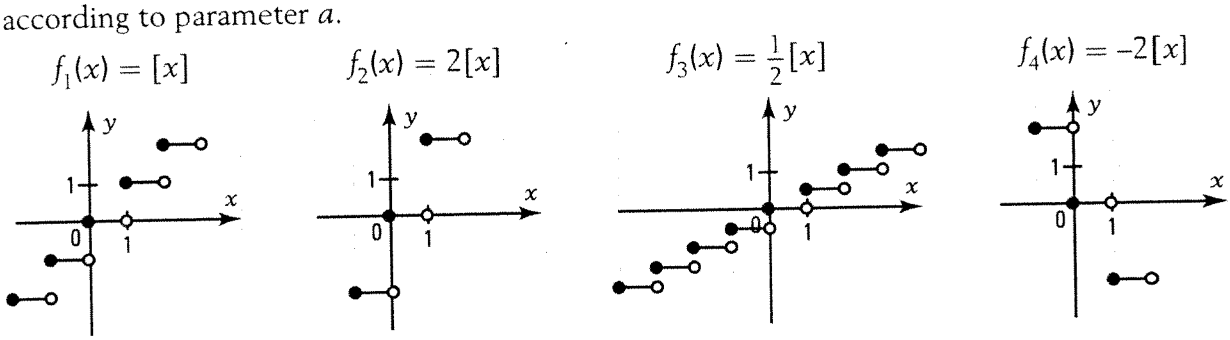


4.4 Transformed greatest integer function

ACTIVITY 1 Transformed greatest integer function

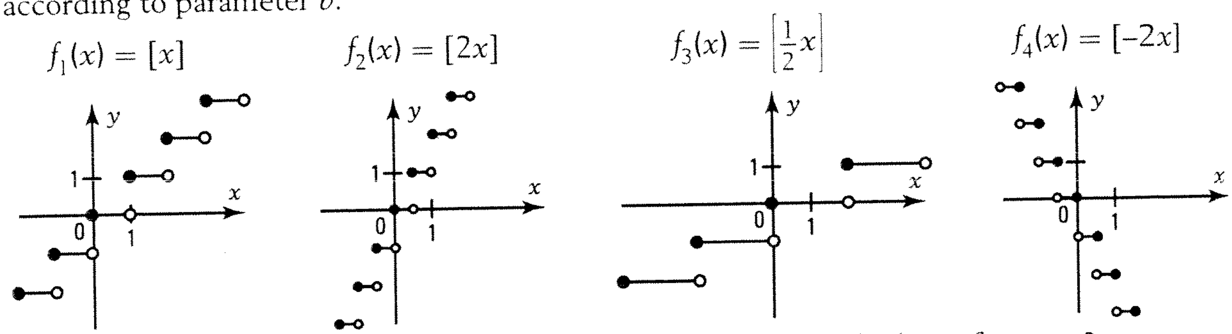
Consider the transformed greatest integer function $f(x) = a[b(x - h)] + k$.

a) We have represented below the basic function f_1 and the functions f_2, f_3, f_4 which differ according to parameter a.



1. Which geometric transformation does parameter a have on the basic function? ***A vertical scale change***
2. Describe the scale change when
   1) $|a| > 1$. ***A stretch*** 2) $|a| < 1$. ***A reduction***
3. What happens when $a < 0$? ***A vertical scale change followed by a reflection about the x-axis.***
4. What is the counterstep height in each case? ***$|a|$***

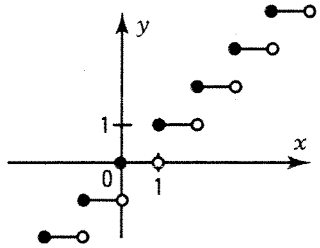
b) We have represented below the basic function f_1 and the functions f_2, f_3, f_4 which differ according to parameter b.



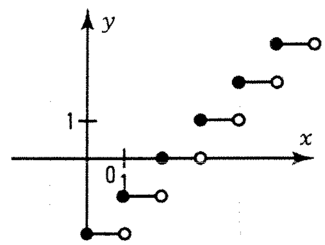
1. Which geometric transformation does parameter b have on the basic function? ***A horizontal scale change***
2. Describe the scale change when
   1) $|b| > 1$. ***a reduction*** 2) $|b| < 1$. ***a stretch***
3. What happens when $b < 0$? ***A horizontal scale change followed by a reflection about the y-axis.***
4. What is the step length in each case? ***$\frac{1}{|b|}$***
5. What is the sign of b when the steps are the following type.
   1) ●—○ ***$b > 0$*** 2) ○—● ***$b < 0$***

c) We have represented below the basic function f_1 and the functions f_2, f_3 which differ according to parameter h.

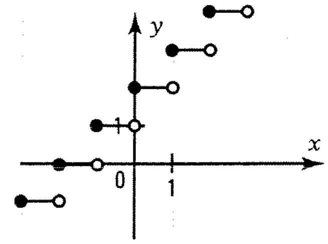
$f_1(x) = [x]$



$f_2(x) = [x - 2]$



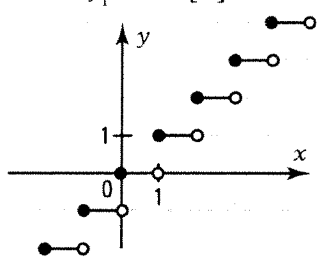
$f_3(x) = [x + 2]$



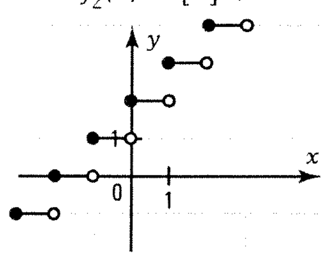
1. Which geometric transformation does parameter h have on the basic function?
   A horizontal translation.
2. Describe the change when
   1) $h > 0$. ***Horizontal translation of h units to the right.***
   2) $h < 0$. ***Horizontal translation of |h| units to the left.***

d) We have represented below the basic function f_1 and the functions f_2, f_3 which differ according to parameter k.

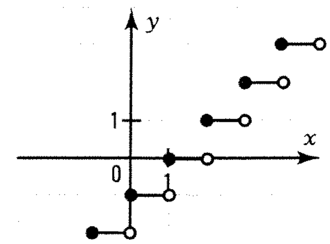
$f_1(x) = [x]$



$f_2(x) = [x] + 2$



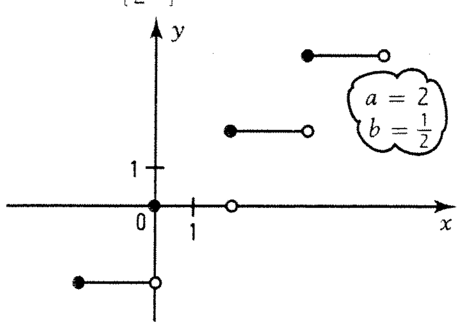
$f_3(x) = [x] - 1$



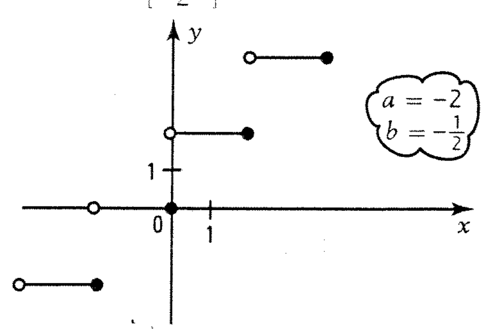
1. Which geometric transformation does parameter k have on the basic function?
   A vertical translation.
2. Describe the change when
   1) $k > 0$. ***Vertical translation of k units upward.***
   2) $k < 0$. ***Vertical translation of |k| units downward.***

e) Here are 4 functions who differ in the signs of a and b.

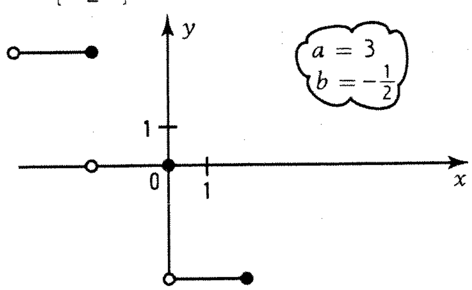
$f_1(x) = 2\left[\frac{1}{2}x\right]$



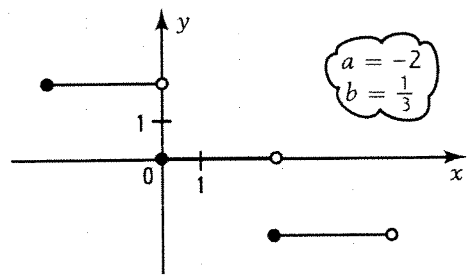
$f_2(x) = -2\left[-\frac{1}{2}x\right]$



$f_3(x) = 3\left[-\frac{1}{2}x\right]$

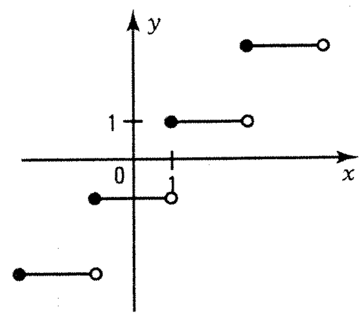


$f_4(x) = -2\left[\frac{1}{3}x\right]$



1. What happens to the variation of $f(x) = a[b(x - h)] + k$ when
   1) $ab > 0$. ***f is increasing*** 2) $ab < 0$. ***f is decreasing***
2. What is the length of each step? $\frac{1}{|b|}$
3. What is the counterstep height? $|a|$
4. Complete the following using the appropriate term "right" or "left".
   1) When b is positive, each step is closed on the ***left*** and open on the ***right***.
   2) When b is negative, each step is closed on the ***right*** and open on the ***left***.

f) On the right, the following function has been represented:
$f(x) = 2\left[\frac{1}{2}(x - 1)\right] + 1$.



1. What is the domain of f? $\mathbb{R}$
2. Explain why the range of $f = \{y \mid y = 2m + 1, m \in \mathbb{Z}\}$.
   $\left[\frac{1}{2}(x - 1)\right]$ ***is an integer that we note as m and each image is of the form 2m + 1.***
3. Explain why the function f has no zeros.
   $2\left[\frac{1}{2}(x - 1)\right] + 1 = 0 \Leftrightarrow \left[\frac{1}{2}(x - 1)\right] = -\frac{1}{2}$. $-\frac{1}{2}$ ***is not an integer.***
4. What is the sign of f? $f(x) < 0, \forall x \in]-\infty, 1]$ $f(x) \geq 0, \forall x \in [1, +\infty[$
5. What is the variation of f? ***f is increasing.***
6. Does the function f have any extrema? ***No***

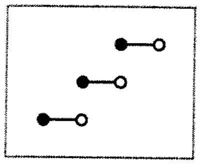
TRANSFORMED GREATEST INTEGER FUNCTION

We represent the transformed greatest integer function by the rule $f(x) = a[b(x - h)] + k$.

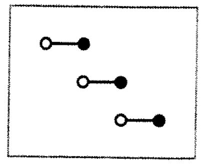
- The Cartesian graph is a step function.
- Each step has a length of $\frac{1}{|b|}$.
  - If $b > 0$, the steps are closed on the left and open on the right (●—○).
  - If $b < 0$, the steps are open on the left and closed on the right (○—●).
- The counterstep height is $|a|$.
- $\text{dom } f = \mathbb{R}$, $\text{ran } f = \{y \mid y = am + k, m \in \mathbb{Z}\}$
- – If $ab > 0$, the function is increasing.
  – If $ab < 0$, the function is decreasing.
- The function f has zeros if and only if k is a multiple of a.

- The signs of a and b determine 4 cases:

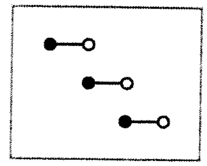
$a > 0$ and $b > 0$



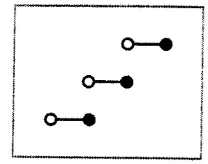
$a > 0$ and $b < 0$



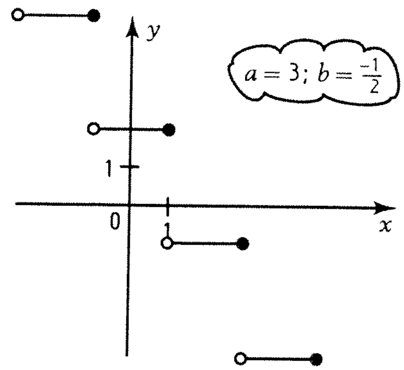
$a < 0$ and $b > 0$



$a < 0$ and $b < 0$



Ex.: $f(x) = 3\left[-\frac{1}{2}(x - 1)\right] + 2$

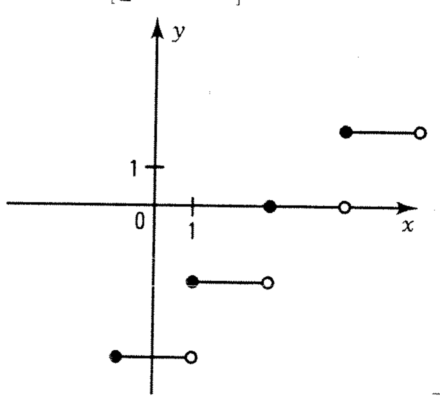


We have: $a = 3$; $b = \frac{-1}{2}$; $h = 1$ and $k = 2$.

- Each step has a length of $\frac{1}{|b|} = 2$.
- The counterstep height is $|a| = 3$.
- dom $f = \mathbb{R}$
- ran $f = \{y \mid y = 3m + 2,\ m \in \mathbb{Z}\}$
- zeros of f: f has no zeros since k is not a multiple of a.
- y-intercept of f: 2.
- $f(x) > 0$ if $x \leqslant 1$; $f(x) < 0$ if $x > 1$
- f is decreasing over $\mathbb{R}$, since $ab < 0$.
- f has no extrema.

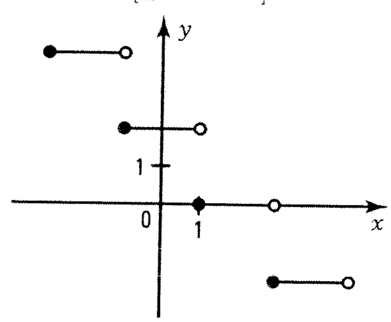
1. Represent the following functions and determine the set S of zeros.

a) $f(x) = 2\left[\frac{1}{2}(x - 1)\right] - 2$



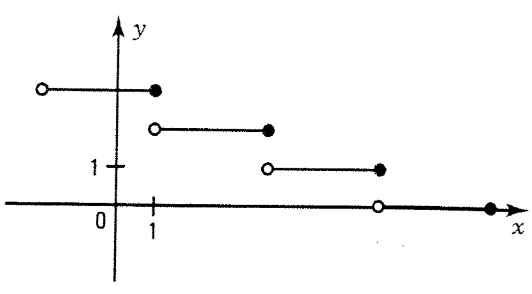
S = [3, 5[

b) $f(x) = -2\left[\frac{1}{2}(x + 1)\right] + 2$



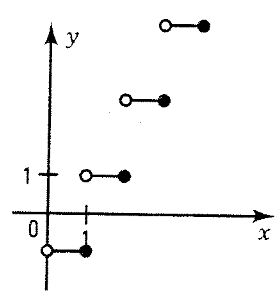
S = [1, 3[

c) $f(x) = \left[-\frac{1}{3}(x - 1)\right] + 3$



S =]7, 10]

d) $f(x) = -2[-(x - 2)] + 1$



S = ∅

2. The following rules define greatest integer functions. Write them in the form $y = a[b(x - h)] + k$.

a) $y = [2x - 1]$ $y = [2(x - \frac{1}{2})]$

b) $y = -2[3x - 6]$ $y = -2[3(x - 2)]$

c) $y = \left[\frac{x-3}{2}\right]$ $y = [\frac{1}{2}(x - 3)]$

d) $y = \left[\frac{x}{3} - 1\right]$ $y = [\frac{1}{3}(x - 3)]$

3. Which geometric transformations apply the basic greatest integer function to the following functions?

a) $f(x) = [x - 5]$ ***Horizontal translation of 5 units to the right.***

b) $f(x) = [x] + 2$ ***Vertical translation of 2 units upward.***

c) $f(x) = [x + 3] - 1$ ***Horizontal translation of 3 units to the left, followed by a vertical translation of one unit downward.***

d) $f(x) = -[x - 2]$ ***Horizontal translation of 2 units to the right, followed by a reflection about the x-axis.***

e) $f(x) = [-2x]$ ***Horizontal scale change followed by a reflection about the y-axis.***

4. For each of the following greatest integer functions, determine

1. the length of one step and its type (●—○) or (○—●).
2. the counterstep height.
3. the set S of zeros.
4. the y-intercept.
5. the variation of the function.

a) $y = 2[-3(x - 1)] + 4$

1. ***length:*** $\frac{1}{3}$; ○—●
2. ***height: 2***
3. $S = \left]\frac{4}{3}, \frac{5}{3}\right]$
4. ***y-int.: 10***
5. ***decreasing function***

b) ~~$y = [\frac{1}{2}(x + 1)] + 6$~~ $-2[\frac{1}{2}(x+4)]$

1. ***length: 2;*** ●—○
2. ***height: 2***
3. $S = [-1, 1[$
4. ***y-int.: 0***
5. ***increasing function***

c) $y = 3[2(x + 1)] - 5$

1. ***length:*** $\frac{1}{2}$; ●—○
2. ***height: 3***
3. $S = \varnothing$
4. ***y-int.: 1***
5. ***increasing function***

d) $y = \frac{1}{2}[-4(x + 1)] + 2$

1. ***length:*** $\frac{1}{4}$; ○—●
2. ***height:*** $\frac{1}{2}$
3. $S = \left]-\frac{1}{4}, 0\right]$
4. ***y-int.: 0***
5. ***decreasing function***

5. Determine the domain and range of the following functions.

a) $y = -3[5(x + 2)] - 7$

dom $= \mathbb{R}$; ***ran*** $= \{y \mid y = -3m - 7, m \in \mathbb{Z}\}$

b) $y = \frac{1}{2}[-3(x - 1)] + 4$

dom $= \mathbb{R}$; ***ran*** $= \{y \mid y = \frac{1}{2}m + 4, m \in \mathbb{Z}\}$

6. Find the set of values of x for which

1. $f(x) \geq 0$.
2. $f(x) < 0$.

a) $f(x) = 3[3(x -$ [redacted]$)] + 2$

1. $x \in [$[redacted]$, +\infty[$

2. $x \in [-\infty,$ [redacted]$[$

b) $f(x) = -3\left[\frac{1}{3}x + 2\right] + 6$

1. $x \in]-\infty, 3[$

2. $x \in [3, +\infty[$

c) $f(x) = -2\left[\frac{1}{4}(x - 1)\right] - 4$

1. $x \in]-\infty, -3[$

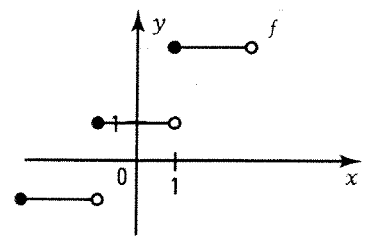
2. $x \in [-3, +\infty[$

d) $f(x) = 3\left[\frac{-1}{2}(x + 2)\right] - 4$

1. $x \in]-\infty, -6]$

2. $x \in]-6, +\infty[$

7. Consider the function $f(x) = 2\left[\frac{1}{2}(x - 1)\right] + 3$ represented on the right.



a) Represent the function $g(x) = 2\left[\frac{1}{2}(x + 1)\right] + 1$.

What do you notice?

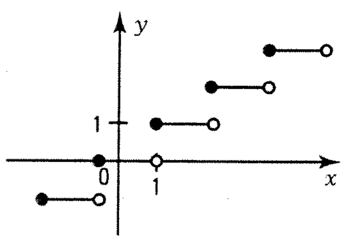
The graph of g coincides with the graph of f.

b) Find the rule of a function h with the same graph as f.

For example, $h(x) = 2\left[\frac{1}{2}(x + 3)\right] - 1$

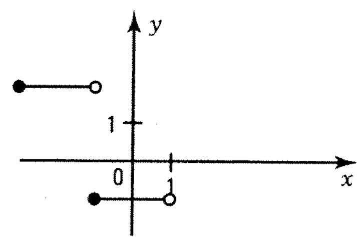
8. For each of the step functions represented below, find a rule corresponding to the function.

a)



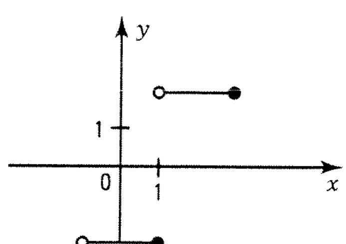
$y = \left[\frac{2}{3}(x - 1)\right] + 1$

b)



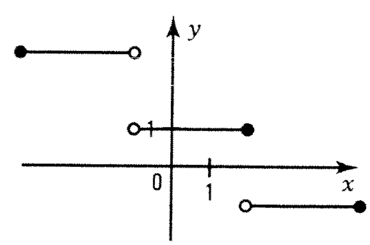
$y = -3\left[\frac{1}{2}(x + 1)\right] - 1$

c)



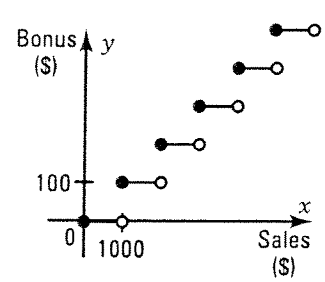
$y = -4\left[-\frac{1}{2}(x - 1)\right] - 2$

d)



$y = 2\left[-\frac{1}{3}(x - 2)\right] + 1$

9. To motivate his salesmen, a sales manager awards a bonus of \$100 for every \$1000 in sales.



a) Draw the Cartesian graph of the function which gives the awarded bonus y as a function of the amount of sales x.

b) What is the rule of the function? $y = 100\left[\frac{x}{1000}\right] \quad (x \geq 0)$

10. A salesman receives a weekly base salary of \$150 and a \$50 bonus for every \$1000 in weekly sales.

a) Find the rule of the function which gives the weekly salary y as a function of the amount x of weekly sales.

$y = 50\left[\frac{x}{1000}\right] + 150$

b) A salesman sells \$12 480 of merchandise in a week. What will his salary be?

\$750

c) For what amount of sales will the salesman receive a salary of \$1000?

An amount within the interval [17 000, 18 000[

d) Is it possible for a salesman to receive a salary equal to \$825?

$50\left[\frac{x}{1000}\right] + 150 = 825 \Leftrightarrow \left[\frac{x}{1000}\right] = 13.5$ can't be a decimal

The last equation has no solution since 13.5 ∉ ℕ.

11. The cost y, in dollars, of mailing a package depends on its mass x, in grams. This cost is defined by the rule $y = -2.5\left[-\frac{x}{100}\right]$.

a) What is the cost of mailing a package with a mass of 260 g? ***\$7.50***

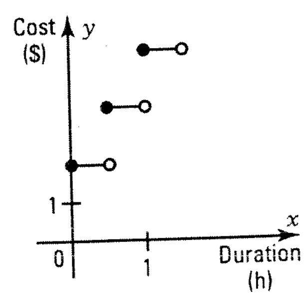
b) What is the mass of a package that costs \$12.50 to mail?

$x \in]400, 500]$

c) Explain in your words how to calculate the cost of mailing a package.

It costs \$2.50 for 100 g or less and \$2.50 more for every additional 100 g.

12. At a parking lot, the cost y of parking is calculated as follows: a minimum cost of \$2 for a parking time of less than 30 min. In addition, \$1.50 is charged for every 30 minute interval of parking time.



a) Draw the graph of the function which gives the total cost y, in dollars, of parking as a function of the parking time x, in hours.

b) Find the rule of the function.

$y = 1.5[2x] + 2$

c) What is the parking time corresponding to a cost of \$8?

$x \in [2, 2.5[$

$8 = 1.5[2x] + 2$
$6 = 1.5[2x]$
$4 \le 2x < 5$
$2 \le x < 2.5$

Evaluation 4

1. For each of the following step functions, determine

1. the step length and its type (●—○) or (○—●).
2. the counterstep height
3. the variation of the function.
4. the set S of zeros.

a) $y = -3\left[\frac{1}{4}(x - 1)\right] + 6$

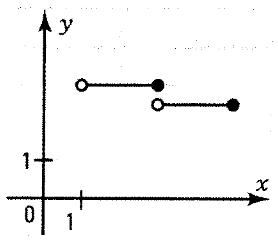
1. ***length: 4; ●—○***
2. ***height: 3***
3. ***decreasing function***
4. ***S = [9, 13[***

b) $y = -2\left[-\frac{2}{3}(x + 1)\right] + 5$

1. ***length: $\frac{3}{2}$; ○—●***
2. ***height: 2***
3. ***increasing function***
4. ***S = ∅***

2. For each of the step functions below, only two steps are represented. Find a rule and the zeros for each function.

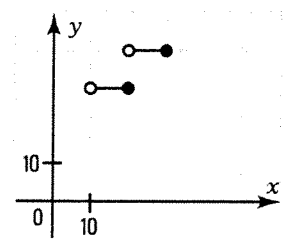
a)



y = 0.5[−0.5(x − 3)] + 3

S =]13, 15]

b)



y = −10[−0.1(x − 20)] + 30

S =]−20, −10]

3. The cost of parking in a garage is described by the rule $y = 3 + 2.5\left[\frac{x}{60}\right]$ where x represents the parking time, in minutes, and y represents the cost in dollars.

a) Describe in your words how the cost is calculated.

A minimal cost of $3 is charged for a parking time of less than 1 h. Then, there is an additional charge of $2.50 for every 60 minutes of parking time.

b) What is the cost of parking 2 h 15 min? ***$8*** $3 + 2.5\left[\frac{135}{60}\right] = 8$

c) What could the parking duration be if it costs $10.50? ***180 ≤ x < 240***

4. A sugar refinery sets the price y, in dollars, according to the quantity x, in kilograms, of sugar ordered. The price is described by the rule $y = -0.1\left[\frac{x}{100}\right] + 2$.

a) What is the selling price for an order of 325 kg?

$1.70

b) Explain in your words how the selling price is calculated.

There is a charge of $2 for an order of less than 100 kg. The base price is then decreased by 10¢ for every additional 100 kg of sugar ordered.

3c) $10.50 = 3 + 2.5\left[\frac{x}{60}\right]$

$3 \le x/60 < 4$

$180 \le x < 240$