Chapter 6

Systems of equations

CHALLENGE 6

EVALUATION 6

CHALLENGE 6

1. At a bookstore, Sylvie pays \$13.60 for 4 notebooks and 3 pens whereas Katherine pays \$9.90 for 3 notebooks and 2 pens. How much will Raphaëlle pay for 2 notebooks and 4 pens?

\$9.80

2. At a department store, they sell 50 ml and 100 ml perfume bottles. Seventy bottles give a total of 5 litres of perfume. If each 50 ml bottle is sold for \$45 and each 100 ml bottle is sold for \$72, what is the total revenue from the sale of all 70 bottles?

\$3960

3. Find all common points of the parabola $y = -2x^2 + 4x$ and the line $2x - y - 4 = 0$.

A(−1, −6); B(2, 0)

4. Consider the parabola $y = -x^2 + 6x - 5$ and the line $y = k$.

What can be said about the real number k if the system $\begin{cases} y = -x^2 + 6x - 5 \\ y = k \end{cases}$ has

a) no solution? ***$k > 4$*** b) one unique solution? ***$k = 4$*** c) two solutions? ***$k < 4$***

5. Consider the parabola with vertex V(3, 6) passing through the point A(6, -3) and the line passing through the points B(−1, 0) and C(5, 6). What are the points of intersection P and Q of the parabola and the line?

Parabola: $y = -(x - 3)^2 + 6$; line: $y = x + 1$; P(1, 2); Q(4, 5)

6. Two fitness centres A and B are having a promotion to attract new members.

The rule $y = 4x + 80$ gives the number of members y from centre A as a function of the number of days since the start of the promotion. The rule for the centre B is given by $y = 0.6x^2 + 60$. Determine the required number of days since the start of the promotion for

a) the two centres to have the same number of members. ***10 days***

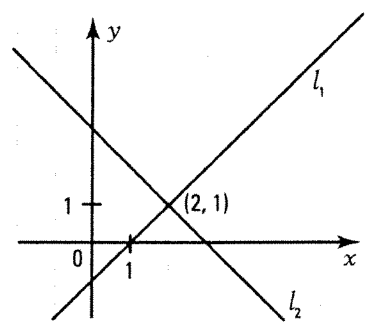
b) the number of members from centre A to be greater than the number of members from centre B.

$x \in [0, 10[$

6.1 System of two first degree equations in two variables

ACTIVITY 1 Graphic representation of a system

a) Given the line l_1 with the equation $x - y - 1 = 0$.
This equation is a first degree equation in two variables. The ordered pair (4, 3) is a solution to this equation.
1. Find another ordered pair that is a solution to the equation $x - y - 1 = 0$. ***(1, 0)***
2. How many ordered pairs are solutions to this equation? ***An infinite number of ordered pairs***
3. Complete: The graphic representation of the set of all solutions to the equation $x - y - 1 = 0$ is ***the set of all points on the line*** l_1 ***.***
4. Represent, in the Cartesian plane, the set of all solutions to the equation $x - y - 1 = 0$.



b) In the same Cartesian plane, draw the line l_2 representing the set of all solutions to the equation $x + y - 3 = 0$.

c) 1. Explain how, using the graph on the right, to find the common solution to the equations $x - y - 1 = 0$ and $x + y - 3 = 0$.
Find the point of intersection of the two lines.
2. What is this common solution? ***(2, 1)***
3. Verify that this is the common solution.
2 - 1 - 1 = 0 and 2 + 1 - 3 = 0

SYSTEM OF TWO-VARIABLE FIRST DEGREE EQUATIONS

- A system of two first degree equations in two variables is any system that can be written in the form:

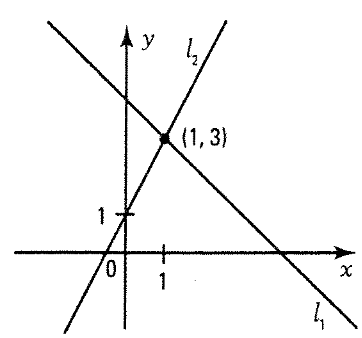
$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

where x and y represent the variables and $a_1, b_1, c_1, a_2, b_2, c_2$ are real constants.

Solving a system: Graphical method

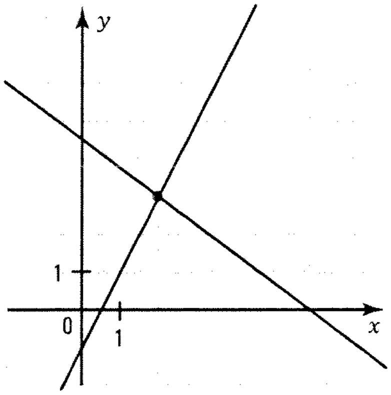
- To solve graphically a system of two-variable first degree equations, we need to represent, in the same Cartesian plane, the solution set of each equation and determine the set S of all ordered pairs (x, y) that verify simultaneously both equations.

Ex.: The system $\begin{cases} x + y = 4 \\ 2x - y = -1 \end{cases}$
is represented in the Cartesian plane on the right by two lines l_1 and l_2 with equations l_1: $x + y = 4$ and l_2: $2x - y = -1$.
The point (1, 3) is common to both lines l_1 and l_2 who are intersecting.
The point (1, 3) therefore simultaneously verifies both equations of the system.
The solution set of the system is therefore: $S = \{(1, 3)\}$.



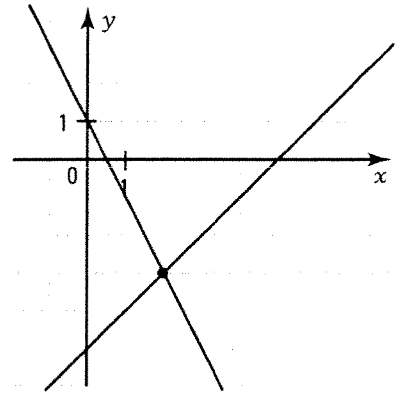
1. Solve the following systems graphically.

a) $\begin{cases} 3x + 4y = 18 \\ -2x + y = -1 \end{cases}$



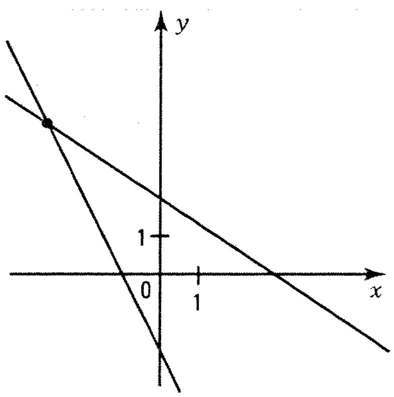
S = {(2, 3)}

b) $\begin{cases} y = -2x + 1 \\ y = x - 5 \end{cases}$



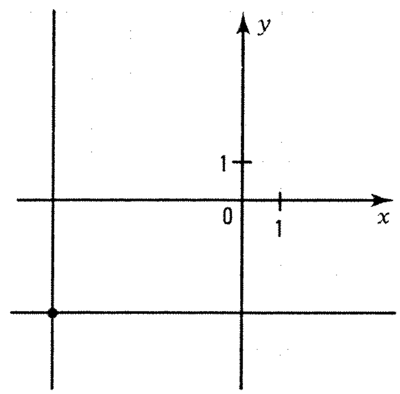
S = {(2, −3)}

c) $\begin{cases} \frac{x}{3} + \frac{y}{2} = 1 \\ 2x + y = -2 \end{cases}$



S = {(−3, 4)}

d) $\begin{cases} x + 5 = 0 \\ y + 3 = 0 \end{cases}$



S = {(−5, −3)}

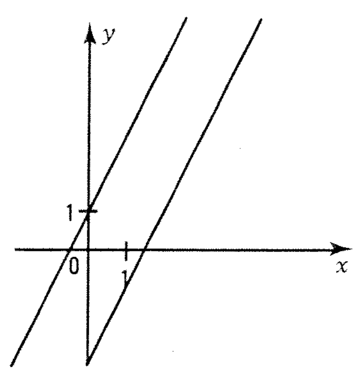
ACTIVITY 2 Particular systems

a) Consider the system $\begin{cases} 2x - y = 3 \\ 4x - 2y = -2 \end{cases}$ of the form $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$.

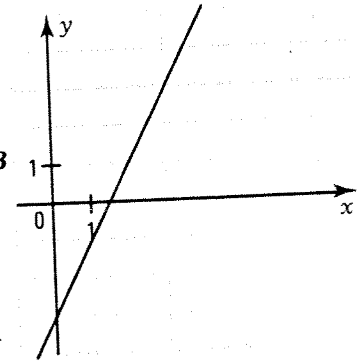
1. Solve this system graphically. *S = Ø*
2. Justify your answer. *The lines are parallel and distinct.*
3. Identify the coefficients:

   $a_1 =$ **2** $a_2 =$ **4** $b_1 =$ **−1** $b_2 =$ **−2** $c_1 =$ **3** $c_2 =$ **−2**

4. Verify that $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$. $\frac{2}{4} = \frac{-1}{-2} \neq \frac{3}{-2}$



b) Consider the system $\begin{cases} 2x - y = 3 \\ 4x - 2y = 6 \end{cases}$ of the form $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$.

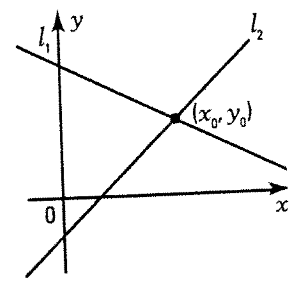


1. Represent this system graphically. What do you notice?
   The lines are parallel and coincident.
2. How many solutions does this system have? ***An infinite number***
3. Where are the solutions to this system located? ***On the line 2x − y = 3***
4. Identify the coefficients:
   $a_1 =$ **2** $a_2 =$ **4** $b_1 =$ **-1** $b_2 =$ **-2** $c_1 =$ **3** $c_2 =$ **6**
5. Verify that $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$. $\mathbf{\frac{2}{4} = \frac{-1}{-2} = \frac{3}{6}}$

SYSTEM OF TWO-VARIABLE FIRST DEGREE EQUATIONS

When solving the system $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$, we distinguish the following 3 cases:

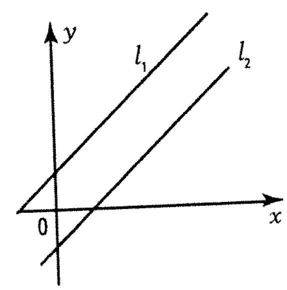
- 1st case: $\boxed{\frac{a_1}{a_2} \neq \frac{b_1}{b_2}}$.



  The lines l_1 and l_2 are intersecting.
  The system has one unique solution.
  The system is called compatible.

  $\boxed{S = \{(x_0, y_0)\}}$

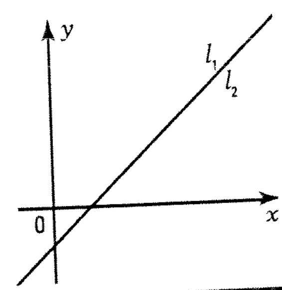
- 2nd case: $\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}}$.



  The lines l_1 and l_2 are parallel and distinct.
  The system has no solution.
  The system is called incompatible.

  $\boxed{S = \varnothing}$

- 3rd case: $\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$.



  The lines l_1 and l_2 are coincident.
  The system has an infinite number of solutions.
  The system is called indeterminate. $\boxed{S = \{(x, y) \mid a_1x + b_1y = c_1\}}$

2. Indicate, for each case, the number of solutions to the system. Justify your answer.

a) $\begin{cases} -2x + 3y = 6 \\ 2x - y = 2 \end{cases}$

one solution

$\mathbf{\frac{-2}{2} \neq \frac{3}{-1}}$

b) $\begin{cases} 2x - 5y = 12 \\ 4x - 10y = 24 \end{cases}$

infinite number of solutions

$\mathbf{\frac{2}{4} = \frac{-5}{-10} = \frac{12}{24}}$

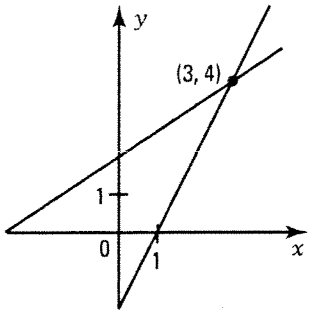
c) $\begin{cases} 2x - 5y = 12 \\ 4x - 10y = 10 \end{cases}$

no solution

$\mathbf{\frac{2}{4} = \frac{-5}{-10} \neq \frac{12}{10}}$

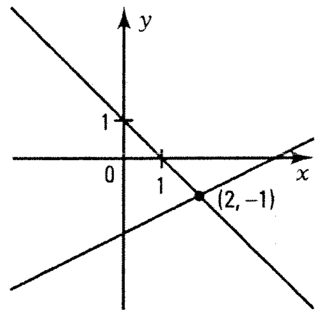
3. Solve the following systems graphically.

a) $\begin{cases} -2x + 3y = 6 \\ 2x - y = 2 \end{cases}$



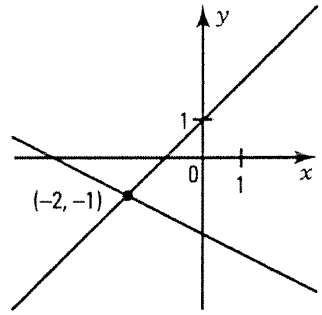
S = {(3, 4)}

b) $\begin{cases} x + y = 1 \\ x - 2y = 4 \end{cases}$



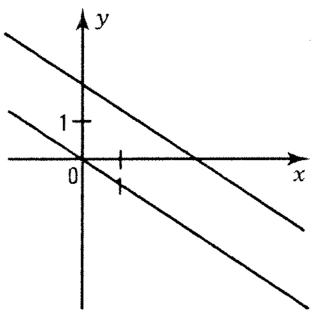
S = {(2, –1)}

c) $\begin{cases} y = x + 1 \\ y = -\frac{1}{2}x - 2 \end{cases}$



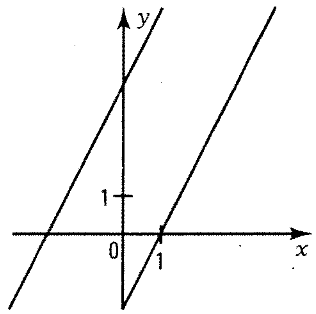
S = {(–2, –1)}

d) $\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 0 \end{cases}$



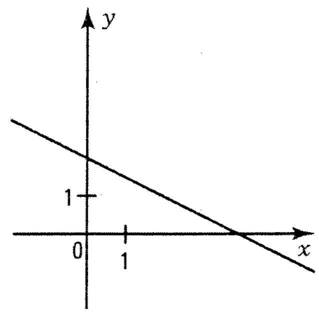
S = ∅

e) $\begin{cases} -2x + y = 4 \\ 2x - y = 2 \end{cases}$



S = ∅

f) $\begin{cases} x + 2y = 4 \\ 3x + 6y = 12 \end{cases}$



S = {(x, y) | x + 2y = 4}

ACTIVITY 3 Solving a problem using a system of equations

To finance their graduation activities, the senior students of a high school have decided to sell t-shirts and long sleeve shirts. The table below indicates their profit according to the number of shirts sold.

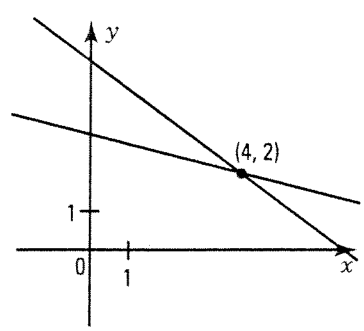
Number of shirts sold		Profit
t-shirts	long sleeve	
150	200	$1000
100	400	$1200

Establish a procedure for calculating their profit if the students sell 120 t-shirts and 300 long sleeve shirts.

1° Let x represent the profit per t-shirt sold and y represent the profit per long sleeve shirt sold.
2° We set up a system of two equations with two variables.

$\begin{cases} 150x + 200y = 1000 \\ 100x + 400y = 1200 \end{cases}$ or $\begin{cases} 3x + 4y = 20 \\ x + 4y = 12 \end{cases}$

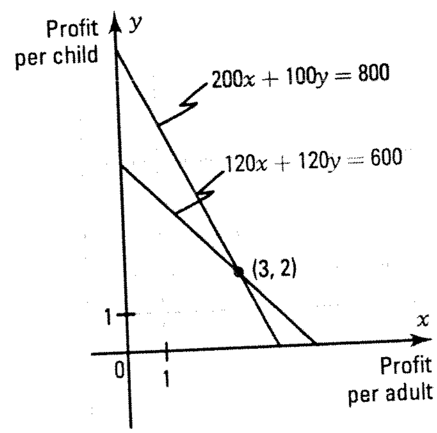
3° We solve the system. S = {(4, 2)}
4° We answer the question.
Profit per t-shirt: $4; Profit per long sleeve shirt: $2.
Total profit: 120 × 4 + 300 × 2 = $1080.



SOLVING A PROBLEM USING A SYSTEM OF EQUATIONS

Problem: The profit made by a travel agency organizing tours depends on the number of adults and children taking the tour.

Number of adults	Number of children	Profit ($)
200	100	800
120	120	600



We want to calculate the agency's profit for a tour with 150 adults and 60 children.

Procedure:

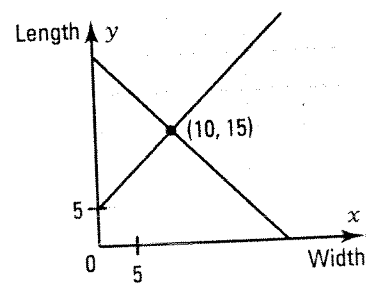
1. **We define the variables.**
   x: profit (in \$) per adult.
   y: profit (in \$) per child.
2. **We set up the system.**
   $$\begin{cases} 200x + 100y = 800 \\ 120x + 120y = 600 \end{cases}$$
3. **We solve the system** to determine the values of x and y.
   S = {(3, 2)}. The agency makes a profit of \$3 per adult and \$2 per child.
4. **We answer the question.**
   The profit made for a tour with 150 adults and 120 children is equal to \$690.

4. A rectangular field has a perimeter of 50 hm. The length of the field is 5 hm greater than the width. What is the area of the field?

x: width; y: length

$$\begin{cases} 2x + 2y = 50 \\ y = x + 5 \end{cases} \Rightarrow S = \{(10, 15)\}$$

Area = 10 × 15 = 150 hm²

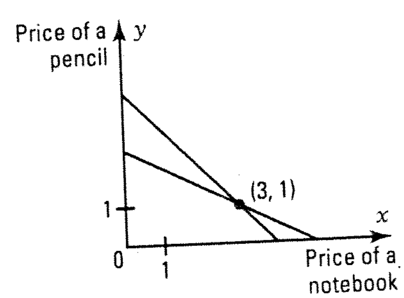


5. Karen buys 2 notebooks and 4 pencils for \$10 whereas Valerie buys 2 notebooks and 2 pencils for \$8. How much will David pay for 3 notebooks and 4 pencils?

x: price of a notebook; y: price of a pencil

$$\begin{cases} 2x + 4y = 10 \\ 2x + 2y = 8 \end{cases} \Rightarrow S = \{(3, 1)\}$$

David will pay \$13.



6.2 Algebraic solving of a two-variables first degree system

ACTIVITY 1 Solving by addition

Consider the system $\begin{cases} 3x + 2y = 7 & (1) \\ 2x - 3y = -4 & (2) \end{cases}$ ← The coefficients of x are not opposite.

a) By multiplying both sides of the 1st equation by 2 and both sides of the 2nd equation by -3, we obtain the following system $\begin{cases} 6x + 4y = 14 & (3) \\ -6x + 9y = 12 & (4) \end{cases}$ ← The coefficients of x are opposite.

Explain why the resulting system is equivalent to the initial system and therefore has the same solution set.

The equations (1) and (3) and the equations (2) and (4) are equivalent.

b) Add the same sides of the equations of the second system to determine the value of the variable y.

$13y = 26 \Rightarrow y = 2$

c) What is the value of the variable x? ***$x = 1$***

d) What is the solution set of the system? ***$S = \{(1, 2)\}$***

ALGEBRAIC SOLVING OF A SYSTEM: ADDITION METHOD

The addition method for solving a system (also called elimination method) is illustrated in the following example.

Given the system $\begin{cases} 2x + 5y = -4 \\ 3x - 2y = 13 \end{cases}$

$$\begin{matrix} \times 3 \\ \times -2 \end{matrix} \begin{cases} 2x + 5y = -4 \\ 3x - 2y = 13 \end{cases}$$

- We multiply the sides of each equation by a non-zero real number in order to get opposite coefficients of the variable x (or the variable y). $\begin{cases} 6x + 15y = -12 \\ -6x + 4y = -26 \end{cases}$
- We add the same sides of the equations of the resulting system to obtain an equation in only one variable. $19y = -38$
- We determine the value of this variable. $y = -2$
- We substitute this value into one of the system's equations to deduce the value of the other variable. $2x + 5(-2) = -4$, $x = 3$
- Establish the solution set S of the system. $S = \{(3, -2)\}$

1. Solve the following systems by addition.

a) $\begin{cases} 3x + 5y = 9 \\ 2x + y = -1 \end{cases}$ ***$S = \{(-2, 3)\}$***

b) $\begin{cases} 5x + 3y = -3 \\ 3x + 2y = -1 \end{cases}$ ***$S = \{(-3, 4)\}$***

c) $\begin{cases} -3x + 10y = 2 \\ x - 5y = 1 \end{cases}$ ***$S = \{(-4, -1)\}$***

d) $\begin{cases} x + 6y = 6 \\ x - 4y = 1 \end{cases}$

$S = \left\{\left(3, \frac{1}{2}\right)\right\}$

e) $\begin{cases} 4x + y = -1 \\ 8x + 3y = 0 \end{cases}$

$S = \left\{\left(-\frac{3}{4}, 2\right)\right\}$

f) $\begin{cases} 3x + 2y = -1 \\ 6x - 4y = 10 \end{cases}$

$S = \left\{\left(\frac{2}{3}, -\frac{3}{2}\right)\right\}$

2. In each of the following situations,

1. identify the variables.
2. translate the situation into a system of two-variables first degree equations.
3. solve the system by addition and give a complete answer.

a) The sum of two numbers is equal to 20 and their difference is equal to 4.

What is the product of these two numbers?

1. *x: first number*
   y: second number
2. $\begin{cases} x + y = 20 \\ x - y = 4 \end{cases}$
3. *S = {(12, 8)}*
   The product is equal to 96.

b) The perimeter of a rectangular yard is equal to 60 m. If we double its length and triple its width, the perimeter is then equal to 144 m. What is the area of the initial yard?

1. *x: length of yard*
   y: width of yard
2. $\begin{cases} 2x + 2y = 60 \\ 4x + 6y = 144 \end{cases}$
3. *S = {(18, 12)}*
   The area is equal to 216 m^2.

c) Julia buys two sweaters and three pairs of pants for $220 in a store. Evelyn buys three sweaters and two pairs of pants in the same store for $230.

How much will Sandra pay for four sweaters and two pairs of pants in this same store?

1. *x: price of a sweater*
   y: price of a pair of pants
2. $\begin{cases} 2x + 3y = 220 \\ 3x + 2y = 230 \end{cases}$
3. *S = {(50, 40)}*
   Sandra will pay $280.

d) Raphael buys a certain number of 50¢ and 10¢ stamps. If he pays $6.30 for a total of 19 stamps, how many of each type of stamp did he buy?

1. *x: number of 50¢ stamps*
   y: number of 10¢ stamps
2. $\begin{cases} 50x + 10y = 630 \\ x + y = 19 \end{cases}$
3. *S = {(11, 8)}*
   Raphael bought eleven 50¢ stamps and eight 10¢ stamps.

ACTIVITY 2 Solving by substitution

Consider the system $\begin{cases} -x + y = 1 & (1) \\ 2x + 3y = 13 & (2) \end{cases}$

a) Express y as a function of x using the equation (1). $y = x + 1$

b) Replace the variable y in equation (2) by the expression in x obtained in a). $2x + 3(x + 1) = 13$

c) Solve the resulting equation obtained in b) to determine the value of x. $x = 2$

d) Deduce the value of the other variable y. $y = 3$

e) What is the solution set of the system? $S = \{(2, 3)\}$

ALGEBRAIC SOLVING OF A SYSTEM: SUBSTITUTION METHOD

The substitution method for solving a system is illustrated in the following example.

Given the system $\begin{cases} 3x + 4y = -6 \\ 2x + y = 1 \end{cases}$

- We isolate one of the variables using one of the system's equations. $y = -2x + 1$
- In the other equation, we substitute the isolated variable by the obtained expression. $3x + 4(-2x + 1) = -6$
- We solve the equation. $3x - 8x + 4 = -6$; $x = 2$
- Then we substitute this resulting value into one of the system's equations and we deduce the value of the other variable. $3(2) + 4y = -6$; $y = -3$
- We establish the solution set S of the system. $S = \{(2, -3)\}$

3. Solve the following systems by substitution.

a) $\begin{cases} 2x - 3y = -7 \\ y = 2x - 3 \end{cases}$ ***S = {(4, 5)}***

b) $\begin{cases} y = 2x - 5 \\ 2x - 5y = 9 \end{cases}$ ***S = {(2, −1)}***

c) $\begin{cases} x = 3y + 1 \\ 2x - 5y = 3 \end{cases}$ ***S = {(4, 1)}***

4. Solve the systems from exercise n° 1 by substitution.

a) ***S = {(−2, 3)}***

b) ***S = {(−3, 4)}***

c) ***S = {(−4, −1)}***

d) $S = \left\{\left(3, \frac{1}{2}\right)\right\}$

e) $S = \left\{\left(-\frac{3}{4}, 2\right)\right\}$

f) $S = \left\{\left(\frac{2}{3}, -\frac{3}{2}\right)\right\}$

5. In each of the following situations,

1. identify the variables.
2. translate the situation into a system of two-variables first degree equations.
3. solve the system by substitution and give a complete answer.

a) The length of a rectangular plot of land measures 10 m more than twice its width. The plot has a perimeter of 110 m. How much does this plot cost if it is sold for $50 per square metre?

1. ***x: width of yard***
y: length of yard

2. $\begin{cases} y = 2x + 10 \\ 2x + 2y = 110 \end{cases}$

3. ***S = {(15, 40)}***
$30 000

b) A father is 10 years older than twice his son's age.
The sum of their ages is 58. What is the difference between the father's age and his son's?

1. ***x: son's age***
y: father's age

2. $\begin{cases} y = 2x + 10 \\ x + y = 58 \end{cases}$

3. ***S = {(42, 16)}***
The father is 26 years older than his son.

c) Nathalie earns an hourly wage of $7.50 from her employer whereas Eric earns an hourly wage of $6.50 at his job.
Over the course of a weekend, Nathalie worked 4 h less than Eric. Together, they earned a total of $194. How much would they have earned if Nathalie had worked the same number of hours as Eric?

1. ***x: Eric's number of hours***
y: Nathalie's number of hours

2. $\begin{cases} y = x - 4 \\ 6.5x + 7.5y = 194 \end{cases}$

3. ***S = {(16, 12)}***
They would have earned a total of $224.

d) In a class of 30 students, there are six more boys than girls. What percentage of this class are girls?

1. ***x*: number of girls**
   ***y*: number of boys**

2. $\begin{cases} x + y = 30 \\ y = x + 6 \end{cases}$

3. ***S* = {(12, 18)}**
   The girls represent 40% of this class.

ACTIVITY 3 Solving by comparison

Consider the system $\begin{cases} 3x + y = 5 & (1) \\ -2x + y = -5 & (2) \end{cases}$

a) Isolate y in each of the system's equations. What equivalent system is obtained? $\begin{cases} y = -3x + 5 \\ y = 2x - 5 \end{cases}$

b) What equation with the variable x can we deduce by comparing the two equations of the system obtained in a)? $-3x + 5 = 2x - 5$

c) Solve this last equation to determine the value of the variable x. $x = 2$

d) Deduce the value of the variable y. $y = -1$

e) What is the solution set of the system? **S = {(2, −1)}**

ALGEBRAIC SOLVING OF A SYSTEM: COMPARISON METHOD

The comparison method for solving a system is illustrated in the following example.

Given the system $\begin{cases} -2x + y = 1 \\ 3x + 2y = 9 \end{cases}$

- We isolate the same variable in each equation. $\begin{cases} y = 2x + 1 \\ y = -\frac{3}{2}x + \frac{9}{2} \end{cases}$
- We deduce by transitivity an equation in only one variable. $2x + 1 = -\frac{3}{2}x + \frac{9}{2}$
- We Solve the resulting equation. $4x + 2 = -3x + 9$, $x = 1$
- Then we substitute this value into one of the system's equations and we deduce the value of the other variable. $y = 2 \times 1 + 1$, $y = 3$
- We establish the solution set S of the system. $S = \{(1, 3)\}$

6. Solve the following systems by comparison.

a) $\begin{cases} y = 2x + 9 \\ y = -3x - 1 \end{cases}$ **S = {(−2, 5)}**

b) $\begin{cases} x = 2y + 7 \\ x = -4y - 5 \end{cases}$ **S = {(3, −2)}**

c) $\begin{cases} y = \frac{3}{4}x + \frac{1}{2} \\ y = \frac{2}{3}x - 1 \end{cases}$ **S = {(−18, −13)}**

7. Solve the systems from exercise n° 1 by comparison.

a) **S = {(−2, 3)}**
b) **S = {(−3, 4)}**
c) **S = {(−4, −1)}**
d) $S = \left\{\left(3, \frac{1}{2}\right)\right\}$
e) $S = \left\{\left(-\frac{3}{4}, 2\right)\right\}$
f) $S = \left\{\left(\frac{2}{3}, -\frac{3}{2}\right)\right\}$

8. In each of the following situations,

1. identify the variables;
2. translate the situation into a system of two 1st degree equations in two variables;
3. solve the system by comparison and give a complete answer.

a) A school principal has the choice of two transportation companies to organize a field trip for the students.
The first company charges a base amount of $120 plus $1.50 per student. The second company charges a base amount of $80 plus $2 per student. How many students must come for the transportation costs to be the same for both companies?

1. ***x*: number of students**
   ***y*: total cost**
2. $\begin{cases} y = 1.5x + 120 \\ y = 2x + 80 \end{cases}$
3. ***S* = {(80, 240)}**
   For 80 students, they both charge $240.

b) Joseph and Nathan are car salesmen for two different dealerships. Joseph receives a weekly base salary of $350 and a 0.5% commission on his sales. Nathan receives a base salary of $100 and a 1% commission on his sales. What must be the amount of sales for Joseph and Nathan to receive the same weekly salary?

1. ***x*: amount of sales ($)**
   ***y*: salary ($)**
2. $\begin{cases} y = 0.005x + 350 \\ y = 0.01x + 100 \end{cases}$
3. ***S* = {(50 000, 600)}**
   For $50 000 in sales, Joseph and Nathan both receive a salary of $600.

c) A line l_1 has a slope of $\frac{3}{2}$ and a y-intercept of –3. A line l_2, perpendicular to l_1, has a y-intercept of 10. What is the point of intersection of these two lines?

1. ***x*: *x*-coordinate of the intersection point**
   ***y*: *y*-coordinate of the intersection point**
2. $\begin{cases} y = \frac{3}{2}x - 3 \\ y = -\frac{2x}{3} + 10 \end{cases}$
3. ***S* = {(6, 6)}**
   The intersection point is (6, 6).

d) Caroline receives a weekly base salary of $120 plus a $10 commission for every item sold. Her friend Jessica receives a weekly base salary of $150 and an $8 commission for every item sold. How many items must they each sell to earn the same weekly salary?

1. ***x* : number of items sold**
   ***y* : salary ($)**
2. $\begin{cases} y = 10x + 120 \\ y = 8x + 150 \end{cases}$
3. ***S* = {(15, 270)**
   They must each sell 15 items.

SOLVING A SYSTEM: CHOOSING A METHOD

If a system is written in the form:

- $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$, we usually solve it by addition.
- $\begin{cases} a_1x + b_1y = c_1 \\ y = a_2x + b_2 \end{cases}$, we usually solve it by substitution.
- $\begin{cases} y = a_1x + b_1 \\ y = a_2x + b_2 \end{cases}$, we usually solve it by comparison.

9. Solve each of the following systems using the appropriate method.

a) $\begin{cases} y = x - 8 \\ y = -2x + 1 \end{cases}$ **S = {(3, –5)}**

b) $\begin{cases} 3x + 2y = -2 \\ 5x + y = 6 \end{cases}$ **S = {(2, –4)}**

c) $\begin{cases} y = -2x + 7 \\ 5x - 2y = 4 \end{cases}$ **S = {(2, 3)}**

6.3 Semi-linear system of equations: one linear and one quadratic

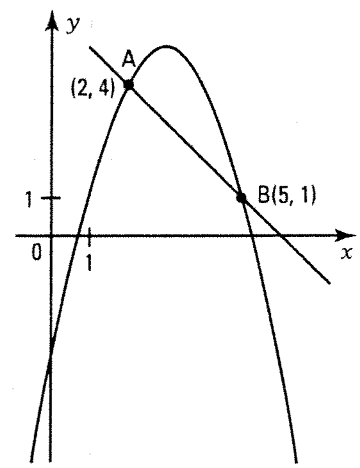
ACTIVITY 1 A parabola and a line

Consider the parabola with equation $y = -x^2 + 6x - 4$ and the line with equation $y = -x + 6$.

To determine the points A and B of intersection of the parabola and the line, we must solve the system $\begin{cases} y = -x^2 + 6x - 4 \\ y = -x + 6 \end{cases}$.

a) Solve the system graphically by drawing the parabola and the line in the same Cartesian plane and determining the intersection points A and B.

A(2, 4); B(5, 1)



b) The following steps enable you to solve algebraically the system $\begin{cases} y = -x^2 + 6x - 4 \\ y = -x + 6 \end{cases}$.

1. What second degree equation in x can you deduce by comparison?

   $-x^2 + 6x - 4 = -x + 6$

2. Solve the resulting equation. How do you interpret the solutions?

   x = 2 and x = 4. The solutions correspond to the x-coordinates of the points of intersection.

3. Find the corresponding y-coordinates. ***y = 4 and y = 1***

4. Therefore, what are the points of intersection A and B?

   A(2, 4); B(5, 1)

ACTIVITY 2 A parabola and a line: The three cases

Consider the following parabola $y = -x^2 + 2x + 3$.

a) Solve the system $\begin{cases} y = -x^2 + 2x + 3 \\ y = -2x + 3 \end{cases}$

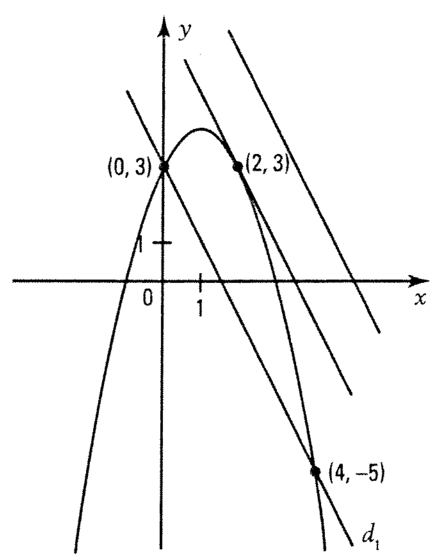
1. using a graph. ***S = {(0, 3), (4, –5)}***
2. by comparison. ***S = {(0, 3), (4, –5)}***

b) Solve the system $\begin{cases} y = -x^2 + 2x + 3 \\ y = -2x + 7 \end{cases}$

1. using a graph. ***S = {(2, 3)}***
2. by comparison. ***S = {(2, 3)}***

c) Solve the system $\begin{cases} y = -x^2 + 2x + 3 \\ y = -2x + 10 \end{cases}$

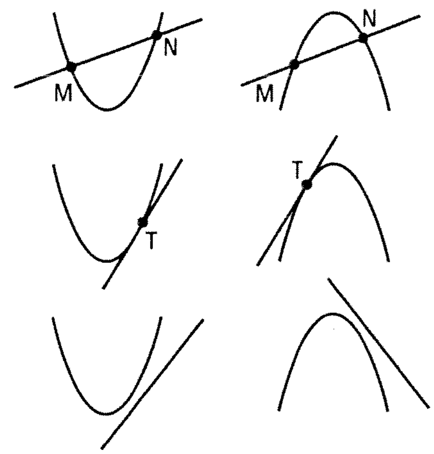
1. using a graph. ***S = Ø***
2. by comparison. ***S = Ø***



SOLVING A SEMI-LINEAR SYSTEM OF EQUATIONS: ONE LINEAR AND ONE QUADRATIC

- To determine the intersection points of the parabola $y = \text{A}x^2 + \text{B}x + \text{C}$ and the line $y = ax + b$, we solve the system

$$\begin{cases} y = \text{A}x^2 + \text{B}x + \text{C} \\ y = ax + b \end{cases}$$



- We distinguish three cases:

1st case: The line intersects the parabola in two points M and N. The system therefore yields two ordered pairs as solutions representing the coordinates of M and N.

2nd case: The line is tangent to the parabola at the point T. The system therefore yields one ordered pair as a solution representing the coordinates of T.

3rd case: The line does not intersect the parabola. The system therefore has no solution.

1. Solve the following systems algebraically.

a) $\begin{cases} y = 3x^2 - 2x + 1 \\ y = -2x + 4 \end{cases}$ ***S = {(-1, 6), (1, 2)}***

b) $\begin{cases} y = -2x^2 + 3x + 1 \\ y = -x + 3 \end{cases}$ ***S = {(1, 2)}***

c) $\begin{cases} y = x^2 + 2x + 3 \\ y = x + 1 \end{cases}$ ***S = Ø***

d) $\begin{cases} y = x^2 - 5x + 6 \\ y = 2 \end{cases}$ ***S = {(1, 2), (4, 2)}***

e) $\begin{cases} y = x^2 + 2x - 3 \\ y = 5 \end{cases}$ ***S = {(-4, 5), (2, 5)}***

f) $\begin{cases} y = -x^2 + 6x - 5 \\ y = -2x + 7 \end{cases}$ ***S = {(2, 3), (6, -5)}***

2. What are the points of intersection A and B of the parabola $y = x^2 - 2x - 3$ and the line $x - y + 1 = 0$?

A(-1, 0) and B(4, 5)

3. Indicate the number of solutions for each system. ($k > 0$)

a) $\begin{cases} y = k \\ y = x^2 \end{cases}$ ***2 solutions***

b) $\begin{cases} y = k \\ y = x^2 + k \end{cases}$ ***1 solution***

c) $\begin{cases} y = k \\ y = -x^2 \end{cases}$ ***0 solution***

d) $\begin{cases} y = k \\ y = -x^2 + k \end{cases}$ ***1 solution***

4. Consider the parabola $y = -x^2 + 6x - 5$ and the line l with equation $2x - y - 1 = 0$.

a) Show that the line is tangent to the parabola.

The equation $-x^2 + 6x - 5 = 2x - 1 \Leftrightarrow x^2 - 4x + 4 = 0$ yields one unique solution $x = 2$.

b) What are the coordinates of the point of tangency P (common point to the parabola and the line)?

P(2, 3)

6.4 Problems on systems

1. Solve the following problems.

a) Caroline works this summer at a grocery store and a pharmacy. The first week, she earns \$138 by working 12 h at the grocery store and 8 h at the pharmacy. The second week, she earns \$142 by working 8 h at the grocery store and 12 h at the pharmacy. How much will she earn in the third week by working 10 h at the grocery store and 14 h at the pharmacy? ***\$170***

b) Evan wants to subscribe to a video club. The first club he goes to charges an annual membership fee of \$10 plus \$3 per rented DVD. The second club he goes to charges a \$25 membership fee and \$2.25 per rented DVD. How many DVD's would he have to rent in one year for both clubs to charge the same total amount? ***20 DVD rentals***

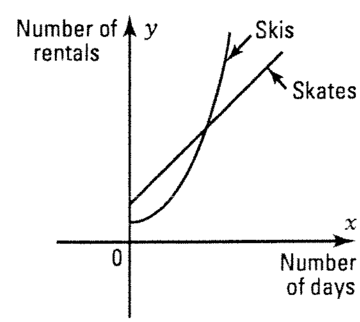
c) A jar contains a total of 20 red, black and blue-coloured marbles. There are two more black marbles than red marbles and four more blue marbles than black marbles. How many marbles of each colour are there? ***4 red, 6 black and 10 blue***

d) A marina contains 52 boats consisting of sail boats and speed boats. There are three times as many sail boats as speed boats. How many sailboats are there and how many speedboats?
39 sail boats and 13 speed boats.

e) In a warehouse, there are a total of 32 boxes. There are small 240 dm^3 boxes and large 320 dm^3 boxes. If the total volume occupied by these boxes is 9280 dm^3, how many small boxes are there? ***12 small boxes***

f) The length of a rectangle measures five times its width. If the perimeter of this rectangle is 144 cm, what is its area? ***720 cm²***

2. A sporting goods store rents skates and skis. The graph on the right represents the number of rentals according to the number of days since the beginning of the winter season. The number of skate rentals is defined by the rule $y = 5x + 60$ and the number of ski rentals is defined by the rule $y = x^2 + 10$.



a) How many days after the beginning of the winter season is the number of skate rentals equal to the number of ski rentals?
10 days after the beginning of the season.

b) Indicate which rented item is more popular according to the number of days since the beginning of the season.
From the start of the season to the 9th day, the skates were more popular. Ten days after the start of the season, the skates and skis were equally popular. The rest of the season, the skis were more popular.

3. The line $y = x + 3$ intersects a parabola at two points A and B. The table on the right indicates the coordinates of different points on this parabola.

x	y
0	7
1	4
2	3
3	4
4	7

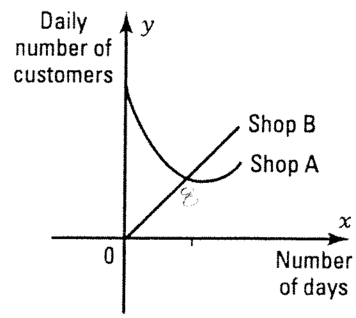
What are the coordinates of the points A and B?

Parabola: ***$y = (x - 2)^2 + 3$***

A(1, 4) and B(4, 7)

4. In a shopping centre, a new coffee shop B just opened up not very far from the existing coffee shop A. The graph on the right shows the evolution of the daily number of customers of each of the two shops during the first 15 days following the opening of coffee shop B. Eight days after the opening of coffee shop B, the two shops received the same number of customers. The rule $y = 3x^2 - 60x + 480$ gives the daily number of customers for coffee shop A since B opened.

How many more customers does coffee shop B have on the day where coffee shop A has its lowest number of customers?

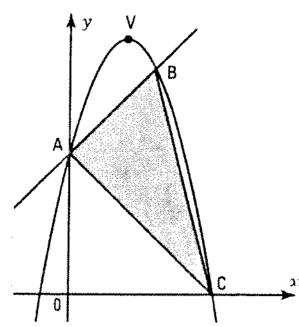


Shop B: $y = 24x$; Vertex of the parabola V(10, 180)

On the tenth day, coffee shop B has 60 more customers than shop A.

5. The line l with equation $x - y + 10 = 0$ intersects the parabola $y = -\frac{1}{2}(x - 4)^2 + 18$ at two points A and B.

Let C represent the positive intersection point of the parabola with the x-axis. Calculate the area of triangle ABC.



A(0, 10); B(6, 16); C(10, 0)

AC: $x + y - 10 = 0$; $m\overline{AC} = 10\sqrt{2}$

$d(B; AC) = \frac{12}{\sqrt{2}}$; Area $\triangle ABC = 60\ u^2$

6. The line $x - y + 1 = 0$ intersects the parabola $y = x^2 - 6x + 11$ at two points A and B. Let V represent the vertex of the parabola. What is the area of the triangle VAB?

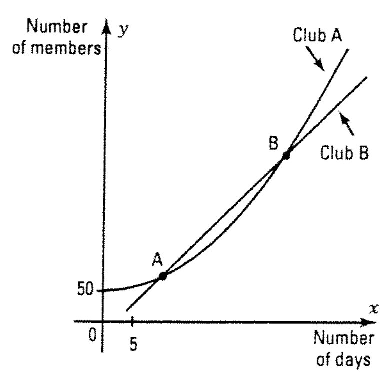
A(2, 3); B(5, 6); V(3, 2); $m\overline{AB} = \sqrt{18}$; AB: $x - y + 1 = 0$; $d(V, AB) = \sqrt{2}$; Area $\triangle ABC = 3u^2$

7. Two competing fitness centres are each having a promotion to attract new members. The rule $= \frac{1}{4}x^2 + 50$ gives the number of members at club A and the rule $y = 10x - 25$ gives the number of members at club B since the start of their promotions.

a) Represent this situation in the Cartesian plane.

b) How many days after the start of the promotion do the two centres have the same number of members? Determine your answer

1. using the graph. ***10 days and 30 days***
2. by solving an equation. ***$\frac{1}{4}x^2 + 50 = 10x - 25$; S = {10, 30}***



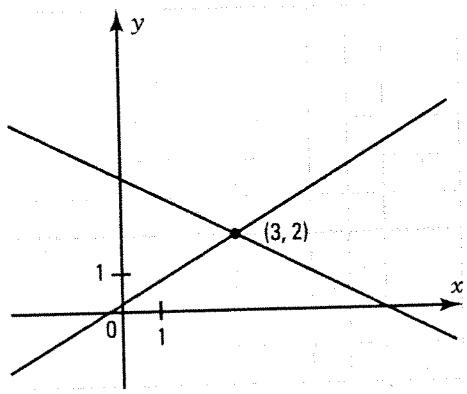
c) How many days after the start of the promotion does centre B have as many or more members than centre A? Determine your answer

1. using the graph. ***When the number of days is in [10, 30].***
2. by solving an inequality. ***S = [10, 30]***

Evaluation 6

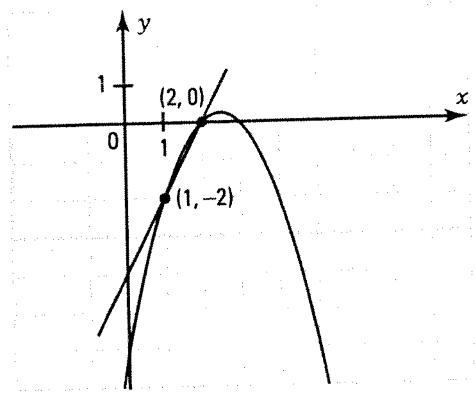
1. Solve the following systems graphically.

a) $\begin{cases} 3x - 5y = -1 \\ x + 2y = 7 \end{cases}$



***S* = {(3, 2)}**

b) $\begin{cases} y = -x^2 + 5x - 6 \\ -2x + y + 4 = 0 \end{cases}$



***S* = {(1, −2), (2, 0)}**

2. Solve the following systems using an appropriate method.

a) $\begin{cases} 5x + 2y = 5 \\ -4x - 3y = 3 \end{cases}$ ***S* = {(3, −5)}**

b) $\begin{cases} 8x + 3y = 5 \\ y = -3x + 1 \end{cases}$ ***S* = {(−2, 7)}**

c) $\begin{cases} y = \frac{3}{4}x - \frac{1}{2} \\ y = \frac{2}{3}y + \frac{1}{5} \end{cases}$ $S = \left\{\left(\frac{42}{5}, \frac{29}{5}\right)\right\}$

d) $\begin{cases} y = x^2 + x - 12 \\ 7x - y = 20 \end{cases}$ ***S* = {(2, −6), (4, 8)}**

e) $\begin{cases} y = 2x^2 - 7x - 4 \\ 3x + y + 6 = 0 \end{cases}$ ***S* = {(1, −9)}**

f) $\begin{cases} y = x^2 - 1 \\ -x + y + 2 = 0 \end{cases}$ ***S* = Ø**

3. In each of the following situations,

1. identify the variables.
2. translate the situation using a system of equations.
3. answer the question in the problem.

a) In the month of June, a bicycle shop spends \$3100 to buy six racing bikes and four mountain bikes. In July, the shop spends \$ 8700 to buy twelve racing bikes and eighteen mountain bikes. How much will the shop spend in August to buy eight racing bikes and ten mountain bikes?

1. ***x*: cost of a racing bike**
   ***y*: cost of a mountain bike**
2. $\begin{cases} 6x + 4y = 3\ 100 \\ 12x + 18y = 8\ 700 \end{cases}$
3. **\$5 300**

b) In an office of 50 employees, there are five more men than twice the number of women. How many men and how many women are there in this office?

1. ***x*: number of women**
   ***y*: number of men**
2. $\begin{cases} x + y = 50 \\ y = 2x + 5 \end{cases}$
3. **35 men and 15 women**

c) The Kandev car rental company charges a basic fee of \$15 per day plus 10¢ per km. The Rak car rental company charges a basic fee of \$25 per day plus 5¢ per km. What distance must be traveled for the two companies to charge the same amount?

1. ***x: number of kilometres***
   y: total cost

2. $\begin{cases} y = 15 + 0.1x \\ y = 25 + 0.05x \end{cases}$

3. ***200 km.***

4. A father is four times older than his son. If you subtract 60 from the square of the son's age, you get the father's age. How old are they?

The son is 10 years old and the father is 40.

5. Consider the line l passing through the points A(0, 7) and B(3, 1) and the parabola with vertex V(1, 1) passing through the point C(−1, 9). Find the points of intersection P and Q of the parabola and the line.

Parabola: $y = 2(x - 1)^2 + 1$; ***Line:*** $y = -2x + 7$; ***P(−1, 9), Q(2, 3)***

6. Two projectiles are simultaneously launched. The heights (in m), $h_1(t)$ and $h_2(t)$ of the first and second projectiles as a function of time t in seconds since the launch are given by the rules $h_1(t) = -2t^2 + 20t + 50$ and $h_2(t) = 2t + 66$.

a) How long after their launch will the two projectiles be at the same height?

$-2t^2 + 20t + 50 = 2t + 66 \Leftrightarrow t^2 - 9t + 8 = 0 \Leftrightarrow t = 1$ or $t = 8$. ***After 1 second and after 8 seconds.***

b) Over what interval of time since the launch is the 1st projectile higher than the 2nd?

$-2t^2 + 20t + 50 > 2t + 66 \Leftrightarrow t^2 - 9t + 8 < 0 \Leftrightarrow t \in]1, 8[$. ***The time since the launch must be greater than 1 sec and less than 8 sec.***

7. The line l with equation $2x - y - 5 = 0$ intersects the parabola $y = x^2 - 12x + 35$ at two points A and B. What distance, to the nearest tenth, separates the vertex of the parabola and point B?

B(10, 15); V(6, −1)

$d(V, B) = \sqrt{272} \simeq 16.5$ ***u***