Chapter 5

Analytic Geometry

CHALLENGE 5

5.1 Distance between two points

5.2 Mid-point of a segment

5.3 Slope of a line

5.4 Intercepts of a line

5.5 Functional form of the equation of a line

5.6 General form of the equation of a line

5.7 Symmetric form of the equation of a line

5.8 Finding the equation of a line

5.9 Distance from a point to a line

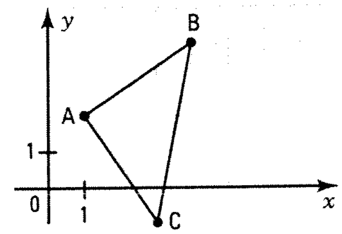
5.10 Regions of the Cartesian plane

5.11 Analytic geometry problems

EVALUATION 5

CHALLENGE 5

1. What kind of triangle is ABC whose vertices are A(1, 2), B(4, 4) and C(3, –1)?

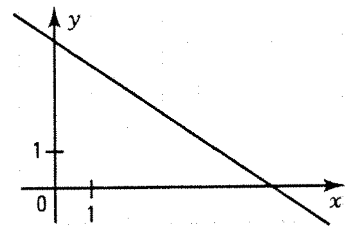


***m*$\overline{AB}$ = $\sqrt{13}$, *m*$\overline{AC}$ = $\sqrt{13}$, *m*$\overline{BC}$ = $\sqrt{26}$**

***m*$\overline{AB}$ = *m*$\overline{AC}$ and $(m\overline{BC})^2 = (m\overline{AB})^2 + (m\overline{AC})^2$**

*⇒ **Triangle ABC is a right isosceles triangle.***

2. Draw the line l whose equation is $2x + 3y - 12 = 0$.

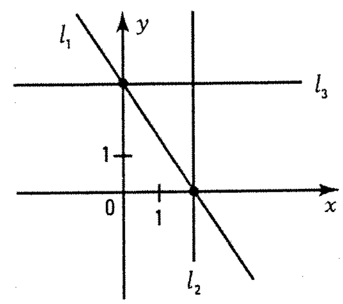


3. Determine the equation of the line

a) l_1. ***3x + 2y – 6 = 0***

b) l_2. ***x – 2 = 0***

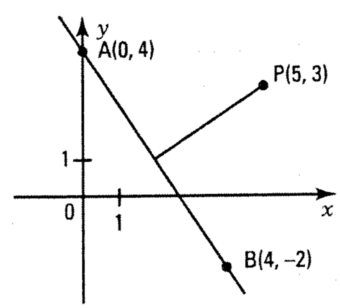
c) l_3. ***y – 3 = 0***



4. Calculate the distance from the point P(5, 3) to the line passing through the points A(0, 4) and B(4, -2).

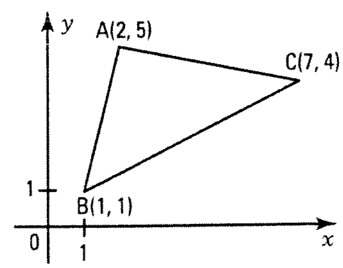
***l*: 3*x* + 2*y* – 8 = 0**

***d*(*P*, *l*) = $\sqrt{13}$**



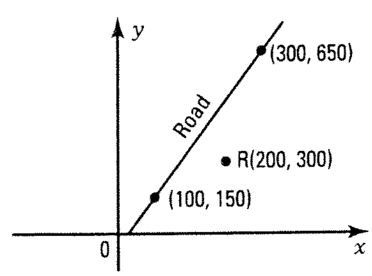
5. Calculate the area of triangle ABC.

10.5 u²



6. A path is to be made perpendicular to a road to gain access to a water reservoir represented by the point R in the Cartesian plane on the right (scaled in metres). What is, to the nearest dollar, the cost of making this path if it costs $200 per metre?

Road: 5x – 2y – 200 = 0; path = 37.14 m; cost: $7428



5.1 Distance between two points

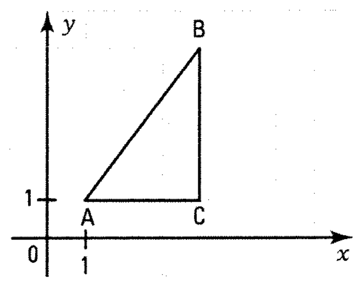
ACTIVITY 1 Distance between two points

a) Consider the following right triangle with vertices A(1, 1), B(4, 5) and C(4, 1). Find a procedure for calculating the distance between A and B and calculate that distance.

1. Calculate m$\overline{BC}$. 2. Calculate m$\overline{AC}$.

3. Deduce m$\overline{AB}$ using the Pythagorean theorem.

m$\overline{BC}$ = 4; m$\overline{AC}$ = 3 ⇒ m$\overline{AB}$ = 5



b) Consider the following right triangle ABC such that $x_A < x_B$ and $y_A < y_B$.

1. Explain why $x_C = x_B$ and $y_C = y_A$.

   Segment BC is vertical and segment AC is horizontal.

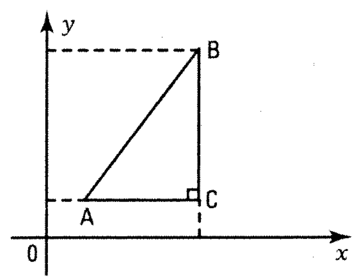
2. Express m$\overline{AC}$ in terms of x_A and x_B. ***m$\overline{AC}$ = $x_B - x_A$***

3. Express m$\overline{BC}$ in terms of y_A and y_B. ***m$\overline{BC}$ = $y_B - y_A$***

4. Deduce m$\overline{AB}$.

   $(m\overline{AB})^2 = (m\overline{AC})^2 + (m\overline{BC})^2 \Rightarrow (m\overline{AB})^2 = (x_B - x_A)^2 + (y_B - y_A)^2$

   $\Rightarrow m\overline{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$



DISTANCE BETWEEN TWO POINTS

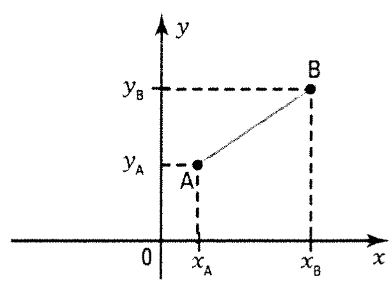
- The distance between two points $A(x_A, y_B)$ and $B(x_B, y_B)$, noted $d(A, B)$, is given by the formula:

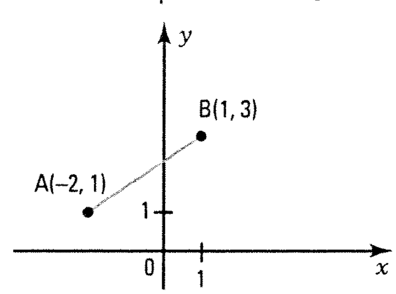
$$d(A, B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

- A distance is always positive or zero: $d(A, B) \geqslant 0$.
- Given two points A and B, we have: $d(A, B) = d(B, A)$.

Ex.: The distance between A(-2, 1) and B(1, 3) is:

$$d(A, B) = \sqrt{(1 + 2)^2 + (3 - 1)^2}$$
$$= \sqrt{9 + 4}$$
$$= \sqrt{13}.$$





1. Calculate the distance between the following points:

a) (1, 5) and (2, 3) $\sqrt{5}$ b) (–1, 2) and (1, 2) ***2***

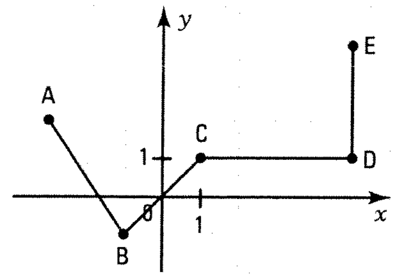
c) (1, –2) and (2, 5) ***$\sqrt{50} = 5\sqrt{2}$*** d) (–1, –3) and (2, 3) ***$\sqrt{45} = 3\sqrt{5}$***

e) (1, 2) and (–2, 1) ***$\sqrt{10}$*** f) (–3, –1) and (2, –3) ***$\sqrt{29}$***

2. What distance separates the following points?

a) A and B ***$\sqrt{13}$*** b) B and C ***$\sqrt{8} = 2\sqrt{2}$***

c) C and D ***4*** d) D and E ***3***

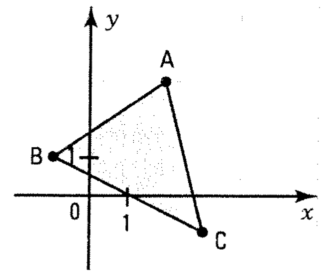


3. Determine the perimeter of triangle ABC on the right.

We have: A(2, 3), B(–1, 1) and C(3, –1), d(A, B) = $\sqrt{13}$

d(B, C) = $\sqrt{20}$ and d(A, C) = $\sqrt{17}$.

Perimeter of △ABC = $\sqrt{20} + \sqrt{13} + \sqrt{17} \approx 12.20$ u.



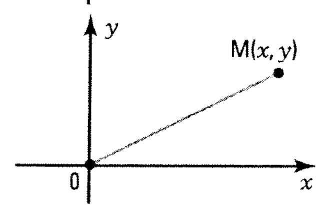
4. a) Given a random point M(*x*, *y*) of the Cartesian plane.

Show that $d(0, \text{M}) = \sqrt{x^2 + y^2}$.

$d(0, M) = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$

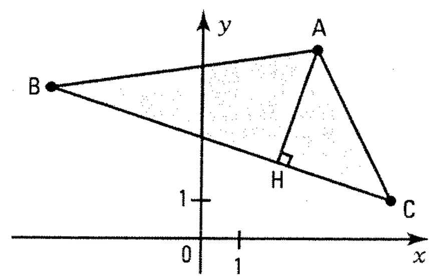
b) Given A(–2, 3), B(1, –2) and C(–3, 4). Calculate:

1. *d*(0, A). ***$\sqrt{13}$*** 2. *d*(0, B). ***$\sqrt{5}$*** 3. *d*(0, C). ***5***



5. Given A(3, 5), B(–4, 4) and C(5, 1) the vertices of triangle ABC on the right, and H(2, 2) the foot of the altitude to side BC, calculate the area of triangle ABC.

$m\overline{BC} = \sqrt{90} = 3\sqrt{10}$, $m\overline{AH} = \sqrt{10}$. Area △ ABC = 15 u²



6. Given three points in the Cartesian plane A(–4, 1), B(1, 6) and C(1, 1).

a) Show that triangle ABC is an isosceles right triangle.

$d(A, B) = m\overline{AB} = \sqrt{50}$, $d(A, C) = m\overline{AC} = 5$; $d(B, C) = m\overline{BC} = 5$

We have: $(m\overline{AB})^2 = (m\overline{AC})^2 + (m\overline{BC})^2$ and $m\overline{AC} = m\overline{BC}$.

We deduce that △ABC is an isosceles right triangle with main vertex C.

b) What is the area of triangle ABC? ***12.5 u²***

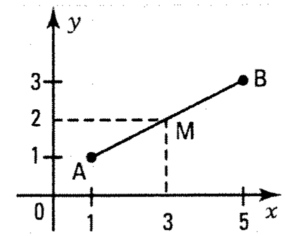
7. Show that the points A(–2, 2), B(5, –5) and C(4, 2) are located on a circle with centre *w*(1, –2). What is the radius of this circle?

It must be shown that d(w, A) = d(w, B) = d(w, C).

d(w, A) = 5; d(w, B) = 5; d(w, C) = 5. The circle has a radius of 5 units.

5.2 Mid-point of a segment

Activity 1 Mid-point of a segment



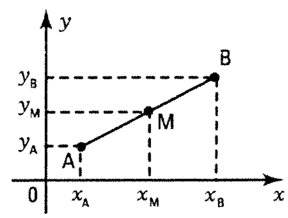
a) Draw in the Cartesian plane a segment AB and locate the mid-point M of this segment.

b) Verify that the x-coordinate of the mid-point of the segment is equal to half the sum of the x-coordinates of the end-points of that segment. In other words: $x_M = \frac{x_A + x_B}{2}$.

A(1, 1); B(5, 3); M(3, 2). ***We have:*** $3 = \frac{1+5}{2}$.

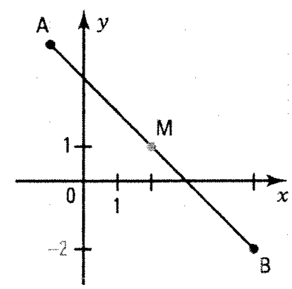
c) What can be said of the y-coordinate of the mid-point? ***We have the same observation.*** $y_M = \frac{y_A + y_B}{2}$.

MID-POINT OF A SEGMENT



- Given M the mid-point of segment AB with end-points $A(x_A, y_A)$ and $B(x_B, y_B)$.
  The coordinates of M are:

$$x_M = \frac{x_A + x_B}{2},\ y_M = \frac{y_A + y_B}{2}$$



Ex.: Given A(–1, 4) and B(5, –2) the end-points of segment AB.
The mid-point M of segment AB has the coordinates:

$$x_M = \frac{-1+5}{2} = 2,\ y_M = \frac{4+-2}{2} = 1.$$

Therefore, we have: M(2, 1).

1. Determine the coordinates of the mid-point M of segment AB in each of the following.

a) A(2, 1); B(–3, 5) ***M***$\left(-\frac{1}{2}, 3\right)$

b) A(–3, –1); B(2, –5) ***M***$\left(-\frac{1}{2}, -3\right)$

c) $A\left(-\frac{1}{2}, \frac{3}{4}\right)$; $B\left(3, -\frac{1}{2}\right)$ ***M***$\left(\frac{5}{4}, \frac{1}{8}\right)$

d) A(2a, –3b); B(4a, –b) ***M(3a, -2b)***

2. Given A(–2, 1) and B(8, 5) the end-points of the diameter of a circle.

a) What are the coordinates of the centre w of this circle? ***w(3, 3)***

b) What is the radius of this circle? $\sqrt{29}$

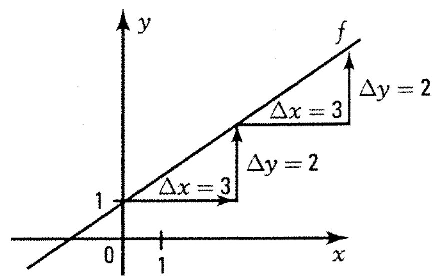
3. Given A(2, 3); B(–1, 1) and C(3, –1) the vertices of triangle ABC. What is the length of the median AM? $\sqrt{10}$

4. Given M(1, 2) the mid-point of segment AB. Determine the coordinates of B if the coordinates of A are A(–3, 4). ***B(5, -8)***

5.3 Slope of a line

ACTIVITY 1 Rate of change and slope

We know that every direct, partial or zero variation function is represented by a non-vertical line in the Cartesian plane.



a) Is the rate of change of a direct, partial or zero variation function constant? ***Yes***

b) Calculate the rate of change of the function f represented on the right and interpret it.

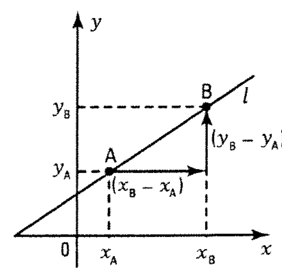
The rate of change is equal to $\frac{2}{3}$. To each positive variation of 3 units of the variable x, there is a corresponding positive variation of 2 units of the variable y.

When a function is represented by a line, the function's rate of change which remains constant is called the **slope** of the line.

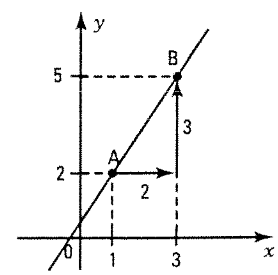
SLOPE OF A LINE

- Given $A(x_A, y_A)$ and $B(x_B, y_B)$ two random points on a line l. The slope of the line l, noted a, is equal to:

$$a = \frac{y_B - y_A}{x_B - x_A}$$

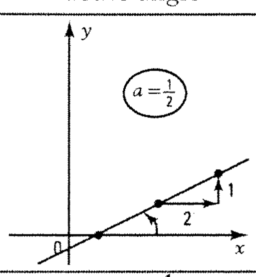
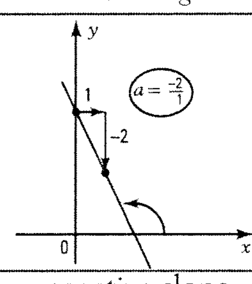
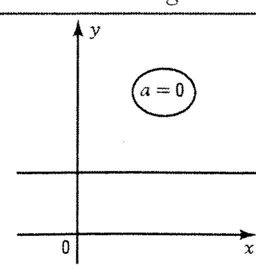
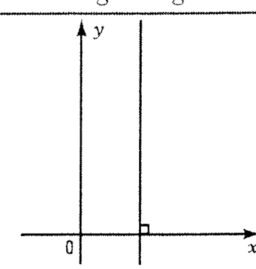


Ex.: Given A(1, 2) and B(3, 5) two points on the line l on the right. The slope a of the line l is equal to $a = \frac{5-2}{3-1} = \frac{3}{2}$.

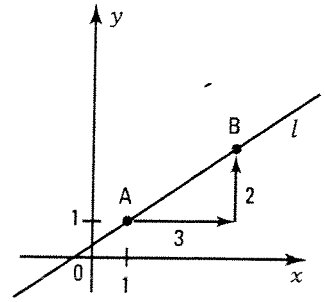


To a positive variation of 2 units on the x-axis, there is a corresponding positive variation of 3 units on the y-axis.

- The slope of a line gives the inclination of the line with respect to the x-axis. We distinguish four cases:

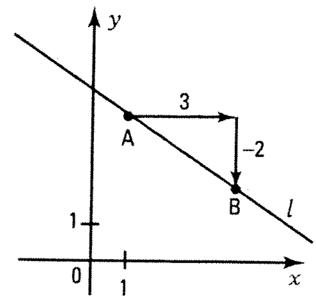
acute angle	obtuse angle	zero angle	right angle
			
positive slope	negative slope	zero slope	undefined slope

1. a) The line l on the right passes through A(1, 1) and B(4, 3).



1. What is the sign of the slope of line l? ***Positive***
2. Calculate the slope of line l. $\frac{2}{3}$
3. Complete the description of the slope of line l:
"For each positive variation of ***3*** units on the x-axis, there is a corresponding ***positive*** variation of ***2*** units on the y-axis."

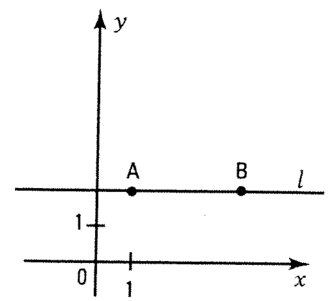
b) The line l on the right passes through A(1, 4) and B(4, 2).



1. What is the sign of the slope of line l? ***Negative***
2. Calculate the slope of line l. $-\frac{2}{3}$
3. Complete the description of the slope of line l:
"For each positive variation of ***3*** units on the x-axis, there is a corresponding ***negative*** variation of ***2*** units on the y-axis."

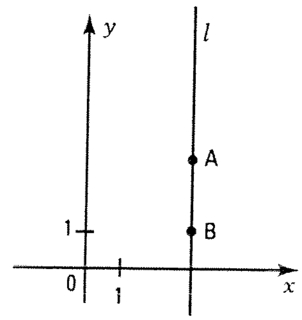
c) The line l on the right is horizontal (parallel to the x-axis). Using any points A and B of your choice, calculate the slope of line l and verify that it is zero.

A(1, 2); B(4, 2); $a = \frac{2-2}{4-1} = \frac{0}{3} = 0$



d) The line l on the right is vertical (parallel to the y-axis). Using any points A and B of your choice, calculate the slope of line l and explain why it is undefined.

A(3, 3); B(3, 1); $a = \frac{1-3}{3-3} = \frac{-2}{0}$. ***Dividing by zero is impossible; therefore the slope of line l is undefined.***



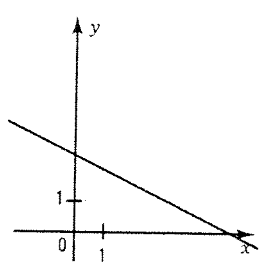
2. Calculate the slope of the line passing through:

a) (2, 1) and (–3, 5). $-\frac{4}{5}$ b) (–3, 1) and (2, –1). $-\frac{2}{5}$ c) (–2, –3) and (1, 5). $\frac{8}{3}$

d) (–2, –4) and (–3, –7). ***3*** e) $\left(\frac{1}{2}, \frac{3}{4}\right)$ and $\left(\frac{4}{5}, \frac{1}{3}\right)$. $-\frac{25}{18}$ f) (0.2; –0.8) and (1; 1.4). $\frac{11}{4}$

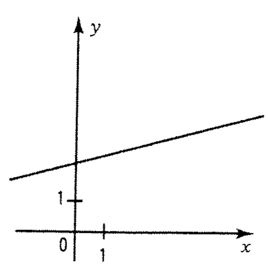
3. What is the slope of each of the following lines

a)



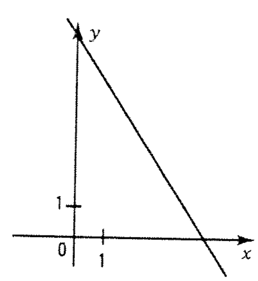
$a = -\frac{1}{2}$

b)



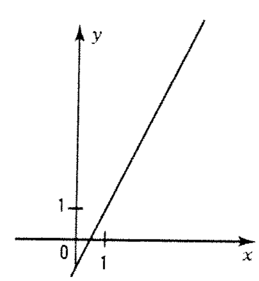
$a = \frac{1}{4}$

c)



$a = -\frac{3}{2}$

d)



$a = 2$

4. Given A(1, 2) and B(5, y). Find the value of y so that the slope of line AB is $\frac{1}{2}$.

$y = 4$

5. Given A(1, $2k$) and B($2k + 1$,5) two points on the line l. Determine the value of parameter k if:

a) the slope of l is equal to $\frac{3}{2}$. ***$k = 1$***

b) the slope of l is equal to $-\frac{3}{2}$. ***$k = -5$***

c) the slope of l is zero. ***$k = \frac{5}{2}$***

d) the slope of l is undefined. ***$k = 0$***

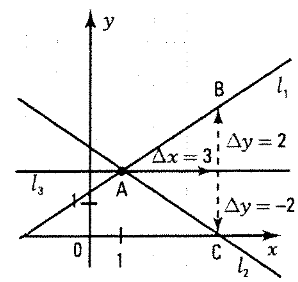
6. Given A(3, 6), B(–4, 4) and C(6, 2) the vertices of triangle ABC. What is the slope of the median AM from vertex A? ***$\frac{3}{2}$***

ACTIVITY 2 Drawing a line given a point and the slope

a) Explain how to draw line l_1 passing through A with a slope of $\frac{2}{3}$. Draw the line.

1. From point A, we move 3 units to the right and 2 units upward to reach the point B.

2. We draw the line AB.



b) Explain how to draw line l_2 passing through A with a slope of $-\frac{2}{3}$. Draw the line.

1. From point A, we move 3 units to the right and 2 units downward to reach the point C.

2. We draw the line AC.

c) Explain how to draw line l_2 passing through A with a slope of zero.

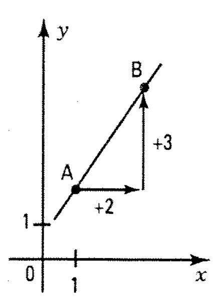
We draw the line parallel to the x-axis passing through A.

DRAWING A LINE GIVEN A POINT AND THE SLOPE

- The method illustrated below enables to draw a line when a point A on the line and the slope a are given.

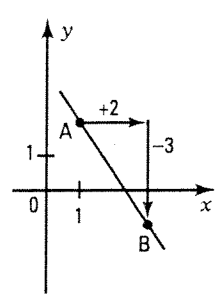
Ex.: A(1, 2) and $a = \frac{3}{2}$

1. From A, we move 2 units to the right and 3 units upward to reach point B.
2. We draw the line passing through A and B.



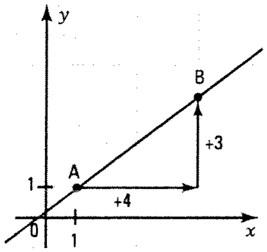
Ex.: A(1, 2) and $a = -\frac{3}{2}$

1. From A, we move 2 units to the right and 3 units downward to reach point B.
2. We draw the line passing through A and B.

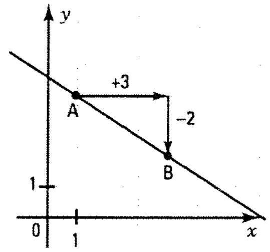


7. Draw the line passing through A with the slope a.

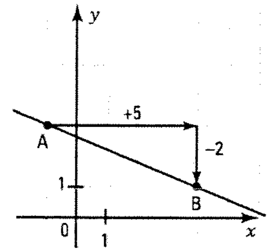
a) A(1, 1) and $a = \frac{3}{4}$



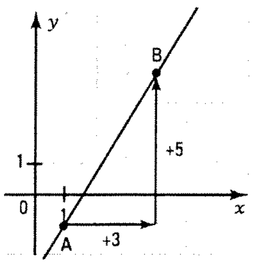
b) A(1, 4) and $a = -\frac{2}{3}$



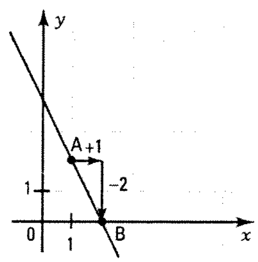
c) A(–1, 3) and $a = -\frac{2}{5}$



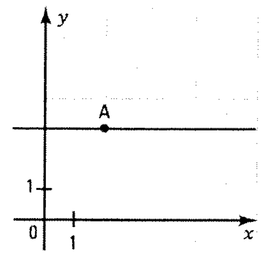
d) A(1, –1) and $a = \frac{5}{3}$



e) A(1, 2) and $a = -2$



f) A(2, 3) and $a = 0$

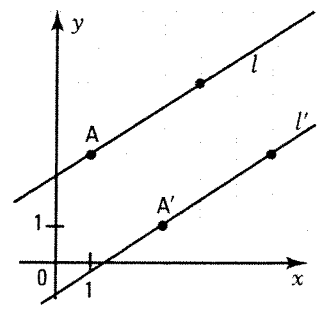


ACTIVITY 3 Parallel lines – Perpendicular lines

a) Given A(1, 3) and A′(3, 1) two points on the Cartesian plane.

1. Draw the line l passing through A with the slope $a = \frac{2}{3}$.
2. Draw the line l' passing through A′ with the slope $a' = \frac{2}{3}$.
3. What can we say about the relative position of the lines l and l'?

They are parallel.



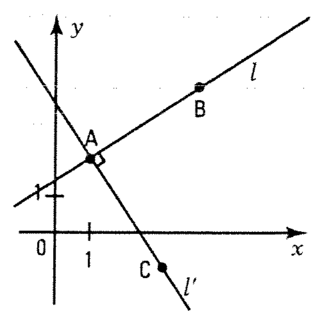
b) Given the point A(1, 2) on the Cartesian plane.

1. Draw the line l passing through A with the slope $a = \frac{2}{3}$.
2. Draw the line l' passing through A with the slope $a' = -\frac{3}{2}$.
3. Show that the lines l and l' are perpendicular. *B(4, 4) ∈ l;*

C(3, –1) ∈ l′. $m\overline{AB} = \sqrt{13}$; $m\overline{AC} = \sqrt{13}$; $m\overline{BC} = \sqrt{26}$

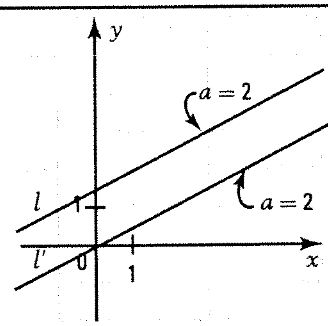
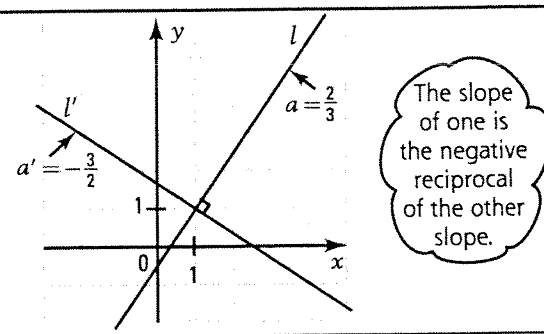
Since $m\overline{AB}^2 + m\overline{AC}^2 = m\overline{BC}^2$ then the △ABC is a right triangle in A.

4. Verify that $a \cdot a' = -1$. $\frac{2}{3} \times \frac{-3}{2} = -1$



PARALLEL LINES – PERPENDICULAR LINES

- Given two oblique lines l and l' with slope a and a' respectively. We have the following equivalencies:

$l // l' \Leftrightarrow a = a'$	$l \perp l' \Leftrightarrow a \cdot a' = -1$
	

8. a) Given A(0, –2), B(4, 1), A′ (0, 2) and B′ (4, 5) four points on the Cartesian plane. What can be said about the lines AB and A′B′? Justify your answer.

$a = \frac{3}{4}$, $a' = \frac{3}{4}$. Since $a = a'$ then AB//A′B′.

b) Given A(5, 4), B(–5, –2), A′(3, –4) and B′ (–3, 6) four points on the Cartesian plane. What can be said about the liness AB and A′B′? Justify your answer.

$a = \frac{3}{5}$, $a' = \frac{-5}{3}$. Since $a \times a' = -1$ then AB $\perp$ A′B′.

9. Given A(2, 4), B(0, 2) and C(6, 8) the vertices of triangle ABC. Verify the following theorem: "The line passing through the mid-points of two sides of a triangle is parallel to the third side of the triangle."

Given M and N the respective mid-points of $\overline{AB}$ and $\overline{AC}$. We have: M(1, 3), N(4, 6).

Slope of MN = 1; slope of BC = 1. Therefore, MN//BC.

10. Given A (1, 2), B(3, 4), C(6, 1) and D(4, –1) the vertices of the quadrilateral ABCD. Show that the quadrilateral ABCD is a rectangle.

We need to show that the opposite sides are parallel and at least one of the angles is a right angle.

Slope of AB = 1, slope of CD = 1, slope of AD = -1, slope of BC = -1.

We have: AB//CD, AD//BC and AB⊥AD.

11. Given A(0, 3), B(1, 0), C(4, 1) and D(3, 4) the vertices of the quadrilateral ABCD. Show that the quadrilateral ABCD is a square.

You need to show that the sides are congruent and at least one of the angles is a right angle.

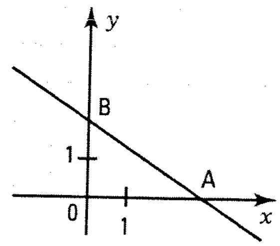
$m\overline{AB} = m\overline{BC} = m\overline{CD} = m\overline{AD} = \sqrt{10}$

Slope of AB = –3, slope of AD = $\frac{1}{3}$. $\overline{AD} \perp \overline{AB}$, since the product of the slopes is –1.

5.4 Intercepts of a line

ACTIVITY 1 *x*-intercept and *y*-intercept

When a line intersects the x-axis at the point A(a, 0) and the y-axis at the point B(0, b), the number a is called the *x-intercept* and the number b is called the *y-intercept* of the line.



a) Draw the line with an x-intercept of 3 and a y-intercept of 2.

b) What is the slope of this line? $-\frac{2}{3}$

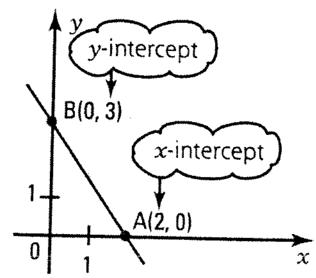
c) What is the position of the line if the

1. x-intercept does not exist? ***Horizontal***
2. y-intercept does not exist? ***Vertical***

INTERCEPTS OF A LINE

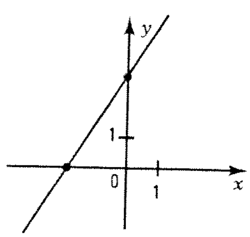
- The x-intercept is the *x-coordinate* of the point of intersection (if it exists) of a line with the x-axis.
- The y-intercept is the *y-coordinate* of the point of intersection (if it exists) of a line with the y-axis.
- The x-intercept and the y-intercept are called the *intercepts of a line*.

  Ex.: Consider the line l on the right.
  - The x-intercept is 2.
  - The y-intercept is 3.
- The x-intercept of a horizontal line does not exist and the y-intercept of a vertical line does not exist.



1. In each of the following cases, determine the intercepts.

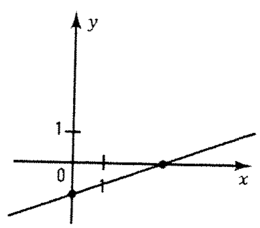
a)



x-intercept: −2

y-intercept: 3

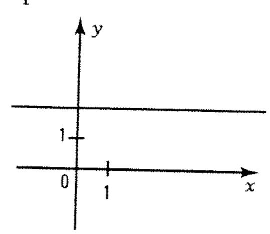
b)



x-intercept: 3

y-intercept: −1

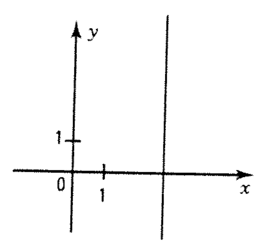
c)



x-intercept: does not exist

y-intercept: 2

d)

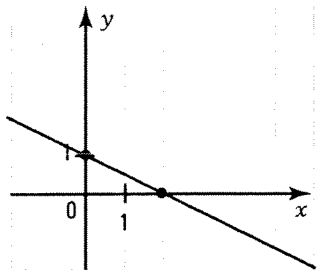


x-intercept: 3

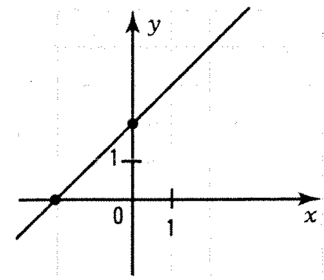
y-intercept: does not exist

2. In each of the following cases, draw the line knowing that

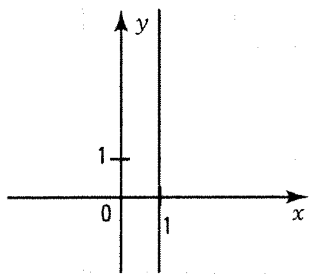
a) – the x-intercept is equal to 2.
– the y-intercept is equal to 1.



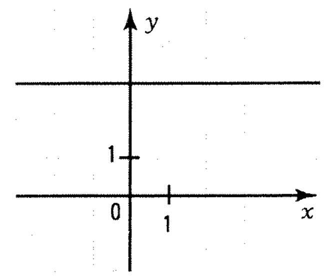
b) – the x-intercept is equal to –2.
– the y-intercept is equal to 2.



c) – the x-intercept is equal to 1.
– the y-intercept does not exist.

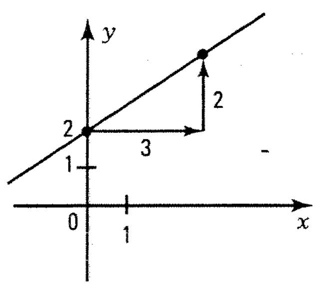


d) – the x-intercept does not exist.
– the y-intercept is equal to 3.

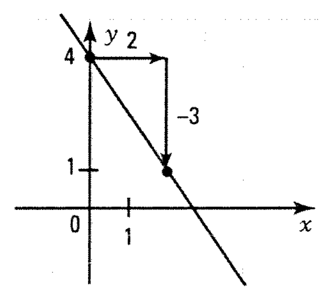


3. In each of the following cases, draw the line knowing that

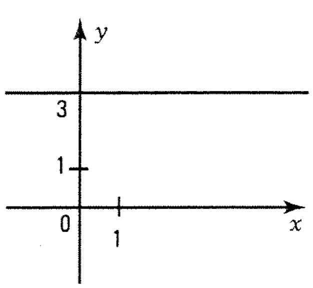
a) – the slope is equal to $\frac{2}{3}$.
– the y-intercept is equal to 2.



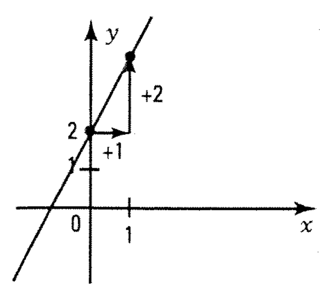
b) – the slope is equal to $-\frac{3}{2}$.
– the y-intercept is equal to 4.



c) – the slope is zero.
– the y-intercept is equal to 3.



d) – the slope is equal to 2.
– the y-intercept is equal to 2.



5.5 Functional form of the equation of a line

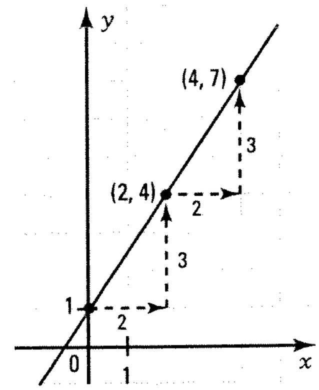
ACTIVITY 1 Functional form

The functional form of the equation of a line is $y = ax + b$ where a represents the slope and b represents the y-intercept of the line.
A line l has the equation: $y = \frac{3}{2}x + 1$. Draw the line in two different ways:

a) Using a point (the y-intercept) and the slope of the line.

b) Using a table of values.

x	0	2	4
y	*1*	*4*	*7*



FUNCTIONAL FORM OF THE EQUATION OF A LINE

- Given a non vertical line l with slope a and y-intercept b. The functional form of the equation of the line l is:

$$y = ax + b$$

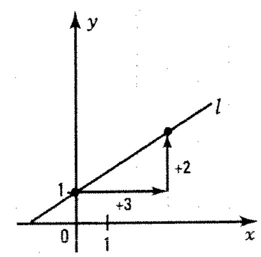
This equation is called the functional equation of the line.

Ex.: Given l: $y = \frac{2}{3}x + 1$.

– The slope of the line l is $a = \frac{2}{3}$.

– The y-intercept of the line l is $b = 1$.

The line l is drawn knowing its slope and y-intercept or by making a table of values:

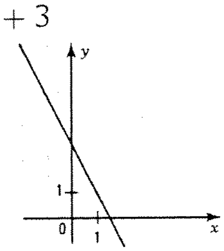
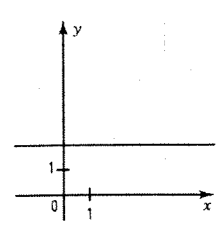
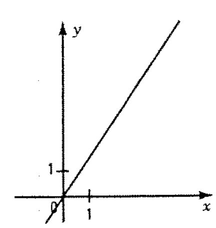


Randomly choose values of x.

x	y
0	1
3	3

We determine the values of y using the equation of the line.

- We have the following cases:

$a \neq 0$ and $b \neq 0$	$a = 0$ and $b \neq 0$	$a \neq 0$ and $b = 0$
Ex.: $y = -2x + 3$ Oblique line not passing through the origin. 	Ex.: $y = 2$ Horizontal line 	Ex.: $y = \frac{3}{2}x$ Oblique line passing through the origin. 

- If l is a vertical line, its Cartesian graph cannot represent a function. It is therefore impossible to represent its equation in functional form.

1. Given the line $l: y = -\frac{3}{4}x + 3$.

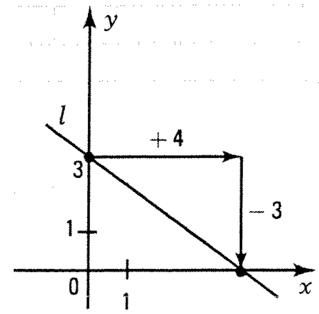
a) Determine:

1. its slope; $-\frac{3}{4}$ 2. its y-intercept. 3

b) Draw the line l in two different ways:

1. using the slope and y-intercept.
2. using a table of values.

x	y
0	3
4	0

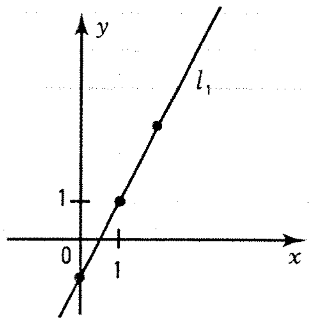


2. Draw each of the following lines.

Determine 3 points on the line to verify that they are aligned.

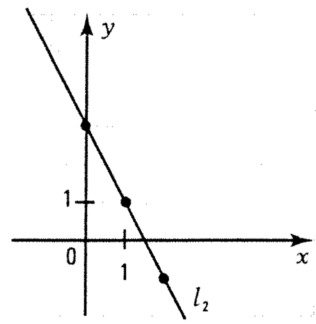
a) $l_1: y = 2x - 1$

x	y
0	-1
1	1
2	3



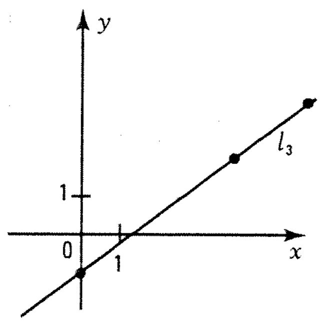
b) $l_2: y = -2x + 3$

x	y
0	3
1	1
2	-1



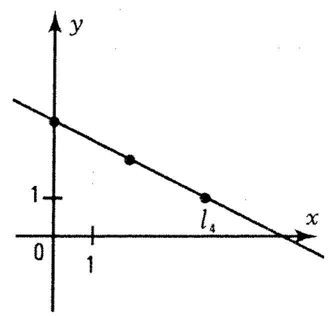
c) $l_3: y = \frac{3}{4}x - 1$

x	y
0	-1
4	2
6	4,5



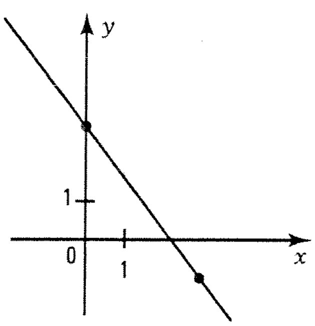
d) $l_4: y = -\frac{1}{2}x + 3$

x	y
0	3
2	2
4	1



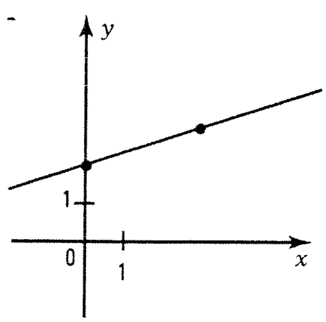
3. For each of the following lines, determine the slope, y-intercept and the functional equation.

a)



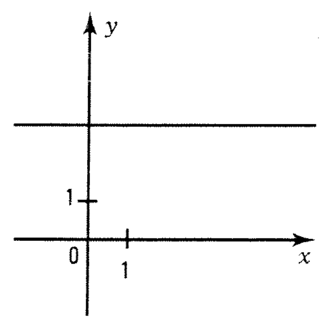
$y = -\frac{4}{3}x + 3$

b)



$y = \frac{1}{3}x + 2$

c)



$y = 0x + 3$

4. Given $l: y = -\frac{3}{4}x + 1$. Determine the slope of the line l in two different ways.

a) Using the equation of the line. $a = -\frac{3}{4}$

b) Using the slope formula with any two points on the line. $A(0, 1), B(4, -2); a = \frac{-2 - 1}{4 - 0} = -\frac{3}{4}$

ACTIVITY 2 Relative position of two lines

a) Two lines l_1 and l_2 have the same slope. What can we say about the relative position of these two lines if

1. the two lines have the same y-intercept? ***They are coincident***
2. the two lines have different y-intercepts? ***They are distinct and parallel***

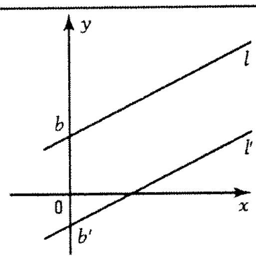
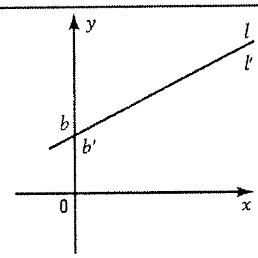
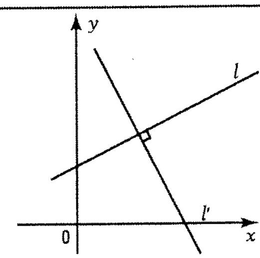
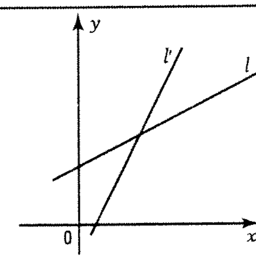
b) Two lines l_1 and l_2 do not have the same slope. What can we say about the relative position of these two lines if

1. the product of their slopes is -1? ***They are intersecting and perpendicular***
2. the product of their slopes is not -1? ***They are intersecting***

RELATIVE POSITION OF TWO LINES

Given two lines $l: y = ax + b$ and $l': y = a'x + b'$.

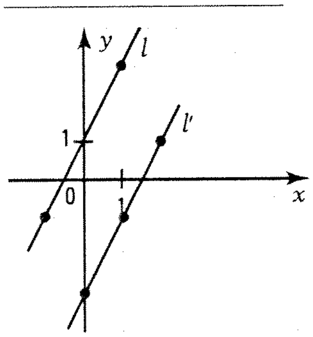
We distinguish the following cases depending on the values of the parameters a, a', b and b'.

$a = a'$ and $b \neq b'$	$a = a'$ and $b = b'$	$a \times a' = -1$	$a \neq a'$ and $a \cdot a' \neq -1$
 Distinct parallel lines	 Coincident parallel lines	 Perpendicular lines	 Non perpendicular intersecting lines

5. In each of the following cases, determine the relative position of the lines l and l' and draw them.

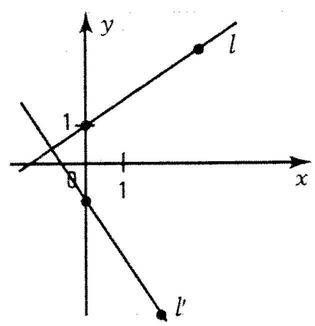
a) $l: y = 2x + 1$

$l': y = 2x - 3$

1. ***distinct parallel***
2. 

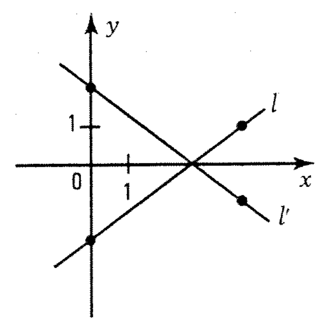
b) $l: y = \frac{2}{3}x + 1$

$l': y = -\frac{3}{2}x - 1$

1. ***perpendicular***
2. 

c) $l: y = \frac{3}{4}x - 2$

$l': y = -\frac{3}{4}x + 2$

1. ***Non perpendicular intersecting***
2. 

6. Given the line $l: y = (2k + 1)x + 1$. Find the value of parameter k if :

a) The line l passes through A(−1, −3). $k = \frac{3}{2}$

b) The line l has a slope of $\frac{2}{3}$. $k = -\frac{1}{6}$

c) The line l is parallel to $l': y = 3x - 2$. $k = 1$

d) The line l is perpendicular to $l': y = 2x - 3$. $k = -\frac{3}{4}$

5.6 General form of the equation of a line

ACTIVITY 1 From general form to functional form

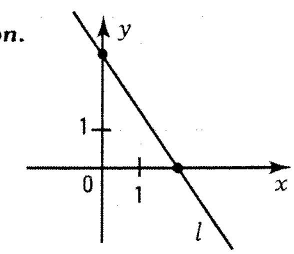
The general form of the equation of a line is: $ax + by + c = 0$.

a) Given the line l with the equation: $3x + 2y - 6 = 0$.
Justify the steps which enable you to get the functional form of the equation of line l.

$3x + 2y - 6 = 0$

$\Leftrightarrow \quad 2y = -3x + 6$ *Subtract 3x and add 6 to each side of the equation.*

$\Leftrightarrow \quad y = \frac{-3}{2}x + 3$ *Divide each side by 2.*



b) For this line l, identify

1. the slope. $\frac{-3}{2}$ 2. the y-intercept. 3

c) Draw the line l.

GENERAL FORM OF THE EQUATION OF A LINE

- The general form of the equation of a line is:

$$ax + by + c = 0$$

where a and b are not simultaneously zero.

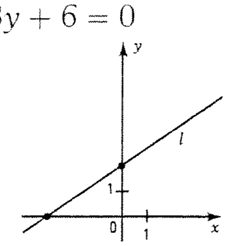
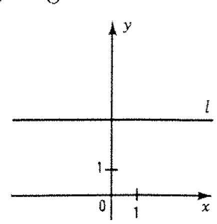
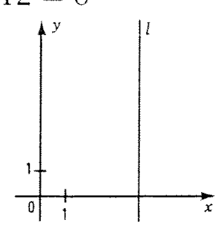
Ex.: Given the line l: $2x - 3y + 6 = 0$. (general form)
Find the functional form of l:
$2x - 3y + 6 = 0 \Leftrightarrow -3y = -2x - 6$
$\Leftrightarrow y = \frac{2}{3}x + 2$. (functional form)

The equation $ax + by + c = 0$ is called general equation of the line.

Ex.: Given the line l: $y = \frac{3}{4}x + 1$. (functional form)
Find the general form of l:
$y = \frac{3}{4}x + 1 \Leftrightarrow -\frac{3}{4}x + y - 1 = 0$ (general form)
$\Leftrightarrow -3x + 4y - 4 = 0$. (other general form)

Note that the functional form (when it exists) of a line is unique.
However, a line can have many equations in general form that are all equivalent.

- We distinguish the following cases:

$a \neq 0$ and $b \neq 0$	$a = 0$ and $b \neq 0$	$a \neq 0$ and $b = 0$
Ex.: l: $2x - 3y + 6 = 0$ The line is oblique 	Ex.: l: $2y - 6 = 0$ The line is horizontal 	Ex.: l: $3x - 12 = 0$ The line is vertical 

- The general form of a line always exists regardless of its position in the Cartesian plane.

1. Explain why each of the following equations is not an equation of a line.

a) $2x - 3y^2 + 1 = 0$ ***The y term is not first degree.***

b) $5x^2 - 2y + 1 = 0$ ***The x term is not first degree.***

c) $\frac{2}{x} - 3y + 1 = 0$ ***The x term is not first degree.*** $\left(\frac{2}{x} = 2x^{-1}\right)$.

d) $2x - 3\sqrt{y} + 1 = 0$ ***The y term is not first degree.*** $\left(\sqrt{y} = y^{\frac{1}{2}}\right)$.

2. Write each equation in general form and indicate the position of the line.

a) $2x - 6y = 3x + 2y - 5$ ***$x + 8y - 5 = 0$ oblique line***

b) $2(x - 1) + 3y = 3(y - 2) + x$ ***$x + 4 = 0$ vertical line***

c) $3(x + 1) - 2y = 3x - 3y$ ***$y + 3 = 0$ horizontal line***

d) $2(x - 3) + 3y = -6$ ***$2x + 3y = 0$ oblique line***

3. Write each of the following equations in functional form ($y = ax + b$).

a) $3x - 5y + 15 = 0$ $y = \frac{3}{5}x + 3$

b) $2x + 3y - 6 = 0$ $y = -\frac{2}{3}x + 2$

c) $2(3x - 2y) - 5(2x + 4y) = 10$ $y = -\frac{1}{6}x - \frac{5}{12}$

d) $\frac{x}{3} + \frac{y}{2} = 1$ $y = -\frac{2}{3}x + 2$

e) $\frac{y-2}{x-1} = \frac{2}{3}$ $y = \frac{2}{3}x + \frac{4}{3}$

4. Write each of the following equations in general form ($ax + by + c = 0$) where $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $c \in \mathbb{Z}$.

a) $y = 2x - 3$ $2x - y - 3 = 0$

b) $y = \frac{2}{3}x + 6$ $2x - 3y + 18 = 0$

c) $2(3x - 2y) + 5(2x + 4y) = 10$ $16x + 16y - 10 = 0$

d) $\frac{x}{5} + \frac{y}{3} = 1$ $3x + 5y - 15 = 0$

e) $\frac{y+2}{x-1} = \frac{2}{3}$ $2x - 3y - 8 = 0$

5. Given l: $2x - 5y + 10 = 0$. Determine if the following points are on l.

a) A(0, 2) ***Yes*** b) B(–5, 0) ***Yes*** c) C(5, 4) ***Yes*** d) D(10, –6) ***No*** e) E(–10, –2) ***Yes***

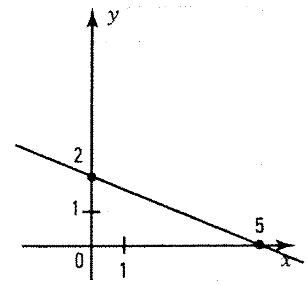
6. Consider the line with equation $3x + 5y - 2 = 0$.

Determine the slope of this line. $-\frac{3}{5}$

7. The line l: $-3x + 4y + 12 = 0$ passes through A(m, –6) and B(8, n). Determine m and n.

$m = -4$; $n = 3$

ACTIVITY 2 Finding the intercepts of a line



Given l: $2x + 5y - 10 = 0$.

a) 1. Explain how to find the x-intercept of the line l.

Replace y by 0 in the equation and deduce the value of x.

2. Find the x-intercept of the line l.

$2x - 10 = 0 \Rightarrow x = 5$. The x-intercept is 5.

b) 1. Explain how to find the y-intercept of the line l.

Replace x by 0 in the equation and deduce the value of y.

2. Find the y-intercept of the line l.

$5y - 10 = 0 \Rightarrow y = 2$. The y-intercept is 2.

c) Use the intercepts of line l to draw the line.

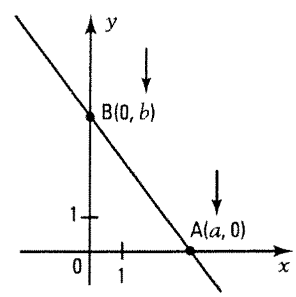
CALCULATING THE INTERCEPTS OF A LINE

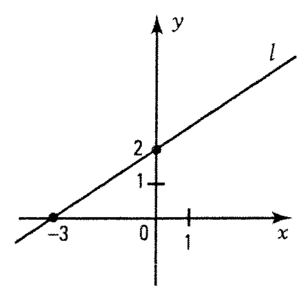
Given the equation of a line:

- To determine the x-intercept of the line, replace y by 0 in the equation and deduce the value of x.
- To determine the y-intercept of the line, replace x by 0 in the equation and deduce the value of y.

Ex.: l: $2x - 3y + 6 = 0$

Calculating the x-intercept	Calculating the y-intercept
$2x - 3y + 6 = 0$	$2x - 3y + 6 = 0$
$y = 0 \Rightarrow 2x + 6 = 0$	$x = 0 \Rightarrow -3y + 6 = 0$
$2x = -6$	$-3y = -6$
$x = -3$	$y = 2$
The x-intercept is –3.	The y-intercept is 2.





8. Find, if they exist, the intercepts of each of the following lines.

a) $4x - 5y + 20 = 0$ ***a = –5, b = 4***

b) $y = -\frac{2}{3}x + 6$ ***a = 9, b = 6***

c) $\frac{x}{6} + \frac{y}{2} = 1$ ***a = 6, b = 2***

d) $2x - 10 = 0$ ***a = 5, b does not exist***

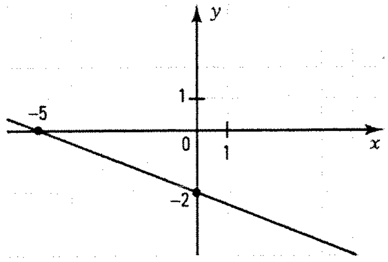
e) $3y + 12 = 0$ ***a does not exist, b = –4***

f) $x - y = 0$ ***a = 0, b = 0***

9. Find the intercepts of the line and then draw the line.

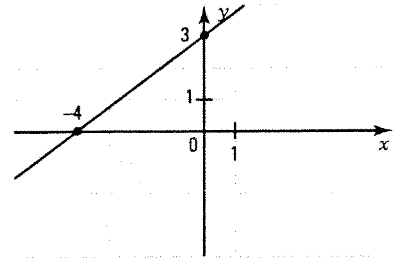
a) $l_1: 2x + 5y + 10 = 0$

x	y
-5	0
0	**-2**



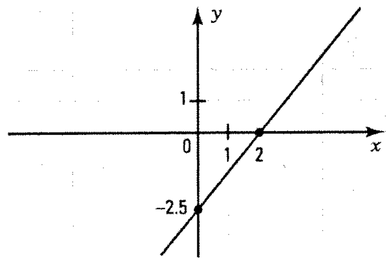
b) $l_2: -3x + 4y - 12 = 0$

x	y
-4	0
0	**3**



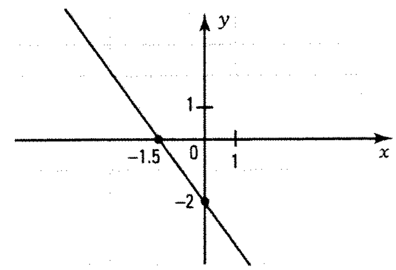
c) $l_3: -5x + 4y + 10 = 0$

x	y
2	0
0	**-2.5**



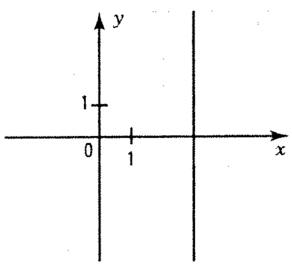
d) $l_4: 4x + 3y + 6 = 0$

x	y
-1.5	0
0	**-2**

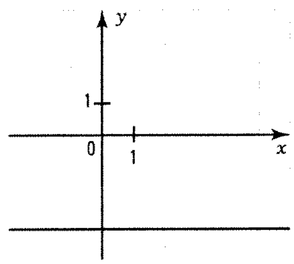


10. Draw the following lines given their equations.

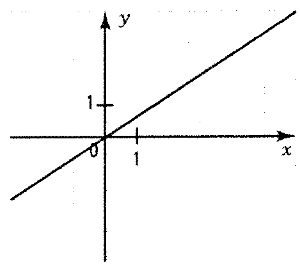
a) $2x - 6 = 0$.



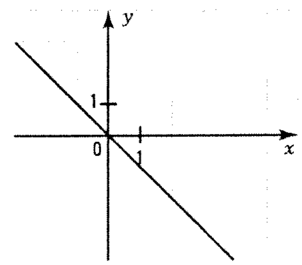
b) $3y + 9 = 0$.



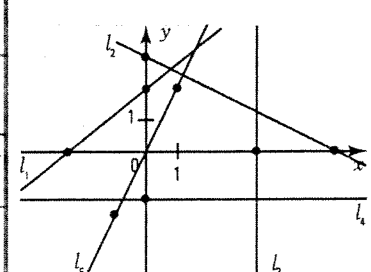
c) $2x - 3y = 0$.



d) $x + y = 0$.

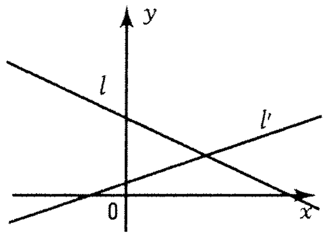
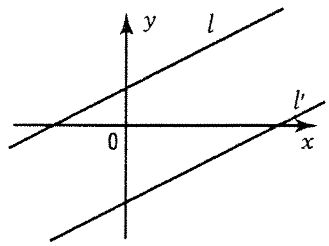
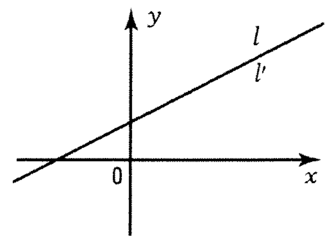


11. Complete the following table.

General form	Functional form	Slope	y intercept	x intercept	Graph
$l_1: 4x - 5y + 10 = 0$	$y = \frac{4}{5}x + 2$	$\frac{4}{5}$	$\frac{-5}{2}$	**2**	
$l_2: x + 2y - 6 = 0$	$y = -\frac{x}{2} + 3$	$-\frac{1}{2}$	**6**	**3**	
$l_3: 2x - 7 = 0$	**does not exist**	**does not exist**	$\frac{7}{2}$	**does not exist**	
$l_4: 2y + 3 = 0$	$y = -\frac{3}{2}$	**0**	**does not exist**	$-\frac{3}{2}$	
$l_5: 2x - y = 0$	$y = 2x$	**2**	**0**	**0**	

RELATIVE POSITION OF TWO LINES

- Given two lines l: $ax + by + c = 0$ and l': $a'x + b'y + c' = 0$.
  The relative position of the lines l and l' is determined by comparing the ratios $\frac{a}{a'}$, $\frac{b}{b'}$ and $\frac{c}{c'}$.

$\frac{a}{a'} \neq \frac{b}{b'}$	$\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$	$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$
 intersecting lines	 distinct parallel lines	 coincident parallel lines

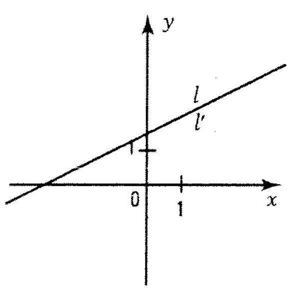
12. In each of the following cases:
1. indicate the relative position of lines l and l' and justify your answer.
2. draw the lines l and l'.

a) l: $2x - 4y + 6 = 0$
l': $3x - 6y + 9 = 0$

They are coincident

1. $\frac{2}{3} = \frac{-4}{-6} = \frac{6}{9}$

2.

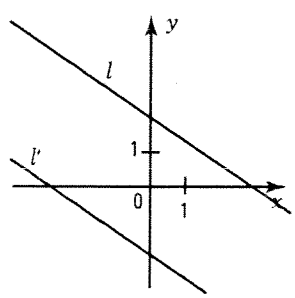


b) l: $2x + 3y - 6 = 0$
l': $3x + 4.5y + 9 = 0$

They are distinct parallel

1. $\frac{2}{3} = \frac{3}{4.5} \neq \frac{-6}{9}$

2.

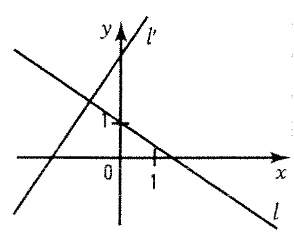


c) l: $2x + 3y - 3 = 0$
l': $3x - 2y + 6 = 0$

They are intersecting

1. $\frac{2}{3} \neq \frac{3}{-2}$

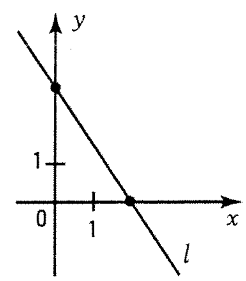
2.



5.7 Symmetrical form of the equation of a line

Activity 1 Symmetrical form

The symmetrical form of the equation of a line is: $\frac{x}{a} + \frac{y}{b} = 1$ where a and b represent the x and y-intercepts respectively.



a) Determine the symmetrical form of the equation of the line l on the right.

$\frac{x}{2} + \frac{y}{3} = 1$

b) 1. Find the slope of the line l. ***$-\frac{3}{2}$***

2. Find the functional equation of the line l. ***$y = -\frac{3}{2}x + 3$***

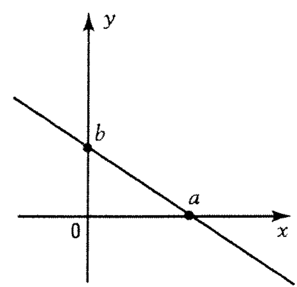
SYMMETRICAL FORM OF THE EQUATION OF A LINE

The symmetrical form of the equation of an oblique line not passing through the origin is:

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

This equation is called symmetrical equation of the line.

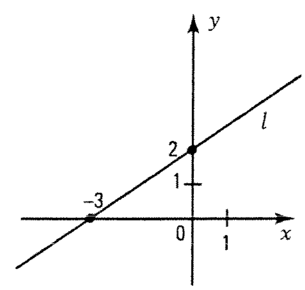
where a is the x-intercept and b is the y-intercept.



Ex.: The line l on the right has the symmetrical equation: $\frac{x}{-3} + \frac{y}{2} = 1$.

The x-intercept of line l is –3.

The y-intercept of line l is 2.



1. Determine the intercepts of each of the following lines.

a) $\frac{x}{2} + \frac{y}{-3} = 1$ ***a = 2, b = -3***

b) $\frac{x}{5} - \frac{y}{2} = 1$ ***a = 5, b = -2***

c) $\frac{-x}{4} + \frac{y}{2} = 1$ ***a = -4, b = 2***

d) $\frac{2x}{5} + \frac{3y}{2} = 1$ ***$a = \frac{5}{2}$, $b = \frac{2}{3}$***

2. Justify the steps which enable you to write each equation in symmetrical form.

a) $2x + 5y + 10 = 0 \Leftrightarrow 2x + 5y = -10$ ***We subtract 10 from each side.***

$\Leftrightarrow \frac{2x}{-10} + \frac{5y}{-10} = 1$ ***We divide each side by –10 to get 1 on the right side of the equation.***

$\Leftrightarrow \frac{x}{-5} + \frac{y}{-2} = 1$ ***We get the symmetrical form.***

b) $y = \frac{3}{4}x + 2 \Leftrightarrow -\frac{3}{4}x + y = 2$ ***We subtract $\frac{3}{4}x$ from each side.***

$\Leftrightarrow -\frac{3x}{8} + \frac{y}{2} = 1$ ***We divide each side by 2 to get 1 on the right side of the equation.***

$\Leftrightarrow \frac{x}{-\frac{8}{3}} + \frac{y}{2} = 1$ ***We get the symmetrical form.***

3. Given the line l with the symmetrical equation $\frac{x}{2} + \frac{y}{3} = 1$.

a) Justify the steps which enable you to deduce the functional equation of line l.

$\frac{x}{2} + \frac{y}{3} = 1 \Leftrightarrow \frac{y}{3} = -\frac{x}{2} + 1$ ***We subtract $\frac{x}{2}$ from each side.***

$\Leftrightarrow y = -\frac{3}{2}x + 3$ ***We multiply each side by 3.***

b) Justify the steps which enable you to deduce a general form equation of line l.

$\frac{x}{2} + \frac{y}{3} = 1 \Leftrightarrow \frac{x}{2} + \frac{y}{3} - 1 = 0 \quad (1)$ ***We subtract 1 from each side.***

$\Leftrightarrow 3x + 2y - 6 = 0 \ (2)$ ***We multiply each side by 6.***

c) The two equations (1) and (2) of the line l obtained in b) are equivalent.

Which one is easier to work with? ***$3x + 2y - 6 = 0$***

4. Explain why you cannot determine the symmetrical form of a line's equation that is

a) vertical; ***The y-intercept b does not exist.***

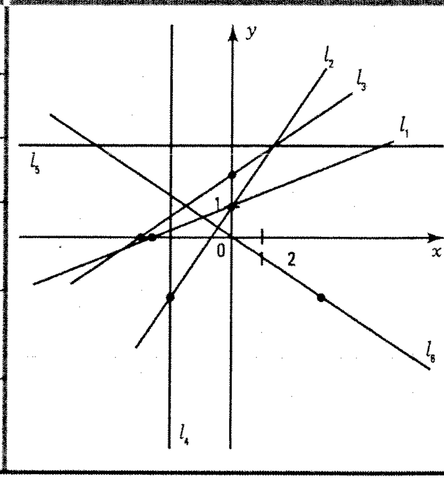
b) horizontal; ***The x-intercept a does not exist.***

c) passing through the origin. ***The intercepts a and b are equal to zero. Division by 0 is impossible.***

5. Complete the following table. Justify your answer when a form does not exist.

	General form $ax + by + c = 0$	Functional form $y = ax + b$	Symmetric form $\frac{x}{a} + \frac{y}{b} = 1$
l_1	$2x - 5y + 5 = 0$	***$y = \frac{2}{5}x + 1$***	***$\frac{x}{-\frac{5}{2}} + \frac{y}{1} = 1$***
l_2	***$3x - 2y + 2 = 0$***	$y = \frac{3}{2}x + 1$	***$\frac{x}{\frac{-2}{3}} + \frac{y}{1} = 1$***
l_3	***$-2x + 3y - 6 = 0$***	***$y = \frac{2}{3}x + 2$***	$\frac{x}{-3} + \frac{y}{2} = 1$
l_4	$3x + 6 = 0$	***No, vertical line***	***No, vertical line***
l_5	$2y - 6 = 0$	***$y = 3$***	***No, horizontal line***
l_6	$2x + 3y = 0$	***$y = -\frac{2}{3}x$***	***No, line passes through origin***

6. Use the preceding table to complete the following table. Justify your answer if a line's characteristic does not exist.

	Slope	x-intercept	y-intercept	Cartesian graph
l_1	$\frac{2}{5}$	$-\frac{5}{2}$	1	
l_2	$\frac{3}{2}$	$-\frac{2}{3}$	1	
l_3	$\frac{2}{3}$	-3	2	
l_4	No, vertical line	-2	No, vertical line	
l_5	0	No, horizontal line	3	
l_6	$-\frac{2}{3}$	0	0	

7. Consider the line with the equation $\frac{x}{7} + \frac{y}{3} = 1$.

What is the slope of this line? $\frac{-3}{7}$

8. A line l has an x-intercept of 2 and a y-intercept of 3.

a) What is the symmetrical equation of l? $\frac{x}{2} + \frac{y}{3} = 1$

b) Write the equation of line l in general form. $3x + 2y - 6 = 0$

c) Write the equation of line l in functional form. $y = -\frac{3}{2}x + 3$

5.8 Finding the equation of a line

ACTIVITY 1 Finding the equation of a line

a) A line has a slope of $\frac{2}{3}$ and a y-intercept of 2.

1. Find its functional equation. $y = \frac{3}{2}x + 2$
2. Find its general equation. $2x - 3y + 6 = 0$

b) A line has a slope of 2 and passes through the point A(2, 5).

1. Find its functional equation. $y = 2x + 1$
2. Find its general equation. $2x - y + 1 = 0$

c) A line passes through the points A(2, 4) and B(4, 7).

1. Find the slope of this line. $\frac{3}{2}$
2. Find its y-intercept. 1
3. Find its functional equation. $y = \frac{3}{2}x + 1$
4. Find its general equation. $3x - 2y + 2 = 0$

FINDING THE EQUATION OF A NON VERTICAL LINE

Situation	Method	Example
[1] The slope a and y-intercept b are known.	Replace a and b by their respective values in the equation $y = ax + b$.	The line l has a slope of $\frac{2}{3}$ and a y-intercept of -1. Its equation is $l: y = \frac{2}{3}x - 1$.
[2] The slope a and a point (h, k) on the line are known.	1. Calculate the y-intercept b knowing that the point (h, k) verifies the equation: $y = ax + b$. 2. Proceed as in situation [1].	Find the equation of the line l with a slope of 2 and passing through the point A(3, 5). 1. $A(3, 5) \in l: y = 2x + b$ $5 = 2(3) + b$ $b = -1$ 2. Deduce $l: y = 2x - 1$.
[3] Two points on the line are known.	1. Calculate the slope a. 2. Proceed as in situation [2].	Find the equation of the line passing through A(1, 3) and B(4, −6). 1. $a = \frac{y_B - y_A}{x_B - x_A} = \frac{-6 - 3}{4 - 1} = -3$ 2. $A(1, 3) \in l: y = -3x + b$ $3 = -3(1) + b$ $b = 6$. Hence, $l: y = -3x + 6$.

1. Line l has a slope of –2 and a y-intercept of 5. Find the equation of line l in the following forms:

a) functional; $y = -2x + 5$ b) general; $2x + y - 5 = 0$ c) symmetric. $\frac{x}{\frac{5}{2}} + \frac{y}{5} = 1$

2. Line l has a slope of $\frac{3}{2}$ and passes through the point A(4, 1). Find the equation of line l in the following forms:

a) functional; $y = \frac{3}{2}x - 5$ b) general; $3x - 2y - 10 = 0$ c) symmetric. $\frac{x}{\frac{10}{3}} + \frac{y}{-5} = 1$

3. Line l passes through the points A(1, 1) and B(3, –3). Find the equation of line l in the following forms:

a) functional; $y = -2x + 3$ b) general; $2x + y - 3 = 0$ c) symmetric. $\frac{x}{\frac{3}{2}} + \frac{y}{3} = 1$

4. Line l has an x-intercept of 2 and a y-intercept of 3. Find the three equation forms of l.

$\frac{x}{2} + \frac{y}{3} = 1$ (symmetric form); $3x + 2y - 6 = 0$ (general form); $y = -\frac{3}{2}x + 3$ (function form)

5. A line passes through the origin and the point A(3, 4). Find the equation of line l in:

a) functional form; $y = \frac{4}{3}x$ b) general form. $4x - 3y = 0$

6. Find the general form equation of the

a) vertical line passing through A(2, 1). $x - 2 = 0$

b) horizontal line passing through B(2, –3). $y + 3 = 0$

7. Find the equation of the line with slope a and passing through the point P.

a) $a = -2$ and P(3, –1) $y = -2x + 5$ b) $a = -\frac{3}{4}$ and P(8, –5) $y = -\frac{3}{4}x + 1$

c) $a = 0$ and P(–2, 5) $y = 5$ d) a does not exist and P(–2, 3) $x = -2$

8. Find the equation of the line passing through A and B.

a) A(4, –2) and B(–4, 4) $y = -\frac{3}{4}x + 1$ b) A(–1, 8) and B(2, –7) $y = -5x + 3$

c) A(2, 1) and B(2, –1) $x = 2$ d) A(2, –3) and B(5, –3) $y = -3$

9. Find the equation of the line with x-intercept a and y-intercept b.

a) $a = -2$ and $b = 3$ $\frac{x}{-2} + \frac{y}{3} = 1$ b) $a = -\frac{3}{4}$ and $b = \frac{2}{3}$ $-\frac{4}{3}x + \frac{3y}{2} = 1$

c) a does not exist and $b = 1$ $y = 1$ d) $a = -2$ and b does not exist $x = -2$

10. Find the equation of the line l passing through A(3, 1) and parallel to the line l': $2x - 3y + 6 = 0$.

l: $y = \frac{2}{3}x - 1$

11. Find the equation of the line passing through the point A(–2, 12) and parallel to $\frac{x}{3} + \frac{y}{9} = 1$.

$y = -3x + 6$

12. Find the equation, in general form, of the line l passing through A(6, –5) and perpendicular to the line l': $3x - 4y + 8 = 0$.

l: $4x + 3y - 9 = 0$

13. Find the equation of the line passing through the point A(9, –5) and perpendicular to $\frac{x}{-8} + \frac{y}{6} = 1$.

l: $y = -\frac{4}{3}x + 7$

14. Given A(2, 5), B(–1, 4) and C(3, 2) the vertices of triangle ABC. Find the equation of the line AM if the segment AM is the median from vertex A.

$y = 2x + 1$

15. Given A(1, 5), B(–2, 1) and C(6, 3) the vertices of triangle ABC. Find the equation of the line AH if the segment AH is the altitude from vertex A.

$y = -4x + 9$

16. Find the equation of the line l that is the perpendicular bisector of the segment with end-points A(–1, 3) and B(5, –1).

$y = \frac{3}{2}x - 2$

17. Answer true or false.

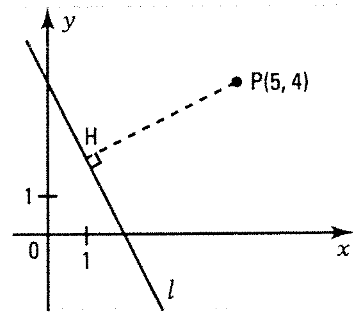
a) The functional form of the equation of a line does not exist when the line is vertical. ***True***

b) The symmetrical form of the equation of a line only exists if it is an oblique line not passing through the origin. ***True***

c) The general form of the equation of a line exists regardless of the line's position in the Cartesian plane. ***True***

5.9 Distance from a point to a line

ACTIVITY 1 Calculating the distance from a point to a line



Consider the line l on the right with the equation $y = -2x + 4$ and the point P(5, 4).

a) Using a straight edge, locate the point H on line l such that PH is perpendicular to the line l.

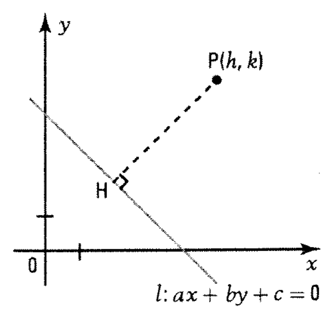
b) 1. Find the slope of the line PH. $\frac{1}{2}$

2. Find the equation of the line PH. $y = \frac{1}{2}x + \frac{3}{2}$

3. Show that the coordinates of the point H(1, 2) verify simultaneously the equations of the lines l and PH. $2 = -2(1) + 4;\ 2 = \frac{1}{2}(1) + \frac{3}{2}$

c) The length of segment PH is the distance from point P to the line l. Calculate this distance.
$\sqrt{20}$ or $2\sqrt{5}$

DISTANCE FROM A POINT TO A LINE



- The distance from a point P to a line is the length of the segment PH where H is the orthogonal projection of point P on the line.
- The distance from a point P(h, k) to a line l: $ax + by + c = 0$ is:

$$d(\text{P}, l) = \frac{|ah + bk + c|}{\sqrt{a^2 + b^2}}.$$

Ex.: The distance from point P(2, –1) to the line l: $3x + 4y - 12 = 0$ is:

$$d(\text{P}, l) = \frac{|3(2) + 4(-1) - 12|}{\sqrt{3^2 + 4^2}} = \frac{|-10|}{5} = \frac{10}{5} = 2.$$

1. In each of the following cases, calculate the distance from the point P to the line l.

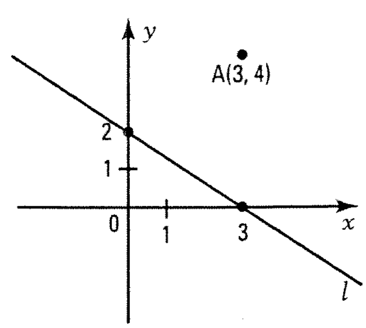
a) A(–2, 1) and l: $3x - 2y + 1 = 0$ $\frac{7\sqrt{13}}{13}$

b) A(4, 2) and l: $3x - 4y - 2 = 0$ 0.4

c) A(2, –1) and l: $y = 2x + 1$ $\frac{6\sqrt{5}}{5}$

d) A(1, 2) and l: $y = x$ $\frac{\sqrt{2}}{2}$

2. Refer to the figure on the right to calculate:



a) the distance from point A to the line l;
$\frac{12}{\sqrt{13}}$

b) the distance from the origin 0 to the line l.
$\frac{6}{\sqrt{13}}$

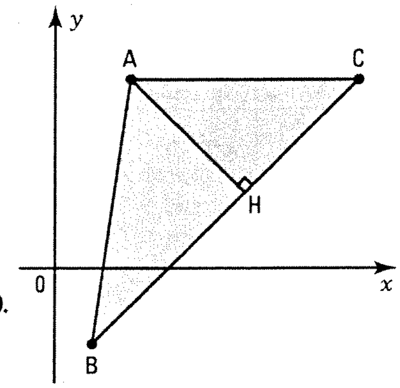
3. Given A(5, 7), B(−2, 3) and C(8, −2), the vertices of triangle ABC. Calculate the length of the altitude AH from vertex A.

BC: $x + 2y - 4 = 0$. $m\overline{AH} = 3\sqrt{5}$

4. Calculate the distance between the parallel lines l_1 and l_2 if l_1: $2x + 3y - 6 = 0$ and l_2: $4x + 6y + 3 = 0$.

$\frac{15}{2\sqrt{13}}$

5. Given A(2, 5), B(1, −2) and C(8, 5), the vertices of triangle ABC.



a) Calculate the perimeter of triangle ABC.

$m\overline{AC} = 6$; $m\overline{BC} = 7\sqrt{2}$; $m\overline{AB} = 5\sqrt{2}$

Perimeter $\triangle ABC = 6 + 12\sqrt{2}$ units

b) Explain how you would calculate the area of triangle ABC.

– Calculate the length of one of the bases ($m\overline{BC}$ for example).

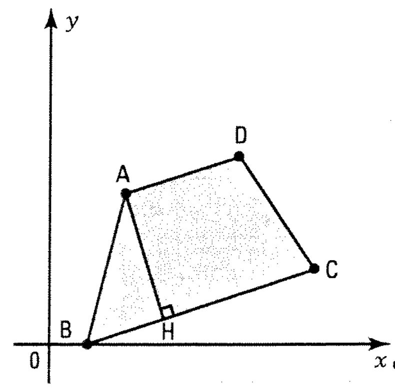
– Calculate the length of the altitude relative to that base ($m\overline{AH}$).

– Deduce the area of the triangle using the formula

$A = \frac{1}{2} m\overline{BC} \times m\overline{AH}$.

c) Calculate the area of triangle ABC. ***$m\overline{BC} = 7\sqrt{2}$; BC: $x - y - 3 = 0$; $m\overline{AH} = 3\sqrt{2}$; Area $\triangle ABC = 21\ u^2$***

6. A quadrilateral ABCD has vertices A(2, 4), B(1, 0), C(7, 2) and D(5, 5).



a) What type of quadrilateral is ABCD? Justify your answer.

A trapezoid, since AD // BC.

In fact, $a_{\overline{AD}} = a_{\overline{BC}} = \frac{1}{3}$.

b) Calculate the perimeter of quadrilateral ABCD.

$m\overline{AB} = \sqrt{17}$, $m\overline{BC} = 2\sqrt{10}$, $m\overline{CD} = \sqrt{13}$, $m\overline{AD} = \sqrt{10}$

Perimetrer ABCD = $\sqrt{17} + 3\sqrt{10} + \sqrt{13}$ units

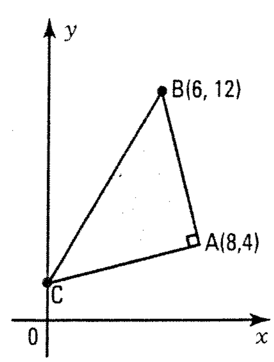
c) Calculate the area of quadrilateral ABCD.

BC: $x - 3y - 1 = 0$; $m\overline{AH} = \frac{11}{\sqrt{10}}$; $m\overline{BC} = 2\sqrt{10}$; $m\overline{AD} = \sqrt{10}$

Area ABCD = 16.5 u^2

7. What is, to the nearest hundredth, the length of the hypotenuse BC in triangle ABC on the right?

$a_{\overline{AB}} = -4$; AC: $y = \frac{1}{4}x + 2$; C(0,2); $m\overline{BC} = 2\sqrt{34}$



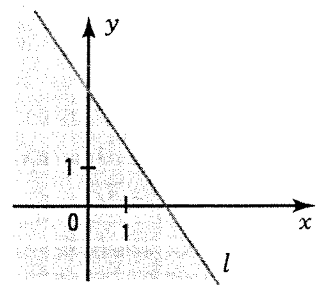
5.10 Regions of the Cartesian plane

ACTIVITY 1 Two-variable inequalities

a) The line $l: 3x + 2y - 6 = 0$ is represented on the right. This line separates the Cartesian plane in two half-planes: the purple half-plane containing the origin 0 and the grey half-plane not containing the origin.

In which half-plane are the points (x, y) of the plane that verify the following inequalities?

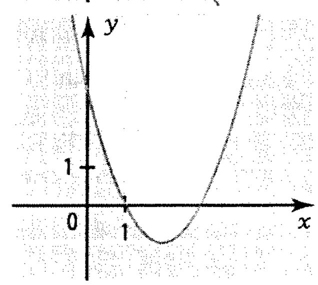
1. $3x + 2y - 6 \leqslant 0$ *purple half-plane* 2. $3x + 2y - 6 \geqslant 0$ *grey half-plane*



b) The parabola $y = x^2 - 4x + 3$ is represented on the right. This parabola separates the Cartesian plane in two regions: the purple region containing the origin 0 and the grey region not containing the origin.

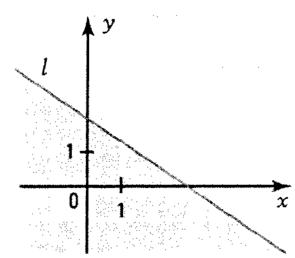
In which region are the points (x, y) of the plane that verify the following inequalities?

1. $y \leqslant x^2 - 4x + 3$ *purple region* 2. $y \geqslant x^2 - 4x + 3$ *grey region*

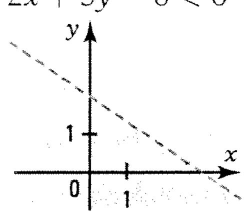


TWO-VARIABLE INEQUALITIES – REPRESENTING THE SOLUTION SET

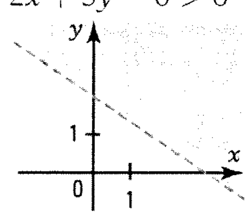
- The solution set of the first degree two-variable inequality $ax + by + c > 0$ is represented by a half-plane whose border is the line $l: ax + by + c = 0$.
  Ex.: To determine the solution set of the inequality $2x + 3y - 6 \leqslant 0$,
  1. We draw the line $l: 2x + 3y - 6 = 0$ which is the border of the desired half-plane.
  2. If the origin O(0, 0) verifies the inequality, the solution is the half-plane containing the origin.
     The border is drawn as a solid line to indicate that the points on the border are solutions, and dotted if they are not.
  3. We shade the half-plane corresponding to the solution set.



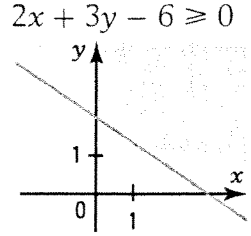
$2x + 3y - 6 < 0$



$2x + 3y - 6 > 0$

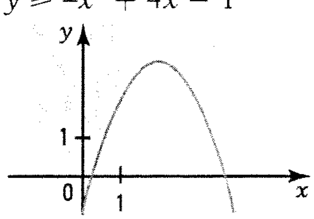


$2x + 3y - 6 \geqslant 0$

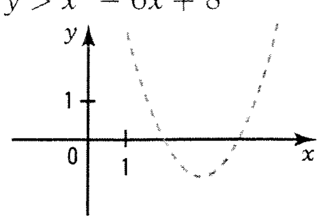


- The solution set to the inequality $y > ax^2 + bx + c$ is a region of the Cartesian plane with the parabola $y = ax^2 + bx + c$ as the border.

Ex.: $y \geqslant -x^2 + 4x - 1$

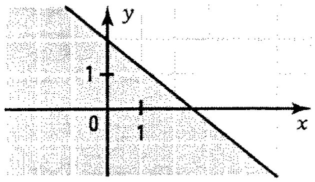


Ex.: $y > x^2 - 6x + 8$

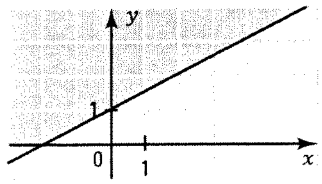


1. Represent the solution set of the following inequalities.

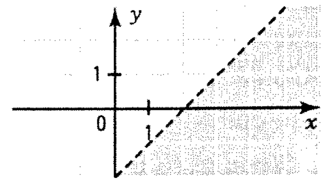
a) $4x + 5y - 10 \leqslant 0$



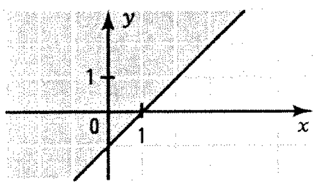
b) $-x + 2y \geqslant 2$



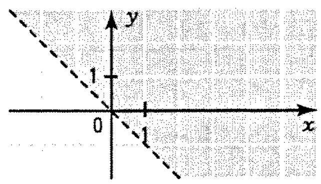
c) $y < x - 2$



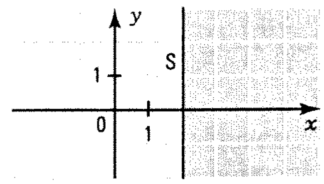
d) $y \geqslant x - 1$



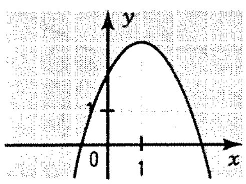
e) $x + y > 0$



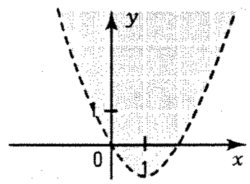
f) $x \geqslant 2$



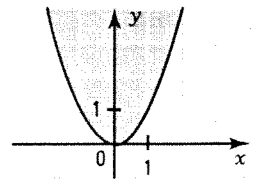
g) $y \geqslant -x^2 + 2x + 2$



h) $y > x^2 - 2x$



i) $y \geqslant x^2$

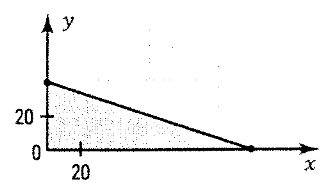


2. For each of the following situations:
1. define the variables used in the situation;
2. translate the situation using an inequality;
3. represent the situation in the Cartesian plane and shade the solution region.

a) A parking lot has a surface area of 700 m^2. Each car occupies an area of 6 m^2 and each bus occupies an area of 18 m^2.

***x*: number of cars** ***y*: number of buses**

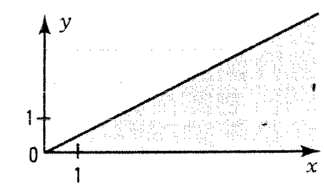
$6x + 18y \leqslant 720$



b) On an outing organized by Scouts, there are at least twice as many boys as girls.

***x*: number of boys** ***y*: number of girls**

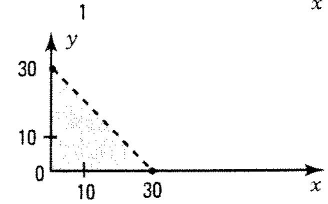
$x \geqslant 2y$



c) The perimeter of a rectangular field is less than 60 m.

***x*: length** ***y*: width**

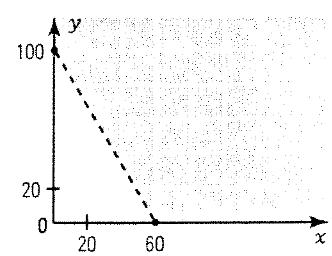
$x + y < 30$



d) Tickets to a concert cost \$10 per adult and \$6 per child. The total revenue from ticket sales is greater than \$600.

***x*: number of adults** ***y*: number of children**

$10x + 6y > 600$



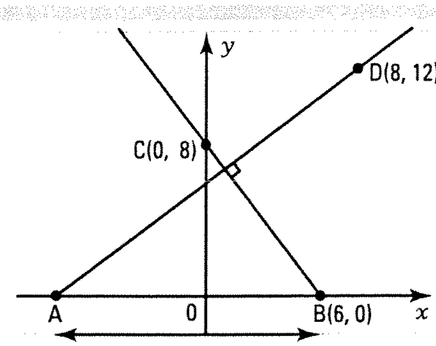
5.11 Analytic geometry problems

1. Consider the figure on the right.
Calculate the distance between A and B.

Slope of BC: $-\frac{4}{3}$***; slope of AP*** $= \frac{3}{4}$

Equation of the line AP: $y = \frac{3}{4}x + 6$

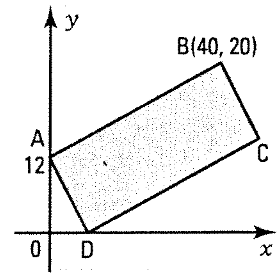
x-intercept of the line AP: -8 ; d(A, B) = 14.



2. Consider the rectangle on the right. Find the area of this rectangle to the nearest square unit.

$m\overline{AB} = \sqrt{1664}$***; AD:*** $y = -5x + 12$

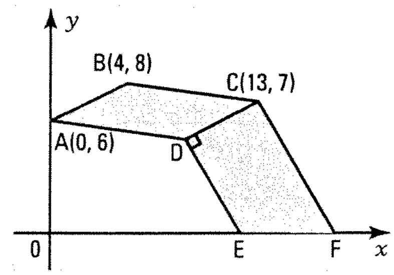
$D\left(\frac{12}{5}, 0\right)$***;*** $m\overline{AD} = \frac{\sqrt{3744}}{5}$***; Area = 499 u²***



3. Consider the parallelogram ABCD and the right trapezoid CDEF represented on the right. Determine the x-coordinate of point F.

$a_{\overline{AB}} = \frac{1}{2}$; $a_{\overline{CF}} = -2$

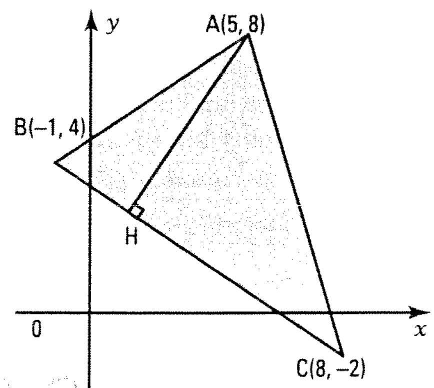
CF: $y = -2x + 33$***; x-coordinate of point F: 16.5***



4. Consider the figure on the right. Determine the length of segment HC if $\overline{AH}$ is an altitude of triangle ABC. (Round your answer to the nearest unit.)

BC: $y = -\frac{2}{3}x + \frac{10}{3}$

$m\overline{AH} = \frac{24}{\sqrt{13}}$; $m\overline{AC} = \sqrt{109}$; $m\overline{AH} = 8$ ***u***



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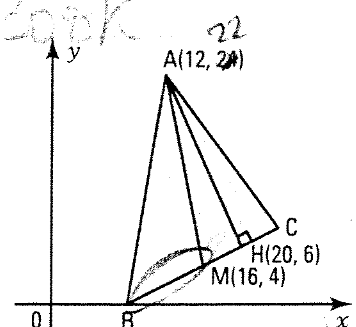
5. In the triangle ABC on the right, the altitude AH and the median AM are drawn. Calculate the area of triangle ABC to the nearest tenth.

MH: $y = \frac{1}{2}x - 4$***; B(8, 0); C(24, 8);*** $m\overline{BC} = \sqrt{320}$

$m\overline{AH} = \sqrt{388}$***; Area = 176.2 u²***



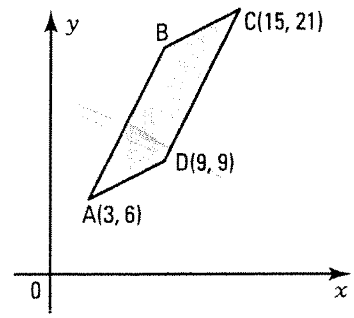
6. Find the area of the parallelogram ABCD represented on the right.

CD: $2x - y - 9 = 0$

Height of the parallelogram = $\frac{9}{\sqrt{5}}$

Base CD = $\sqrt{180}$

Area = 54 u^2

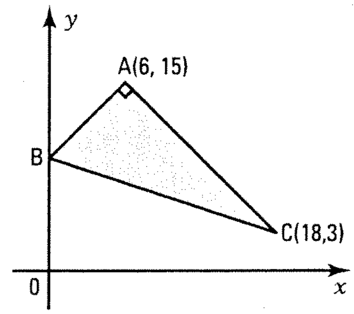


7. Calculate the area of the right triangle ABC.

AB: $y = x + 9 \Rightarrow B(0,9)$

$m\overline{AB} = \sqrt{72}$; $m\overline{AC} = \sqrt{288}$

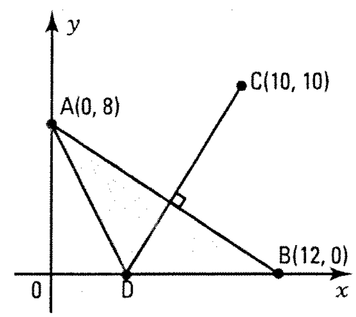
Area $\triangle ABC$ = 72 u^2



8. The segments AB and CD on the right are perpendicular. Calculate the area of triangle ABD.

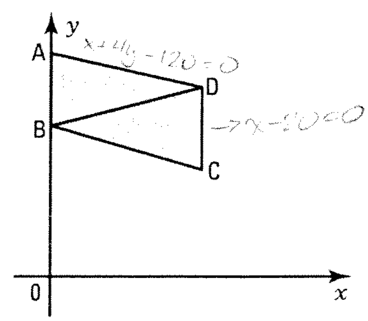
CD: $y = \frac{3}{2}x - 5$; D(4, 0); d(B, D) = 8 u

Area of triangle ABD = 32 u^2



9. Consider the parallelogram ABCD on the right. The lines AD and CD have the equation $x + 4y - 120 = 0$ and $x - 20 = 0$ respectively. What is the length of diagonal BD if the side CD measures 10 units?

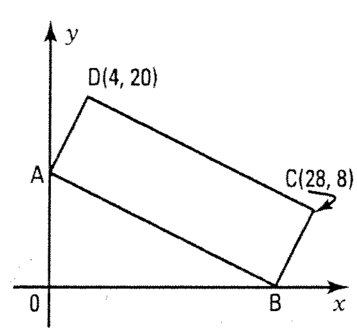
D(20, 25); A(0, 30); B(0, 20); d(B, D) = $5\sqrt{17}$



10. Calculate the area of the rectangle ABCD on the right.

AD: $y = 2x + 12$; A(0, 12); $m\overline{AD} = \sqrt{80}$

$m\overline{CD} = \sqrt{720}$; Area = 240 u^2



Evaluation 5

1. Given two points A(–2, 1) and B(4, –3), determine

a) the distance between A and B. ***$\sqrt{52}$***

b) the coordinates of point M, mid-point of segment AB. ***M(1, –1)***

c) the slope of the line AB. ***$\frac{-2}{3}$***

d) the equation of the line AB in

1. functional form; ***$y = -\frac{2}{3}x - \frac{1}{3}$***
2. general form. ***$2x + 3y + 1 = 0$***

2. What kind of triangle is ABC with vertices A(1, 2); B(2, 4) and C(3, 1)? Justify your answer.

$m\overline{AB} = \sqrt{5}$; $m\overline{AC} = \sqrt{5}$; $m\overline{BC} = \sqrt{10}$

$(m\overline{BC})^2 = (m\overline{AB})^2 + (m\overline{AC})^2$. △ABC is an isosceles right triangle.

3. What kind of quadrilateral is ABCD with vertices A(–4, 6), B(2, 12), C(12, 2) and D(6, –4)? Justify your answer.

Slope of AB = slope of CD = 1; slope of BC = slope of AD = –1.

The opposite sides of the quadrilateral are parallel (same slope).

The non-parallel sides are perpendicular (product of slopes = –1).

The quadrilateral ABCD is therefore a rectangle.

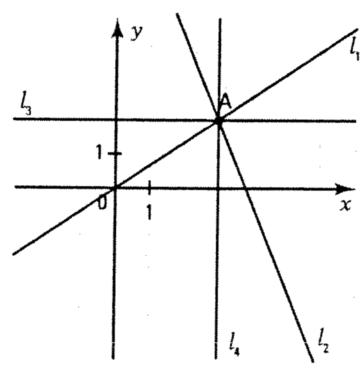
4. Using the point A(3, 2)

a) draw the line l_1 passing through A with a slope of $\frac{2}{3}$.

b) draw the line l_2 passing through A with a slope of $-\frac{5}{2}$.

c) draw the line l_3 passing through A with a slope of zero.

d) draw the line l_4 passing through A with an undefined slope.



5. Draw the following lines in the Cartesian plane on the right.

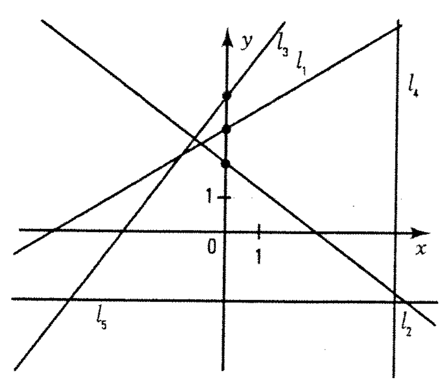
l_1: $-3x + 5y - 15 = 0$.

l_2: $y = \frac{-3}{4}x + 2$.

l_3: $\frac{x}{-3} + \frac{y}{4} = 1$.

l_4: $2x - 10 = 0$.

l_5: $3y + 6 = 0$.



6. For each of the lines in exercise n° 5, complete the following table:

	slope	x-intercept	y-intercept
l_1	$\frac{3}{5}$	-5	3
l_2	$\frac{-3}{4}$	$\frac{8}{3}$	2
l_3	$\frac{4}{3}$	-3	4
l_4	does not exist	5	does not exist
l_5	0	does not exist	-2

7. Complete the following table. (Justify your answer if a form does not exist.)

General form	Functional form	Symmetric form
$2x + 5y - 10 = 0$	$y = -\frac{2}{5}x + 2$	$\frac{x}{5} + \frac{y}{2} = 1$
$2x - 3y + 3 = 0$	$y = \frac{2}{3}x + 1$	$\frac{x}{\frac{-3}{2}} + \frac{y}{1} = 1$
$2x - 5y - 10 = 0$	$y = \frac{2}{5}x - 2$	$\frac{x}{5} + \frac{y}{-2} = 1$
$2x + 3 = 0$	does not exist (vertical line)	does not exist (vertical line)
$3y - 12 = 0$	$y = 4$	does not exist (horizontal line)
$-2x + y = 0$	$y = 2x$	does not exist (line passes through the origin)

8. The line l has the equation $2x - 3y + 12 = 0$. Determine

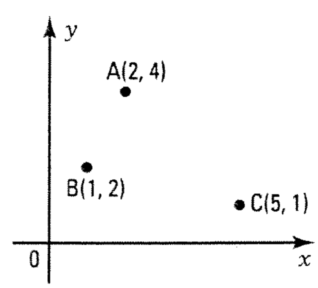
a) the x-intercept. -6

b) the y-intercept. 4

c) the equation in symmetric form. $\frac{x}{-6} + \frac{y}{4} = 1$

d) the equation in functional form. $y = \frac{2}{3}x + 4$

9. Find the general equation of



a) the line passing through A and C. $x + y - 6 = 0$

b) the line passing through B and parallel to the line AC. $x + y - 3 = 0$

c) the line passing through B and perpendicular to the line AC. $x - y + 1 = 0$

d) the vertical line passing through C. $x - 5 = 0$

e) the horizontal line passing through C. $y - 1 = 0$

10. Find the equation of the line passing through A(6, 2) and B(9, 4) in the given forms.

a) functional form $y = \frac{2}{3}x - 2$

b) general form $-2x + 3y + 6 = 0$

c) symmetric form $\frac{x}{3} + \frac{y}{-2} = 1$

11. Given the line l: $3x - 4y + 12 = 0$ and the point A(1, 2).

a) Find the equation of the line l_1 passing through A and parallel to l. ***$3x - 4y + 5 = 0$***

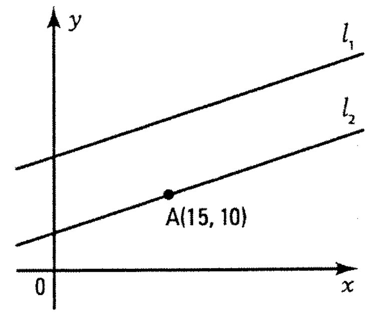
b) Find the equation of the line l_2 passing through A and perpendicular to l.

$4x + 3y - 10 = 0$

c) Calculate the distance from point A to the line l. ***1.4 u***

12. The lines l_1 and l_2 on the right are parallel. Line l_1 has the equation $x - 3y + 45 = 0$.
Find the y-intercept of line l_2 if it passes through the point A(15, 10).

l_2: $y = \frac{2}{3}x + 5$. The y-intercept of l_2 is 5.



13. Consider the points A(3, 4), B(−1, 2) and C(8, −1). Calculate the distance from point A to the line BC.

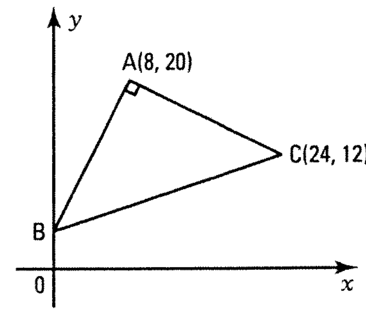
BC: $x + 3y - 5 = 0$; d(A; BC) = $\frac{10}{\sqrt{10}} = \sqrt{10}$

14. Calculate the area of the triangle ABC on the right.

AB: $y = 2x + 4$; B(0,4);

m$\overline{AB}$ = $\sqrt{320}$; m$\overline{AC}$ = $\sqrt{320}$

Area of triangle ABC = 160 u²

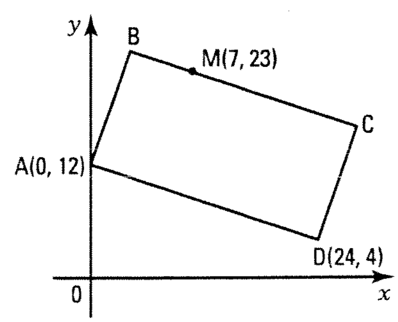


15. Calculate the area of the rectangle ABCD on the right.

AD: $x + 3y - 36 = 0$; m$\overline{AD}$ = 25.3 u

d(M, AD) = $\frac{40}{\sqrt{10}}$

Area of triangle ABCD = 320 u²



16. Consider the figure on the right. Determine the length of segment HC given that $\overline{AH}$ is an altitude of triangle ABC. (Round your answer to the nearest unit.)

BC: $2x + 3y - 10 = 0$

m$\overline{AH}$ = $2\sqrt{13}$

m$\overline{AC}$ = $\sqrt{104} = 2\sqrt{26}$

m$\overline{HC}$ = $2\sqrt{13} \approx 7$ u