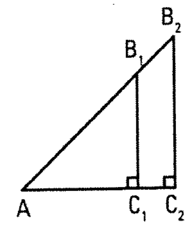
Chapter 8

Trigonometry

CHALLENGE 8

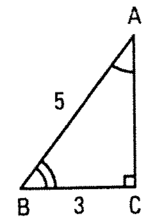
1. Consider the triangles AB_1C_1 and AB_2C_2.



Complete.

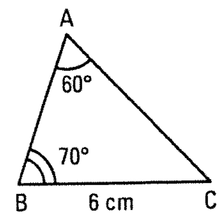
a) $\frac{m\overline{B_1C_1}}{m\overline{AB_1}} = \frac{m\overline{B_2C_2}}{\boldsymbol{m\overline{AB_2}}}$ b) $\frac{m\overline{AC_1}}{m\overline{AB_1}} = \frac{m\overline{AC_2}}{\boldsymbol{m\overline{AB_2}}}$ c) $\frac{m\overline{B_1C_1}}{m\overline{AC_1}} = \frac{m\overline{B_2C_2}}{\boldsymbol{m\overline{AC_2}}}$

2. Calculate



a) m ∠A. ***36.9°*** b) m ∠B. ***53.1°***

3. Consider triangle ABC on the right.



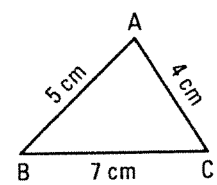
a) Calculate its perimeter to the nearest tenth.

$m\overline{AC}$ = 6.5 cm; $m\overline{AB}$ = 5.3 cm; Perimeter = 17.8 cm

b) Calculate its area to the nearest hundredth.

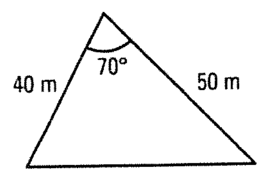
Area = 14.9 cm²

4. Calculate the area of triangle ABC to the nearest tenth.



p = 8; Area = $\sqrt{8 \times 3 \times 1 \times 4} \approx 9.8$ cm²

5. Consider the triangular field on the right.
The field is sold for $25 per square metre.
We want to surround this field with a fence. If the price of the fence costs $12 per metre, calculate, to the nearest dollar, the total cost of the field with the fence.

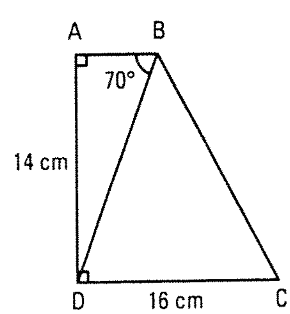


Perimeter ≈ 142 m

Area = $\sqrt{71 \times 31 \times 19 \times 21} \approx 937$ m², p = 71

Total cost ≈ $25 129

6. Consider the trapezoid ABCD on the right. What is the area of this trapezoid?

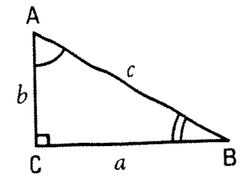


$m\overline{AB}$ = 5.1 cm; Area of trapezoid = 147.7 cm²

8.1 Trigonometric ratios

ACTIVITY 1 Right triangles

Consider triangle ABC on the right.



Let a, b and c represent the respective measures of the sides BC, AC and AB.

a) Write the Pythagorean Theorem associated to triangle ABC. $a^2 + b^2 = c^2$

b) 1. Which angles are the acute angles? $\angle A$ and $\angle B$

2. Explain why the acute angles are complementary.

$m\angle A + m\angle B + m\angle C = 180°$ (sum of interior angles of a triangle)

Since $m\angle C = 90°$ then $m\angle A + m\angle B = 90°$

c) In this triangle, side AB is the hypotenuse, side AC is the opposite side to angle B whereas side BC is the adjacent side to angle B. What is

1. the opposite side to angle A? $\overline{BC}$ 2. the adjacent side to angle A? $\overline{AC}$

RIGHT TRIANGLES

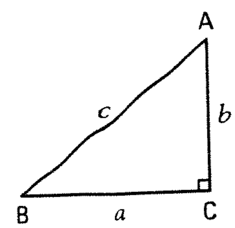
- Any right triangle verify the Pythagorean Theorem:

$$a^2 + b^2 = c^2$$

- The acute angles of a right triangle are complementary.

$$m\angle B + m\angle C = 90°$$

- The opposite side AB to the right angle is the hypotenuse.
  Side AC is the opposite side to angle B.
  Side BC is the adjacent side to angle B.



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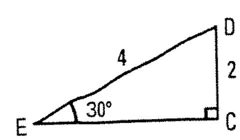
1. Consider the given right triangle on the right.



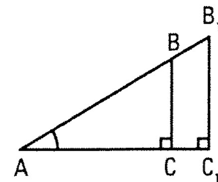
a) Find $m\angle D$. $60°$

b) Find $m\overline{EC}$. $\sqrt{12}$ or $2\sqrt{3}$

c) Identify

1. the hypotenuse. $\overline{ED}$
2. the opposite side to $\angle D$. $\overline{EC}$
3. the adjacent side to $\angle D$. $\overline{DC}$
4. the opposite side to $\angle E$. $\overline{DC}$
5. the adjacent side to $\angle E$. $\overline{EC}$

ACTIVITY 2 Trigonometric ratios

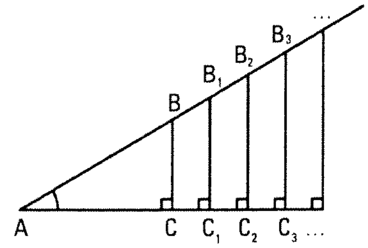


a) Which similarity theorem enables you to justify that triangles ABC and AB_1C_1 on the right are similar?

The AA theorem. Angle A is common to both triangles and angles BCA and B_1C_1A are congruent. The two triangles ABC and AB_1C_1 have two congruent corresponding angles and are therefore similar.

b) Complete: $\frac{m\overline{BC}}{m\overline{B_1C_1}} = \frac{m\overline{AB}}{\boldsymbol{m\overline{AB_1}}}$.

c) By switching the middle terms of the preceding proportion, we get $\frac{m\overline{BC}}{m\overline{AB}} = \frac{m\overline{B_1C_1}}{\boldsymbol{m\overline{AB_1}}}$.



d) 1. What can be said of the right triangles on the right?

They are all similar.

2. Complete: $\frac{m\overline{BC}}{m\overline{AB}} = \frac{m\overline{B_1C_1}}{\boldsymbol{m\overline{AB_1}}} = \frac{m\overline{B_2C_2}}{\boldsymbol{m\overline{AB_2}}} = \frac{m\overline{B_3C_3}}{\boldsymbol{m\overline{AB_3}}} = \ldots$

Therefore, for each triangle, we observe that the ratio of the measure of the opposite side to angle A over the measure of the hypotenuse remains the same. Such a ratio is called sine A.

e) 1. Since triangles ABC and AB_1C_1 are similar, complete: $\frac{m\overline{AC}}{m\overline{AB}} = \frac{m\overline{AC_1}}{\boldsymbol{m\overline{AB_1}}}$.

2. Generalize: $\frac{m\overline{AC}}{m\overline{AB}} = \frac{m\overline{AC_1}}{\boldsymbol{m\overline{AB_1}}} = \frac{m\overline{AC_2}}{\boldsymbol{m\overline{AB_2}}} = \frac{m\overline{AC_3}}{\boldsymbol{m\overline{AB_3}}} = \ldots$

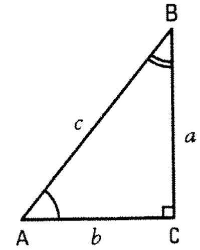
Therefore, for each triangle, we observe that the ratio of the measure of the adjacent side to angle A over the measure of the hypotenuse remains the same. Such a ratio is called cosine A.

f) 1. Since triangles ABC and $A_1B_1C_1$ are similar, complete: $\frac{m\overline{BC}}{m\overline{AC}} = \frac{m\overline{B_1C_1}}{\boldsymbol{m\overline{AC_1}}}$.

2. Generalize: $\frac{m\overline{BC}}{m\overline{AC}} = \frac{m\overline{B_1C_1}}{\boldsymbol{m\overline{AC_1}}} = \frac{m\overline{B_2C_2}}{\boldsymbol{m\overline{AC_2}}} = \frac{m\overline{B_3C_3}}{\boldsymbol{m\overline{AC_3}}} = \ldots$

Therefore, for each triangle, we observe that the ratio of the measure of the opposite side to angle A over the measure of the adjacent side to angle A remains the same. Such a ratio is called tangent A.

TRIGONOMETRIC RATIOS IN A RIGHT TRIANGLE



- The sine of an acute angle is equal to the ratio of the measure of the opposite side to that angle over the measure of the hypotenuse.
  The sine of angle A is written sin A.

$$\sin A = \frac{\text{measure of opposite side}}{\text{measure of hypotenuse}} = \frac{a}{c}$$

- The cosine of an acute angle is equal to the ratio of the measure of the adjacent side to that angle over the measure of the hypotenuse.
  The cosine of angle A is written cos A.

$$\cos A = \frac{\text{measure of adjacent side}}{\text{measure of hypotenuse}} = \frac{b}{c}$$

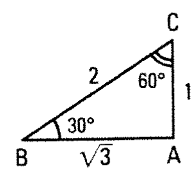
- The tangent of an acute angle is equal to the ratio of the measure of the opposite side to that angle over the measure of the adjacent side to that angle.
  The tangent of angle A is written tan A.

$$\tan A = \frac{\text{measure of opposite side}}{\text{measure of adjacent side}} = \frac{a}{b}$$

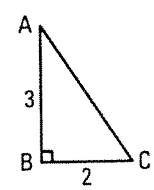
- When writing a trigonometric ratio, we can write the measure of the angle when it is known. Thus, the sine of angle B measuring 30° is written sin 30°.

Ex.: $\sin 30° = \frac{1}{2}$; $\cos 30° = \frac{\sqrt{3}}{2}$; $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin 60° = \frac{\sqrt{3}}{2}$; $\cos 60° = \frac{1}{2}$; $\tan 60° = \sqrt{3}$



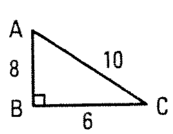
2. Using the triangle on the right, determine



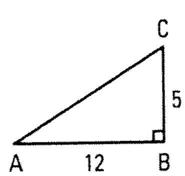
a) sin A. $\frac{2}{\sqrt{13}}$ b) cos A. $\frac{3}{\sqrt{13}}$ c) tan A. $\frac{2}{3}$

d) sin C. $\frac{3}{\sqrt{13}}$ e) cos C. $\frac{2}{\sqrt{13}}$ f) tan C. $\frac{3}{2}$

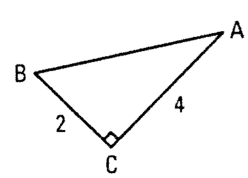
3. In each of the following cases, determine the value of the sine, cosine and tangent of angle A.

a) 

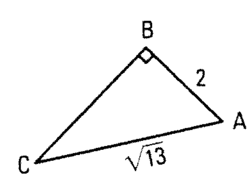
sin A = 0.6
cos A = 0.8
tan A = $\frac{3}{4}$

b) 

sin A = $\frac{5}{13}$
cos A = $\frac{12}{13}$
tan A = $\frac{5}{12}$

c) 

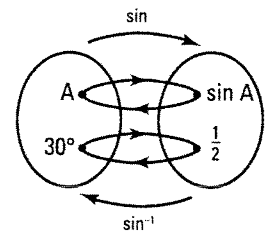
sin A = $\frac{2}{\sqrt{20}}$
cos A = $\frac{4}{\sqrt{20}}$
tan A = $\frac{1}{2}$

d) 

sin A = $\frac{3}{\sqrt{13}}$
cos A = $\frac{2}{\sqrt{13}}$
tan A = $\frac{3}{2}$

CALCULATOR

- The key [sin] on the calculator enables you to calculate the value of sin A knowing the measure of angle A.
- The key [$\sin^{-1}$] on the calculator enables you to calculate the measure of angle A knowing sin A.
  Thus, $\sin 30° = \frac{1}{2}$ and $\sin^{-1}\left(\frac{1}{2}\right) = 30°$.



4. a) Using a calculator, complete the following table.

b) When m ∠A increases from 0° to 90°,

1. sin A increases or decreases? ***sin A increases***
2. cos A increases or decreases? ***cos A decreases***
3. tan A increases or decreases? ***tan A increases***

c) Verify that when two angles are complementary, the sine of one is equal to the cosine of the other.

d) Verify that $\tan A = \frac{\sin A}{\cos A}$.

m ∠A	sin A	cos A	tan A
0°	***0***	***1***	***0***
20°	***0.3420***	***0.9848***	***0.3640***
30°	***0.5***	***0.8660***	***0.5774***
45°	***0.7071***	***0.7071***	***1***
60°	***0.8660***	***0.5***	***1.7321***
80°	***0.9848***	***0.3420***	***5.6713***
90°	***1***	***0***	

5. a) Using a calculator, complete the following table (round m ∠A to the nearest unit).

b) Verify your answers from question 4b).

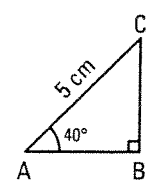
c) Verify question 4c).

d) Verify question 4d).

m ∠A	sin A	cos A	tan A
10°	0.1736	***0.9848***	***0.1763***
50°	***0.7660***	0.6428	***1.1918***
75°	***0.9659***	***0.2588***	3.7321
80°	0.9848	***0.1736***	***5.6713***
89°	***0.9998***	0.0175	***57.2900***

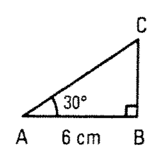
ACTIVITY 3 Find the missing sides

a) Triangle ABC is a right triangle. Angle A measures 40° and the hypotenuse measures 5 cm.



1. Calculate $m\overline{BC}$. $\sin 40° = \frac{m\overline{BC}}{5} \Rightarrow m\overline{BC} = 5 \sin 40° = 3.21$ cm
2. Calculate $m\overline{AB}$. $\cos 40° = \frac{m\overline{AB}}{5} \Rightarrow m\overline{AB} = 5 \cos 40° = 3.83$ cm
3. Verify your results using the Pythagorean Theorem. $(3.21)^2 + (3.83)^2 \simeq 5^2$

b) ABC is a right triangle. Angle A measures 30° and its adjacent side measures 6 cm.



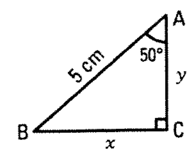
1. Calculate $m\overline{BC}$. $\tan 30° = \frac{m\overline{BC}}{6} \Rightarrow m\overline{BC} = 6 \tan 30° = 3.46$ cm
2. Calculate $m\overline{AC}$. $\cos 30° = \frac{6}{m\overline{AC}} \Rightarrow m\overline{AC} = \frac{6}{\cos 30°} = 6.93$ cm
3. Verify your results using the Pythagorean Theorem. $6^2 + (3.46)^2 \simeq (6.93)^2$

FINDING MISSING SIDES USING TRIGONOMETRIC RATIOS

In a right triangle,

- finding the measure x of side BC opposite to the known angle A, knowing also the measure of the hypotenuse, requires the use of sin A.

  $\sin 50° = \frac{x}{5} \Rightarrow x = 5 \sin 50° \simeq 3.83$ cm

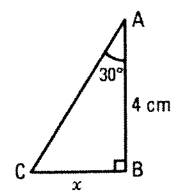


- finding the measure y of side AC adjacent to the known angle A, knowing also the measure of the hypotenuse, requires the use of cos A.

  $\cos 50° = \frac{y}{5} \Rightarrow y = 5 \cos 50° \simeq 3.21$ cm

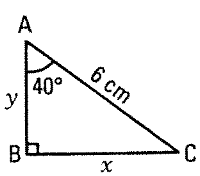
- finding the measure x of side BC opposite to the known angle A, knowing also the measure of the adjacent side to angle A, requires the use of tan A.

  $\tan 30° = \frac{x}{4} \Rightarrow x = 4 \tan 30° \simeq 2.31$ cm

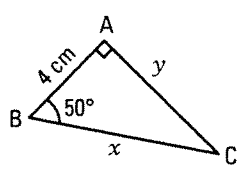


6. Calculate in each case x and y. (Round each answer to the nearest tenth.)

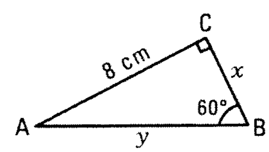
a)



b)



c)



a) $x = 6 \sin 40° = 3.9$ cm

$y = 6 \cos 40° = 4.6$ cm

b) $x = \frac{4}{\cos 50°} = 6.2$ cm

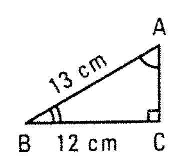
$y = 4 \tan 50° = 4.8$ cm

c) $x = \frac{8}{\tan 60°} = 4.6$ cm

$x = \frac{8}{\sin 60°} = 9.2$ cm

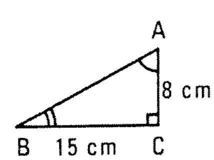
ACTIVITY 4 Finding missing angles

a) Consider the given right triangle ABC. The opposite side to angle A measures 12 cm and the hypotenuse measures 13 cm.



1. Calculate m ∠A. $\sin A = \frac{12}{13} \Rightarrow m\angle A = \sin^{-1}\left(\frac{12}{13}\right) \approx 67.4°$
2. Calculate m ∠B. $\cos B = \frac{12}{13} \Rightarrow m\angle B = \cos^{-1}\left(\frac{12}{13}\right) \approx 22.6°$
3. Verify that angles A and B are complementary. $67.4° + 22.6° = 90°$

b) Consider the given right triangle ABC with the sides of the right angle measuring 8 cm and 15 cm.

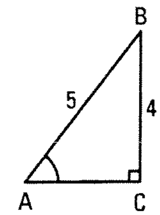


1. Calculate m ∠A. $\tan A = \frac{15}{8} \Rightarrow m\angle A = \tan^{-1}\left(\frac{15}{8}\right) = 61.9°$
2. Calculate m ∠B. $\tan B = \frac{8}{15} \Rightarrow m\angle B = \tan^{-1}\left(\frac{8}{15}\right) = 28.1°$
3. Verify that angles A and B are complementary. $61.9° + 28.1° = 90°$

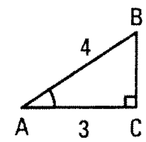
FINDING MISSING ANGLES USING TRIGONOMETRIC RATIOS

In a right triangle,

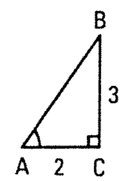
- finding the acute angle A when its opposite side and the hypotenuse are known requires the use of sin A.
  $\sin A = \frac{4}{5} \Rightarrow m\angle A = \sin^{-1}\left(\frac{4}{5}\right) = 53.1°$



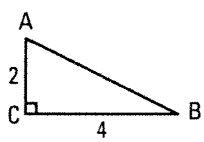
- finding the acute angle A when its adjacent side and the hypotenuse are known requires the use of cos A.
  $\cos A = \frac{3}{4} \Rightarrow m\angle A = \cos^{-1}\left(\frac{3}{4}\right) = 41.4°$



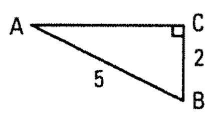
- finding the acute angle A when its opposite side and adjacent side are known requires the use of tan A.
  $\tan A = \frac{3}{2} \Rightarrow m\angle A = \tan^{-1}\left(\frac{3}{2}\right) = 56.3°$



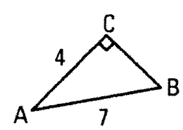
7. Find the measures of angles A and B. (Round each answer to the nearest unit.)

a)


m ∠A = 63°; m ∠B = 27°

b)


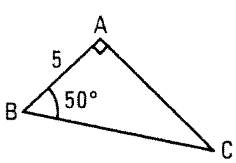
m ∠A = 24°; m ∠B = 66°

c)


m ∠A = 55°; m ∠B = 35°

Solving a triangle consists of determining the measure of all its sides and angles.

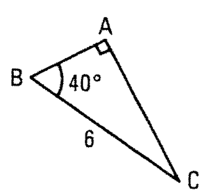
8. Solve the following triangles. (Round your answers to the nearest tenth.)

a)


m ∠C = 40°

m$\overline{AC}$ = 6

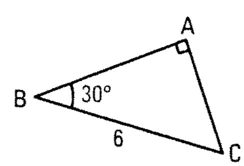
m$\overline{BC}$ = 7.8

b)


m ∠C = 50°

m$\overline{AB}$ = 4.6

m$\overline{AC}$ = 3.9

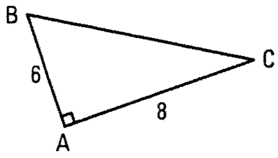
c)


m ∠C = 60°

m$\overline{AB}$ = 5.2

m$\overline{AC}$ = 3

d)

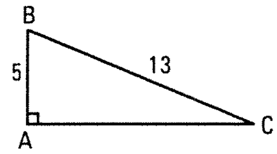


$m\overline{BC} = 10$

$m\angle B = 53.1°$

$m\angle C = 36.9°$

e)

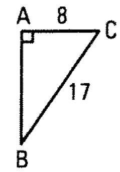


$m\overline{AC} = 12$

$m\angle B = 67.4°$

$m\angle C = 22.6°$

f)



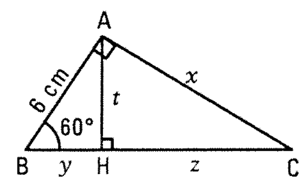
$m\overline{AB} = 15$

$m\angle B = 28.1°$

$m\angle C = 61.9°$

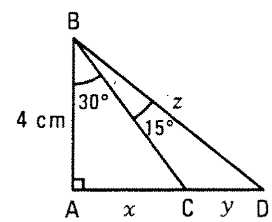
9. In the triangle ABC on the right, the altitude AH is drawn. Determine the value of the unknowns x, y, z, t.

$x = 10.4$ cm; $y = 3.0$ cm; $z = 9.0$ cm; $t = 5.2$ cm



10. Using the figure on the right, determine the measure of the unknowns x, y and z.

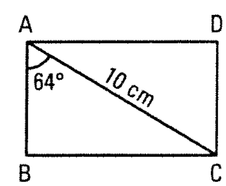
$x = 2.3$ cm; $y = 1.7$ cm; $z = 5.7$ cm



11. Calculate the area of rectangle ABCD on the right. (Round your answer to the nearest tenth.)

$m\overline{AB} = 10 \cos 64° = 4.38$ cm; $m\overline{BC} = 10 \sin 64° = 8.99$ cm

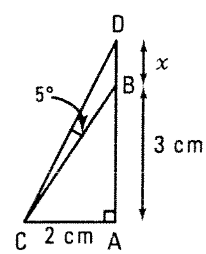
Area $\simeq 39.4$ cm^2



12. Using the figure on the right, determine x.

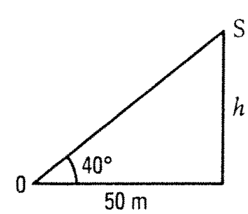
$m\angle ACB = 56.3°$; $m\angle ACD = 61.3°$

$m\overline{AD} = 3.7$ cm; $m\overline{BD} = 0.7$ cm



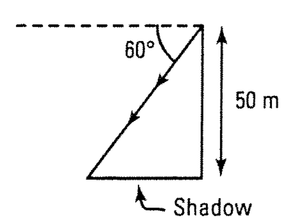
13. An observer 0 is located 50 m away from the base of a building and is looking up to the top of the building S with a 40° angle of elevation. Determine the height h of the building.

$h = 50 \tan 40° = 42$ m

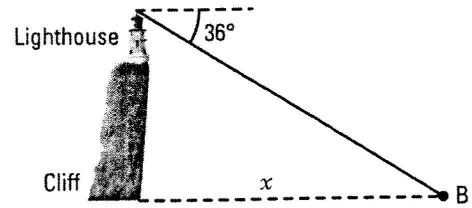


14. What is the length of the shadow cast by a 50 m high building when the sun's rays make a 60° angle of depression?

Length of the shadow $= 50 \tan 30° = 28.9$ m

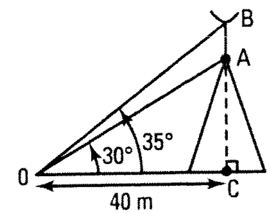


15. From the top of a 20 m high lighthouse, a boat is seen with a 36° angle of depression. If the lighthouse is located at the top of a 45 m cliff, calculate the distance x separating the boat and the base of the cliff.



x = 65 tan 54° = 89.5 m

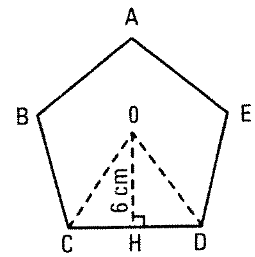
16. A parabolic antenna is located at the top of a tower. An observer located at 0, and 40 m away from the centre of the tower's base, uses a clinometer to measure the angles of elevation to the top of the tower (A) and to the top of the antenna (B). These angles measure 30° and 35° respectively. Calculate the height of the antenna.



$m\overline{AC}$ = 40 tan 30°; $m\overline{BC}$ = 40 tan 35°

$m\overline{AB}$ = $m\overline{BC}$ − $m\overline{AC}$ = 4.9 m

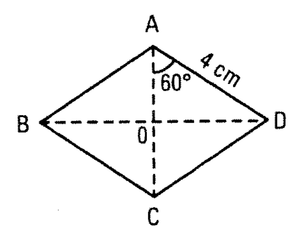
17. What is the area (rounded to the nearest tenth) of a regular pentagon with an apothem equal to 6 cm?



△COD is isosceles; m ∠COD = 72°; m ∠0DH = 54°

$m\overline{HD}$ = 4.36 cm. Area ≃ 130.8 cm²

18. Calculate the area of the rhombus ABCD on the right. (Round the answer to the nearest hundredth.)

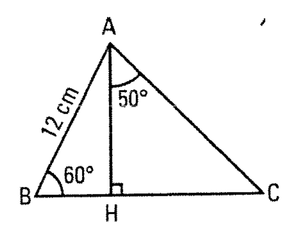


Let O represent the intersection of the diagonals.

The diagonals of a rhombus are perpendicular.

$m\overline{0A}$ = 2cm; $m\overline{0D}$ = 3.46 cm. Area = 13.84 cm²

19. Calculate the area of triangle ABC on the right. (Round the answer to the nearest tenth.)



$m\overline{BH}$ = 6 cm; $m\overline{AH}$ = 10.39 cm;

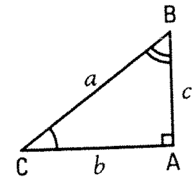
$m\overline{HC}$ = 12.38 cm; $m\overline{BC}$ = 18.38 cm

Area ≃ 95.5 cm²

8.2 Remarkable trigonometric ratios

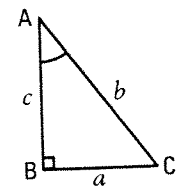
ACTIVITY 1 Properties

a) Using the triangle on the right, show that when two angles are complementary, the sine of one angle is equal to the cosine of the other by comparing



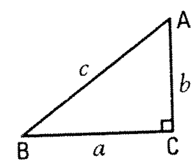
1. sin B and cos C. $\boldsymbol{\sin B = \cos C = \frac{b}{a}}$
2. sin C and cos B. $\boldsymbol{\sin C = \cos B = \frac{c}{a}}$

b) Using the triangle on the right, show that $\tan A = \frac{\sin A}{\cos A}$.



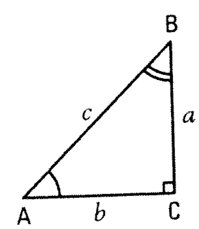
$\boldsymbol{\tan A = \frac{a}{c}; \quad \frac{\sin A}{\cos A} = \frac{\frac{a}{b}}{\frac{c}{b}} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{c}}$

c) Using the triangle on the right, show that $\sin^2 A + \cos^2 A = 1$.



$\boldsymbol{\sin^2 A + \cos^2 A = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1}$

PROPERTIES



- If A and B are two complementary angles, the sine of one is equal to the cosine of the other.

  $\boxed{\sin A = \cos B}$ or $\boxed{\sin B = \cos A}$

- For any angle A: $\boxed{\tan A = \frac{\sin A}{\cos A}}$ $(\cos A \neq 0)$

- For any angle A: $\boxed{\sin^2 A + \cos^2 A = 1}$

  This identity is called the fundamental identity.

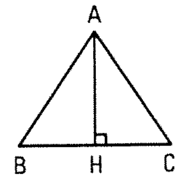
  Ex.: $\sin 30° = \cos 60° = \frac{1}{2}$

  $\tan 60° = \frac{\sin 60°}{\cos 60°} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

  $\sin^2 30° + \cos^2 30° = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$

ACTIVITY 2 Remarkable angles: 0°, 30°, 45°, 60°, 90°

a) Triangle ABC on the right is equilateral, each side measuring 1 unit. The altitude AH is drawn.



1. Explain why $m\overline{BH} = 0.5$ u.

   In an equilateral triangle, the height AH is also a median.

2. Explain why m ∠ABC = m ∠BAC = m ∠ACB = 60°.

   In an equilateral triangle, each angle measures 60°.

3. Explain why m ∠BAH = 30°.

   In an equilateral triangle, the height AH is also a bisector.

4. Refer to triangle ABH to show that $\sin 30° = \frac{1}{2}$.

   $\sin 30° = \frac{0.5}{1} = \frac{1}{2}$

5. Explain why $m\overline{AH} = \frac{\sqrt{3}}{2}$.

   $m\overline{AH}^2 + m\overline{BH}^2 = m\overline{AB}^2$ ***(Pythagorean Theorem)*** $\Rightarrow m\overline{AH}^2 = 1 - (0{,}5)^2 = \frac{3}{4} \Rightarrow m\overline{AH} = \frac{\sqrt{3}}{2}$

6. Refer to triangle ABH to show that $\sin 60° = \frac{\sqrt{3}}{2}$.

   $\sin 60° = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$

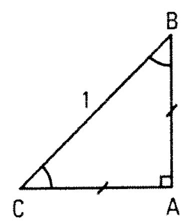
7. Explain why $\cos 60° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.

   $\cos 60° = \sin 30° = \frac{1}{2}$ ***(complementary angles);*** $\cos 30° = \sin 60° = \frac{\sqrt{3}}{2}$ ***(complementary angles)***

8. Explain why $\tan 30° = \frac{\sqrt{3}}{3}$ and $\tan 60° = \sqrt{3}$

   $\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$; $\tan 60° = \frac{\sin 60°}{\cos 60°} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

b) The triangle on the right is isosceles. The hypotenuse measures 1 unit.



1. What is the measure of each side of the right angle?

   Let $m\overline{AB} = m\overline{AC} = x$

   We have: $x^2 + x^2 = 1$ ***(Pythagorean Theorem);*** $2x^2 = 1$; $x = \frac{\sqrt{2}}{2}$.

2. What is the measure of each acute angle? ***45°***

3. Show that $\sin 45° = \frac{\sqrt{2}}{2}$ and that $\cos 45° = \frac{\sqrt{2}}{2}$.

   $\sin 45° = \sin B = \frac{\sqrt{2}}{2}$; $\cos 45° = \cos B = \frac{\sqrt{2}}{2}$

4. Explain why $\tan 45° = 1$.

   $\tan 45° = \frac{\sin 45°}{\cos 45°} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$

c) Use your calculator to verify that sin 0° = 0; sin 90° = 1; cos 0° = 1 and cos 90° = 0.

d) Explain why tan 90° is undefined.

   $\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0}$; ***Division by zero is undefined.***

REMARKABLE ANGLES

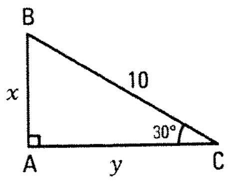
angle	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	

Memorize the row for sine.

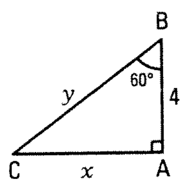
Use the sine row to deduce the row for cosine since $\cos x = \sin(90° - x)$.

Use the sine and cosine rows to deduce the row for tangent since $\tan x = \frac{\sin x}{\cos x}$.

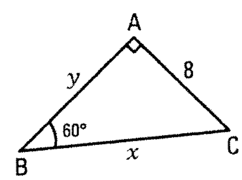
1. Find the exact values of x and y.

a) 

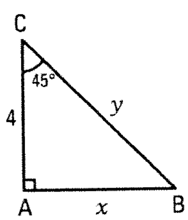
$x = 5;\ y = 5\sqrt{3}$

b) 

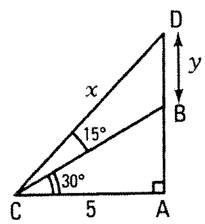
$x = 4\sqrt{3};\ y = 8$

c) 

$x = \frac{16\sqrt{3}}{3};\ y = \frac{8\sqrt{3}}{3}$

d) 

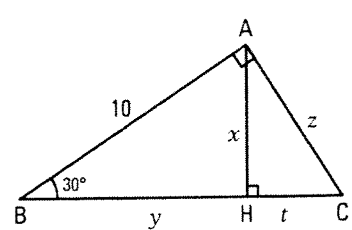
$x = 4;\ y = 4\sqrt{2}$

e) 

$x = 5\sqrt{2};\ y = 5 - \frac{5\sqrt{3}}{3}$

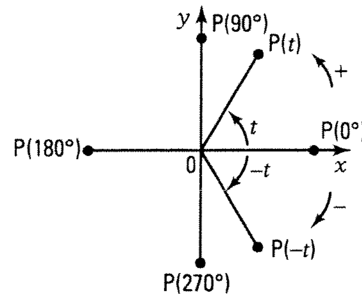
2. In the right triangle ABC on the right, the altitude AH is drawn. Find the exact values of x, y, z and t.

$x = 5;\quad y = 5\sqrt{3};\quad z = \frac{10\sqrt{3}}{3};\quad t = \frac{5\sqrt{3}}{3}$



ACTIVITY 3 The trigonometric circle

The circle on the right centered at 0 (0,0) with radius 1 is called the trigonometric cercle, or unit circle. For each angle t, we associate the point P(t) on the circle called a trigonometric point.

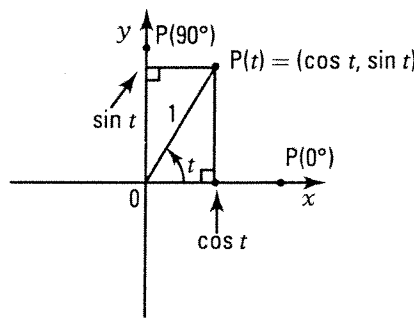


a) Complete the following tables which associate, to each point P(t), its Cartesian coordinates (x, y).

P(t)	(x, y)
P(0°)	(1, 0)
P(90°)	*(0,1)*
P(180°)	*(-1, 0)*
P(270°)	*(0, -1)*
P(360°)	*(1, 0)*

P(t)	(x, y)
P(0°)	*(1, 0)*
P(−90°)	*(0, -1)*
P(−180°)	*(-1, 0)*
P(−270°)	*(0, 1)*
P(−360°)	*(1, 0)*

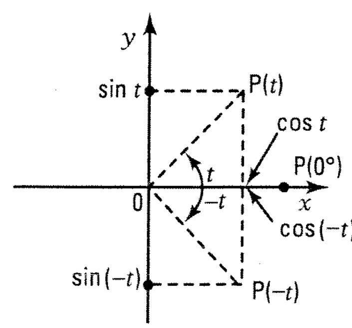
b) If P(t) is a trigonometric point, then its x-coordinate is equal to cos t and its y-coordinate is equal to sin t. Complete the table below which associates, to each remarkable trigonometric point of the 1st quadrant, its exact Cartesian coordinates.



P(t)	P(0°)	P(30°)	P(45°)	P(60°)	P(90°)
(x, y)	*(1, 0)*	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	*(0, 1)*

c) The trigonometric points P(t) and P($-t$) are symmetric about the x-axis. Explain why.

1. cos ($-t$) = cos t. ***P(t) and P(−t) have the same x-coordinate.***
2. sin ($-t$) = −sin t. ***P(t) and P(−t) have opposite y-coordinates.***



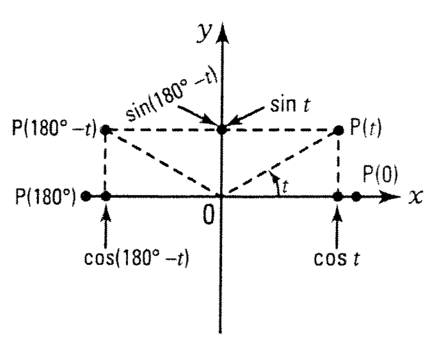
d) The trigonometric points P(t) and P(180° − t) are symmetric about the y-axis. Explain why.

1. cos (180° − t) = −cos t

   P(180° − t) and P(t) have opposite x-coordinates.

2. sin (180° − t) = sin t

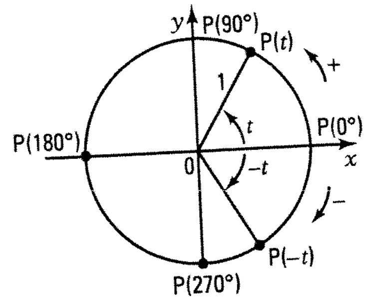
   P(180° − t) and P(t) have the same y-coordinate.



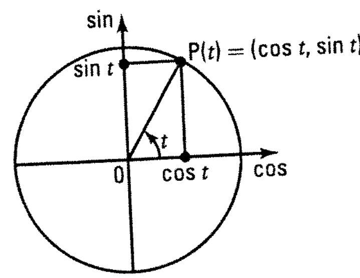
TRIGONOMETRIC CIRCLE

- The trigonometric circle, or unit circle, is the circle centered at the origin O(0,0) with radius 1. Each trigonometric angle t corresponds to a point P(t) on the unit circle called trigonometric point.

  Ex.: P(90°) = (0, 1); P(–90°) = (0, –1)
  P(180°) = (–1, 0); P(–180°) = (–1, 0)



- Given a trigonometric point P(t), cos t represents the x-coordinate of P(t) and sin t represents the y-coordinate.



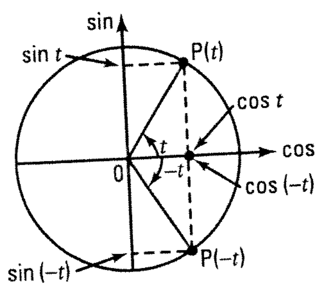
  $$P(t) = (\cos t, \sin t)$$

  The x-axis is thus called the cosine axis and the y-axis is called the sine axis.

  Note:

  $$-1 \leqslant \cos t \leqslant 1 \quad \text{and} \quad -1 \leqslant \sin t \leqslant 1$$

- P(t) and P($-t$) are symmetrical about the cosine axis.

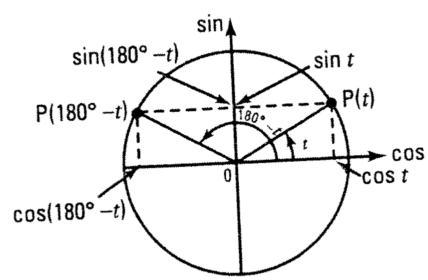


  $$\cos(-t) = \cos t$$
  $$\sin(-t) = -\sin t$$

  Ex.: $\cos(-60°) = \cos 60° = \frac{1}{2}$

  $\sin(-60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$

- P(t) and P(180° – t) are symmetrical about the sine axis.



  $$\cos(180° - t) = -\cos t$$
  $$\sin(180° - t) = \sin t$$

  Ex.: $\cos(150°) = -\cos 30° = -\frac{\sqrt{3}}{2}$

  $\sin(150°) = \sin 30° = \frac{1}{2}$

3. Complete with the appropriate inequality sign ⩾ or ⩽.

a) $0° \leqslant t \leqslant 90° \Rightarrow \cos t$ __⩾__ 0 and $\sin t$ __⩾__ 0

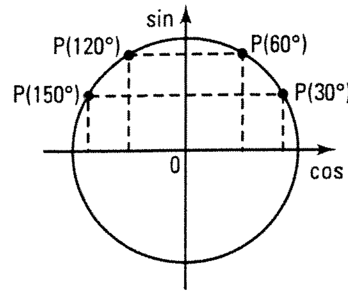
b) $90° \leqslant t \leqslant 180° \Rightarrow \cos t$ __⩽__ 0 and $\sin t$ __⩾__ 0

c) $180° \leqslant t \leqslant 270° \Rightarrow \cos t$ __⩽__ 0 and $\sin t$ __⩽__ 0

d) $270° \leqslant t \leqslant 360° \Rightarrow \cos t$ __⩾__ 0 and $\sin t$ __⩽__ 0

4. a) Knowing that $\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 30° = \frac{1}{2}$, use symmetry about the y-axis to deduce

1. $\cos 150°$. $\frac{-\sqrt{3}}{2}$ 2. $\sin 150°$. $\frac{1}{2}$



b) Knowing that $\cos 60° = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$, use symmetry about the y-axis to deduce

1. $\cos 120°$. $\frac{-1}{2}$ 2. $\sin 120°$. $\frac{\sqrt{3}}{2}$

c) Knowing that $\cos 45° = \frac{\sqrt{2}}{2}$ and $\sin 45° = \frac{\sqrt{2}}{2}$, deduce in the same way

1. $\cos 135°$. $-\frac{\sqrt{2}}{2}$ 2. $\sin 135°$. $\frac{\sqrt{2}}{2}$

5. Use symmetry about the x-axis to find

a) $\cos(-30°)$. $\frac{\sqrt{3}}{2}$ b) $\cos(-45°)$. $\frac{\sqrt{2}}{2}$ c) $\cos(-60°)$. $\frac{1}{2}$

d) $\sin(-30°)$. $\frac{-1}{2}$ e) $\sin(-45°)$. $\frac{-\sqrt{2}}{2}$ f) $\sin(-60°)$. $\frac{-\sqrt{3}}{2}$

6. Knowing that $\tan t = \frac{\sin t}{\cos t}$, find the exact values of

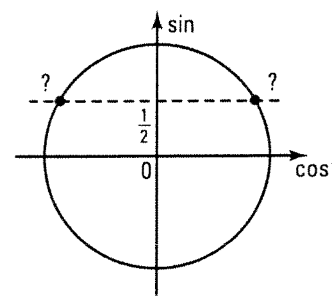
a) $\tan 120°$. $-\sqrt{3}$ b) $\tan 135°$. -1 c) $\tan 180°$. 0

d) $\tan(-30°)$. $-\frac{\sqrt{3}}{3}$ e) $\tan(-45°)$. -1 f) $\tan(-60°)$. $-\sqrt{3}$

ACTIVITY 4 Two angles with the same sine

a) Given $0 \leqslant t \leqslant 180°$, find the two values of t such that $\sin t = \frac{1}{2}$.

$t = 30°$ or $t = 150°$



b) Given $\sin t = 0.6$; the calculator only gives the acute angle t by calculating $\sin^{-1}(0.6)$.

1. What is the solution (acute angle) rounded to the nearest unit? *37°*
2. What is the other solution (obtuse angle)? *143°*

SUPPLEMENTARY ANGLES

- Two supplementary angles have the same sine.

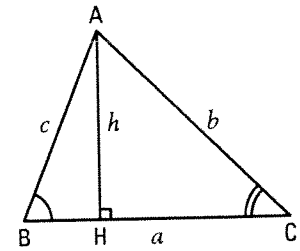
$$\sin t = \sin(180° - t)$$

$\sin 30° = \sin 150° = \frac{1}{2}$; $\sin 60° = \sin 120° = \frac{\sqrt{3}}{2}$; $\sin 40° = \sin 140° = 0.6428$

8.3 Sine and cosine laws

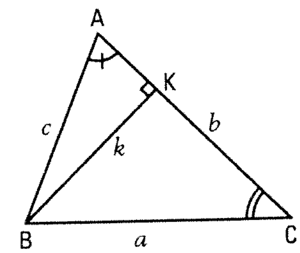
ACTIVITY 1 Sine law

a) In the right triangle ABC on the right, let a, b and c represent the measures of sides BC, AC and AB. The altitude AH is drawn and represented by h. Explain why



1. $h = c \sin B$. ***In the △ABH, we have:*** $\sin B = \frac{h}{c}$.
2. $h = b \sin C$. ***In the △ACH, we have:*** $\sin C = \frac{h}{b}$.
3. $c \sin B = b \sin C$. ***From equations 1 and 2, we deduce the 3rd equation.***
4. $\frac{b}{\sin B} = \frac{c}{\sin C}$. ***Property of cross-products in a proportion.***

b) In the same triangle ABC, the altitude BK is drawn and represented by k. Explain why



1. $k = c \sin A$. ***In the △ABK, we have:*** $\sin A = \frac{k}{c}$.
2. $k = a \sin C$. ***In the △BCK, we have:*** $\sin C = \frac{k}{a}$.
3. $c \sin A = a \sin C$. ***From equations 1 and 2, we deduce the 3rd equation.***
4. $\frac{a}{\sin A} = \frac{c}{\sin C}$. ***Property of cross-products in a proportion.***

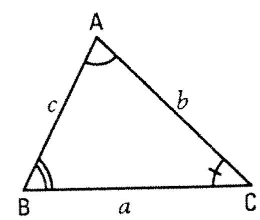
c) Compare the three ratios $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$. What can be deduced?

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

SINE LAW

- The sides in a triangle are directly proportional to the sine of the opposite angles to these sides.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

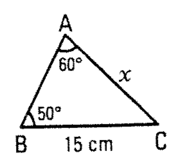


- The sine law can be used to find the measure of a missing side or angle.

1st case: Finding a side when we know two angles and a side.

We calculate the measure x of AC.

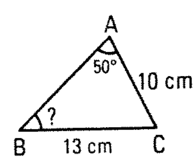
$$\frac{x}{\sin 50°} = \frac{15}{\sin 60°} \Rightarrow x = \frac{15 \sin 50°}{\sin 60°} \approx 13.27 \text{ cm}$$



2nd case: Finding an angle when we know two sides and the opposite angle to one of these two sides.

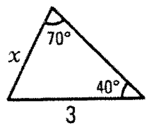
We calculate the measure of angle B.

$$\frac{10}{\sin B} = \frac{13}{\sin 50°} \Rightarrow \sin B = \frac{10 \sin 50°}{13} = 0.5893 \Rightarrow m\angle B \approx 36°$$



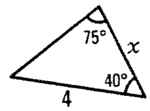
1. Use the sine law to calculate the value of x. (Round to the nearest tenth.)

a)



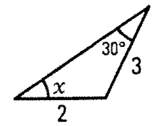
x = 2.1

b)



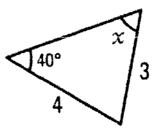
x = 3.8

c)



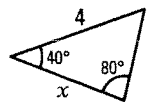
x = 48.6°

d)



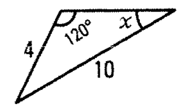
x = 59°

e)



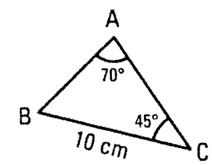
x = 3.5

f)



x = 20.3°

2. Solve the following triangle. (Round each measure to the nearest tenth.)

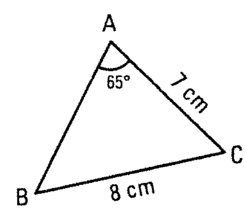


$m\angle B = 65°$

$m\overline{AB} = 7.5$ cm

$m\overline{AC} = 9.6$ cm

3. Solve the following triangle. (Round each measure to the nearest tenth.)

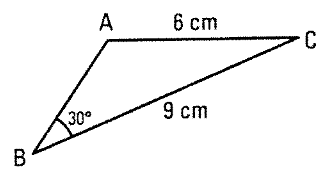


$m\angle B = 52.5°$

$m\angle C = 62.5°$

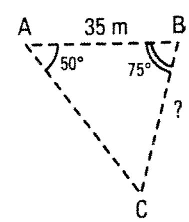
$m\overline{AB} = 7.8$ cm

4. Angle A in the triangle on the right is obtuse. Use the sine law to determine the measure of angle A.



$\sin A = 0.75$; $m\angle A = 180° - 48.6° = 131.4°$

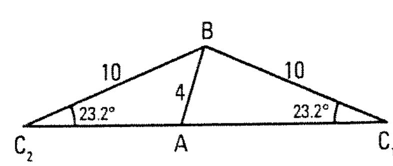
5. Three boats are near each other. What distance separates boats B and C if the distance, to the nearest metre, separating boats A and B is 35m?



$$\frac{\sin 55°}{35} = \frac{\sin 50°}{x} \Rightarrow x = \frac{35 \sin 50°}{\sin 55°} = 32.7 \text{ m}$$

The distance separating B and C is 33 m.

6. There exists two triangles where $m\angle C = 23.2°$, $m\overline{AB} = 4$ and $m\overline{BC} = 10$. These two triangles ABC_1 and ABC_2 are represented on the right.



a) Solve both triangles.

$m\overline{AC}_1 = 9.9$; $m\overline{AC}_2 = 8.5$

$m\angle A_1 = 80°$, $m\angle B_1 = 76.8°$

$m\angle A_2 = 100°$, $m\angle B_2 = 56.8°$

b) Explain why the angles BAC_1 and BAC_2 have the same sine?

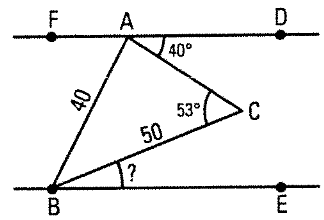
They are supplementary angles.

7. There are two possible ways to represent a triangle ABC with m $\angle C = 31{,}3°$, m$\overline{AB} = 6$ and m$\overline{BC} = 10$, one where angle A is acute and one where angle A is obtuse. Indicate for each case the measure of each angle and each side of the triangle.

1st case: m $\angle A = 60°$; m $\angle B = 89°$; m$\overline{AC} = 11.5$

2nd case: m $\angle A = 120°$; m $\angle B = 29°$; m$\overline{AD} = 5.6$

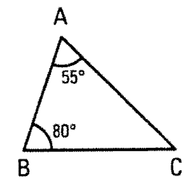
8. Lines AD and BE on the right are parallel, m $\angle CAD = 40°$, m $\angle ACB = 53°$, m$\overline{AB} = 40$, m$\overline{BC} = 50$. What is, to the nearest degree, the measure of angle CBE?



$\dfrac{\sin A}{50} = \dfrac{\sin 53°}{40} \Rightarrow m\angle BAC = 86.7° \Rightarrow m\angle FAB = 53.3°$

$m\ \angle ABE = 53.3°;\ m\ \angle ABC = 40.3°;\ m\ \angle CBE = 13°$

9. The perimeter of triangle ABC on the right is equal to 153 cm. Find, to the nearest unit, each side of this triangle.

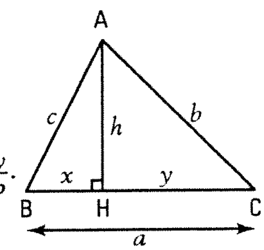


$\dfrac{a}{\sin 55°} = \dfrac{b}{\sin 80°} = \dfrac{c}{\sin 45°} = \dfrac{a+b+c}{\sin 55° + \sin 80° + \sin 45°} = \dfrac{153}{2.511}$

a = 50 cm; b = 60 cm; c = 43 cm

ACTIVITY 2 Cosine law

In the triangle ABC on the right, the altitude AH is drawn.



a) Explain why

1. $h = c \sin B$ and $x = c \cos B$. ***$\triangle ABH$ is a right triangle $\Rightarrow \sin B = \frac{h}{c}$ and $\cos B = \frac{x}{c}$.***
2. $h = b \sin C$ and $y = b \cos C$. ***$\triangle ACH$ is a right triangle $\Rightarrow \sin C = \frac{h}{b}$ and $\cos C = \frac{y}{b}$.***

b) Justify the steps of the following proof.

Statement	Justification
$c^2 = x^2 + h^2$	***Pythagorean Theorem applied to $\triangle ABH$***
$c^2 = (a - y)^2 + h^2$	***$a = x + y \Rightarrow x = a - y$***
$c^2 = a^2 - 2ay + y^2 + h^2$	***Expansion of the square***
$c^2 = a^2 - 2ab \cos C + b^2 \cos^2 C + b^2 \sin^2 C$	***Substitution: $y = b \cos C$ and $h = b \sin C$***
$c^2 = a^2 - 2ab \cos C + b^2 (\cos^2 C + \sin^2 C)$	***Common factor***
$c^2 = a^2 - 2ab \cos C + b^2$	***Fundamental identity ($\sin^2 C + \cos^2 C = 1$)***
$c^2 = a^2 + b^2 - 2ab \cos C$	***Commutative property of addition.***

c) Using a similar proof to b), prove that $b^2 = a^2 + c^2 - 2ac \cos B$.

$b^2 = y^2 + h^2$

$b^2 = (a - x)^2 + h^2$

$b^2 = a^2 - 2ax + x^2 + h^2$

$b^2 = a^2 - 2ac \cos B + c^2 \cos^2 B + c^2 \sin^2 B$

$b^2 = a^2 - 2ac \cos B + c^2(\cos^2 B + \sin^2 B)$

$b^2 = a^2 - 2ac \cos B + c^2$

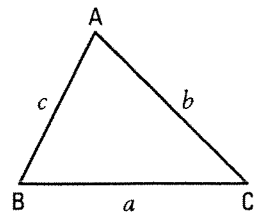
$b^2 = a^2 + c^2 - 2ac \cos B$

d) Use the results obtained in b) and c) to complete the following

$a^2 = b^2 + c^2 -$ ***$2bc \cos A$***

COSINE LAW

- The square of any side in a triangle is equal to the sum of the squares of the other two sides minus twice the product of these sides by the cosine of the angle between these two sides.



$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

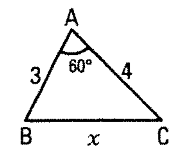
- The cosine law can be used to find the measure of a side or an angle.

1st case: Finding a side when the other two sides, and the angle contained between them, are given.

We calculate the measure x of side BC.

$x^2 = 3^2 + 4^2 - 2(3)(4) \cos 60° = 9 + 16 - 24 \times \frac{1}{2} = 13$

$x = \sqrt{13} \approx 3{,}6$

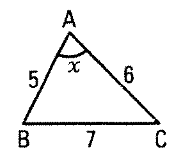


2nd case: Finding an angle when the three sides are given.

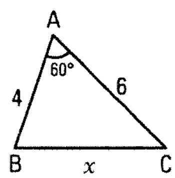
We calculate the measure of angle A.

$7^2 = 5^2 + 6^2 - 2(5)(6) \cos A$

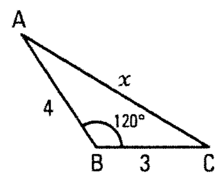
$49 = 61 - 60 \cos A \Rightarrow \cos A = 0.2 \Rightarrow m\,\angle A = 78.5°$



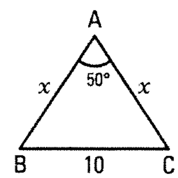
10. Use the cosine law to calculate the value of x. (Round to the nearest tenth.)

a) 

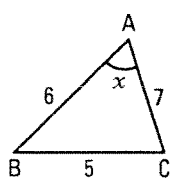
x = 5.3

b) 

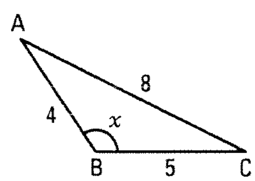
x = 6.1

c) 

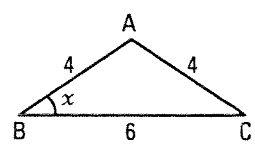
x = 11.8

d) 

x = 44.4°

e) 

x = 125.1°

f) 

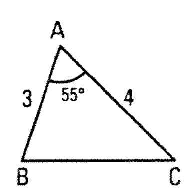
x = 41.4°

11. Solve the following triangle.

m$\overline{BC}$ = 3.35

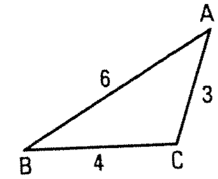
m ∠B = 78°

m ∠C = 47.2°



12. Solve the triangle on the right. (Round each measure to the nearest tenth.)

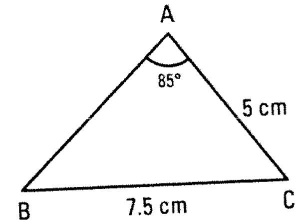
m ∠A = 36.3°; m ∠B = 26.4°; m ∠C = 117.3°



13. In the triangle on the right, we have: $m\overline{AC}$ = 5 cm, $m\overline{BC}$ = 7.5 cm and m ∠A = 85°.
What is, to the nearest tenth, the measure of side AB?

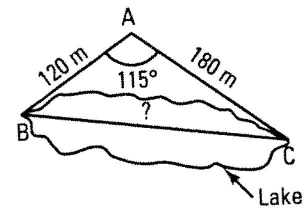
m ∠B = 41.6°; m ∠C = 53.4°; m$\overline{AB}$ = 6.04

The measure of $\overline{AB}$, to the nearest tenth, is 6 cm.



14. An engineer wants to determine the distance between two cottages on opposite sides of a lake. The engineer is located at point A and the cottages are at B and C. What is, to the nearest metre, the distance separating the two cottages?

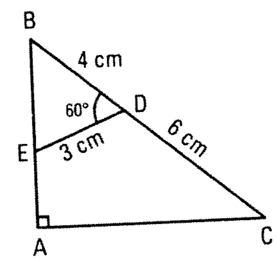
255 m



15. In the triangle ABC on the right, segment DE is drawn.
m ∠BDE = 60°, $m\overline{BD}$ = 4 cm, $m\overline{DE}$ = 3 cm and $m\overline{DC}$ = 6 cm.
What is, to the nearest hundredth, the measure of side AC?

m$\overline{BE}$ = $\sqrt{13}$ ≈ 3.61 cm; m ∠B ≈ 46°;

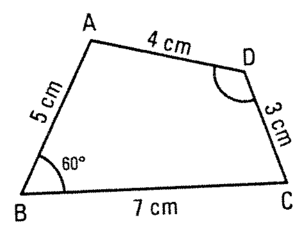
m$\overline{AC}$ = 10 sin 46° = 7.19 cm



16. In the quadrilateral ABCD on the right, we have: $m\overline{AB}$ = 5 cm, $m\overline{BC}$ = 7 cm, $m\overline{CD}$ = 3 cm, $m\overline{AD}$ = 4 cm and m ∠ABC = 60°.
What is, to the nearest degree, the measure of angle ADC?

m$\overline{AC}$ = 6.24 cm; m ∠ADC = 125.7°

Angle D measures 126° to the nearest degree.

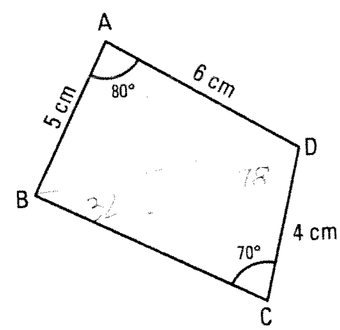


17. In the quadrilateral on the right, we have: $m\overline{AB}$ = 5 cm, $m\overline{AD}$ = 6 cm, $m\overline{CD}$ = 4 cm, m ∠BAD = 80° and m ∠BCD = 70°.
What is, to the nearest tenth, the measure of segment BC?

m$\overline{BD}$ = 7.11 cm; m ∠DBC = 31.9°; m ∠BDC = 78.1°

m$\overline{BC}$ = 7.40 cm.

The measure of $\overline{BC}$, to the nearest tenth, is 7.4 cm.

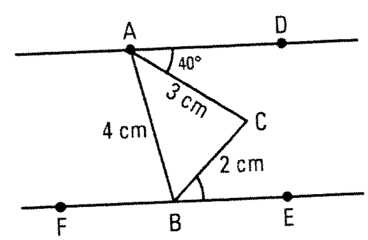


18. In the figure on the right, lines AD and FE are parallel, $m\overline{AC}$ = 3 cm, $m\overline{BC}$ = 2 cm and $m\overline{AB}$ = 4 cm.
What is, to the nearest degree, the measure of angle CBE?

m ∠BAC ≈ 29°, m ∠ABC = 46.7°

m ∠ABF = 69°, m ∠CBE = 64.3°

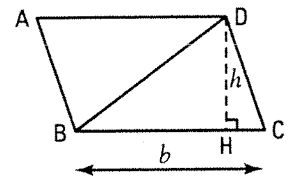
Angle CBE measures 64° to the nearest degree.



8.4 Area of a triangle

Activity 1 General formula

On the right, parallelogram ABCD and its altitude DH are represented. Let b represent the parallelogram's base and h the height.



a) What is the area of the parallelogram? ***$b \times h$***

b) Explain why triangles ABD and BCD are congruent.

$m\angle ADB = m\angle DBC$ (alternate-interior angles)

$m\angle ABD = m\angle BDC$ (alternate-interior angles)

$\overline{BD}$ is a common side to both triangles.

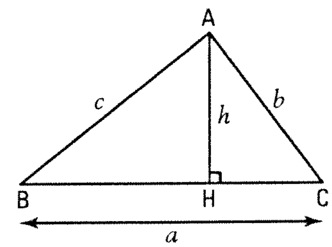
$\triangle ABD \cong \triangle CDB$ (Theorem of congruence ASA)

c) 1. Establish the formula that can be used to calculate the area of triangle BCD. ***Area $= \frac{b \times h}{2}$***

2. Calculate the area of triangle BCD if $b = 6$ cm and $h = 4$ cm. ***$A = 12\ cm^2$***

Activity 2 Trigonometric formula

Triangle ABC and its height AH are represented on the right. Let a, b and c represent the measure of the sides and h represent the height.



a) Justify the steps that establish the formula for calculating the area of triangle ABC.

1. Area $= \frac{a \cdot h}{2}$ ***General formula (activity 1)***

2. Area $= \frac{a \cdot c \sin B}{2}$ or Area $= \frac{a \cdot b \sin C}{2}$ ***$h = c \sin B$ or $h = b \sin C$***

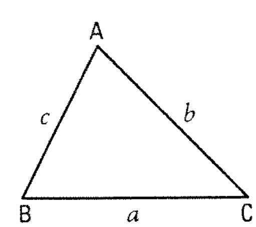
b) Calculate the area of triangle ABC if

1. $a = 6$ cm, $c = 4$ cm and $m\angle B = 30°$. ***Area $= \frac{6 \times 4 \times \sin 30°}{2} = 6\ cm^2$***

2. $a = 8$ cm, $b = 3$ cm and $m\angle C = 60°$. ***Area $= \frac{8 \times 3 \times \sin 60°}{2} = 6\sqrt{3}\ cm^2$***

Activity 3 Hero's formula

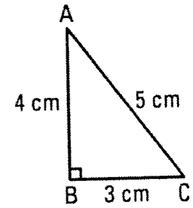
When you are given the measures of all three sides a, b and c of a triangle, Hero's formula enables you to calculate the area of the triangle.



$A = \sqrt{p(p-a)(p-b)(p-c)}$ where p represents half the perimeter of the triangle.

$p = \frac{1}{2}(a + b + c)$

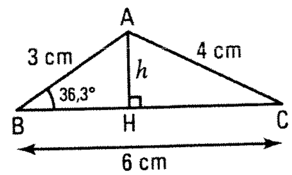
a) Calculate the area of the given right triangle using



1. the general formula. $A = \frac{3 \times 4}{2} = 6\ cm^2$
2. Hero's formula. $p = 6\ cm;\ A = \sqrt{6 \cdot (6-4) \cdot (6-3) \cdot (6-5)}$

$A = \sqrt{36} = 6\ cm^2$

b) Calculate, to the nearest hundredth, the area of the triangle on the right using



1. the trigonometric formula (activity 2).

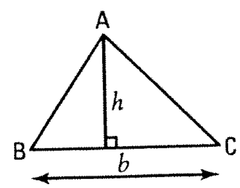
$A = \frac{ac \sin B}{2} = \frac{6 \cdot 3 \sin 36.3°}{2} = 5.33\ cm^2.$

2. Hero's formula. $A = \sqrt{6.5 \times 3.5 \times 0.5 \times 2.5} = \sqrt{28.4375} = 5.33\ cm^2$

AREA OF A TRIANGLE

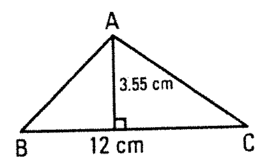
- **General formula**

Given the measure of the base b and the height h relative to that base, the area A of the triangle is given by:



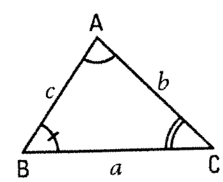
$$A = \frac{b \times h}{2}$$

Ex.: $A = \frac{12 \times 3.55}{2} = 21.3\ cm^2$



- **Trigonometric formula**

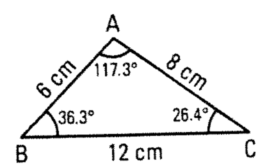
In a triangle, given the measure of an angle and its two sides,



$$A = \frac{ac \sin B}{2} \quad \text{or} \quad A = \frac{ac \sin B}{2} \quad \text{or} \quad A = \frac{bc \sin A}{2}$$

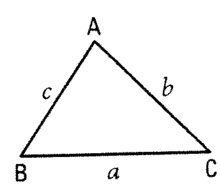
Ex.: $A = \frac{12 \times 6 \times \sin 36.3°}{2}$ or $A = \frac{12 \times 8 \times \sin 26.4°}{2}$ or $A = \frac{6 \times 8 \times \sin 117.3°}{2}$

$A = 21.3\ cm^2$



- **Hero's formula**

In a triangle, given the measures of all three sides,

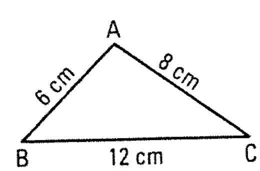


$$A = \sqrt{p(p-a)(p-b)(p-c)}$$

where p represents half of the triangle's perimeter. $p = \frac{a+b+c}{2}$.

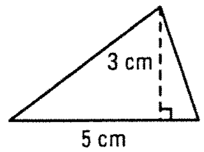
Ex.: $p = \frac{1}{2}(6 + 12 + 8) = 13$ cm

$A = \sqrt{13 \times 7 \times 1 \times 5} = \sqrt{455} \simeq 21.3\ cm^2$



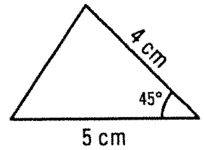
1. Calculate the area of the following triangles using an appropriate formula.

a)



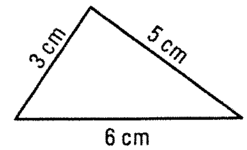
A = 7.5 cm²

b)



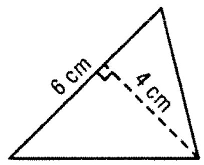
A = 7.07 cm²

c)



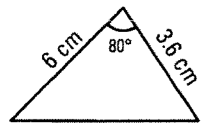
A = 7.48 cm²

d)



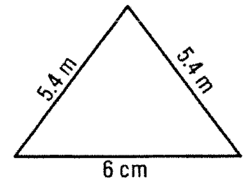
A = 12 cm²

e)



A = 10.64 cm²

f)

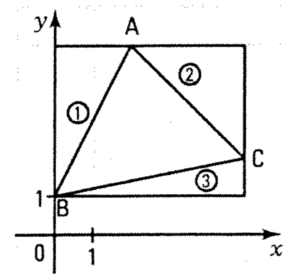


A = 13.47 cm²

2. Calculate the area of triangle ABC with vertices A(2, 5), B(0, 1) and C(5, 2).

Area of rectangle = 20 u²; A_1 = 4 u²; A_2 = 4.5 u²; A_3 = 2.5 u²

Area △ABC = 20 − (4 + 4.5 + 2.5) = 9 u²

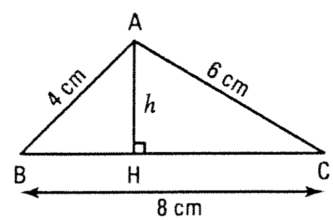


3. Calculate, to the nearest unit, the length of the altitude AH of triangle ABC on the right.

Area = $\frac{8 \times h}{2}$ = 4 h

Area = $\sqrt{9 \times 5 \times 1 \times 3}$ = 11.62 cm²

h = 3 cm

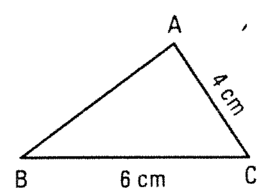


4. The area of the triangle on the right is equal to 11.6 cm². What is, to the nearest degree, the measure of angle C?

Area = $\frac{6 \times 4 \times \sin C}{2}$ = 12 sin C

12 sin C = 11.6 ⇒ m ∠C = 75.2°

Angle C measures 75°.



5. Justify the steps in the proof of the sine law.

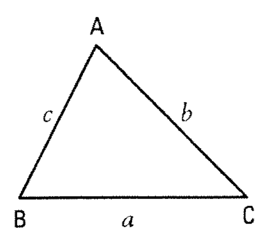
1. $\frac{bc \sin A}{2} = \frac{ac \sin B}{2} = \frac{ab \sin C}{2}$

   The 3 trigonometric formulas for calculating the area of triangle ABC.

2. $bc \sin A = ac \sin B = ab \sin C$ ***Multiply both sides by 2.***

3. $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ ***Divide each side by abc.***

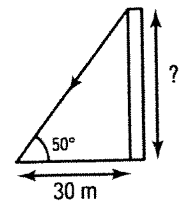
4. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ ***Invert the ratios.***



Evaluation 8

1. The shadow cast by a building measures 30 m when the sun's rays hit the ground at an angle of 50°. What is, to the nearest metre, the height of the building?

36 m



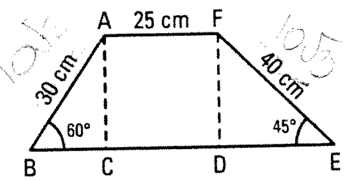
2. Calculate, to the nearest unit, the area of the trapezoid on the right.

m$\overline{BC}$ = 30 cos 60° = 15 cm; m$\overline{DE}$ = 40 cos 45° = 28.28 cm

m$\overline{AC}$ = 30 sin 60° = 25.98 cm

A = (68.28 + 25) × 25.98 ÷ 2 = 1211.71 cm²

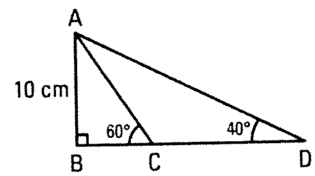
The area is equal to 1212 cm².



3. In the figure on the right, we have:
m$\overline{AB}$ = 10 cm; m ∠ACB = 60°; m ∠ADC = 40°.
Calculate, to the nearest tenth, the measure of segment CD.

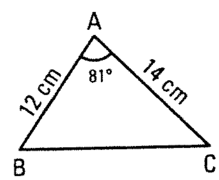
m$\overline{BD}$ = $\frac{10}{\tan 40°}$ = 11.92 cm; m$\overline{BC}$ = $\frac{10}{\tan 60°}$ = 5.77 cm

m$\overline{CD}$ = 6.2 cm



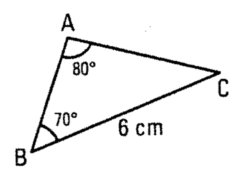
4. Solve the triangle on the right. (Round each measure to the nearest unit.)

$\overline{BC}$ = 17 cm; m ∠B = 54°; m ∠C = 45°



5. Solve the triangle on the right. (Round each measure to the nearest tenth.)

m$\overline{AC}$ = 5.7 cm; m$\overline{AB}$ = 3 cm; m ∠C = 30°

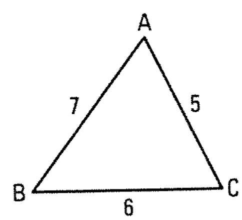


6. Solve the triangle on the right. (Round each measure to the nearest tenth.)

m ∠A = 57.1°

m ∠B = 44.4°

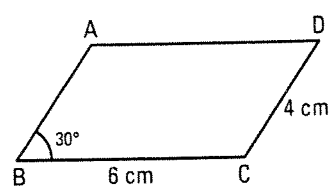
m ∠C = 78.5°



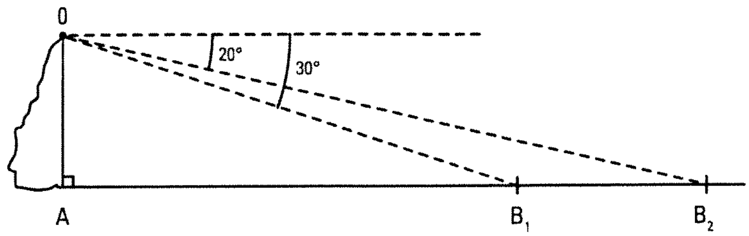
7. The consecutive sides of a parallelogram measure respectively 4 cm and 6 cm. If one of the acute angles measures 30°, calculate, to the nearest tenth, the length of each diagonal.

big diagonal: 9.7 cm

small diagonal: 3.2 cm



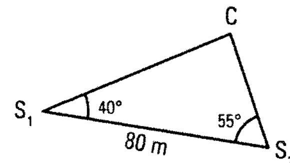
8. From the top of a 40 m cliff, two boats B_1 and B_2 are simultaneously observed, the first under a 30° angle of depression and the second under a 20° angle of depression. What distance, to the nearest metre, separates the two boats?



$m\overline{AB}_2 = 40 \tan 70° = 109.9$ m; $m\overline{AB}_1 = 40 \tan 60° = 69.28$ m

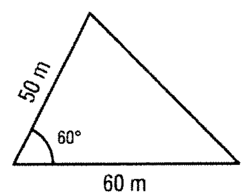
$d(B_1, B_2) = 40.62$ m ≈ 41 m

9. Two skiers S_1 and S_2 are 80 m apart and see the chalet located at C with angles of 40° and 55° respectively. What is the distance, to the nearest metre, of each skier to the chalet?



$d(S_1, C) = 65.78 \approx 66$ m; $d(S_2, C) = 51.62 \approx 52$ m

10. The triangular field on the right sells for $40 per square metre. The field must be surrounded by a fence costing $12 per metre. What is, to the nearest dollar, the total cost of the field and its fence?



Area of the field $= \frac{60 \times 50 \times \sin 60°}{2} = 1299$ m^2

Cost of the field $\approx$ $51 960

Perimeter of the field = 60 + 50 + 55.67 = 165.7 m

Cost of the fence $\approx$ $1988; Total cost = $53 948

11.)