Chapter 6

Vectors

CHALLENGE 6

1. Consider the line segment AB with endpoints A(−1, 3) and B(5, 6).

Find the coordinates of the point P located on line segment AB if $\frac{mAP}{mAB} = \frac{2}{3}$.

$\overrightarrow{AP} = (x + 1, y - 3)$; $\overrightarrow{AB} = (6, 3)$; $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB} = (4, 2) \Rightarrow P(3, 5)$.

2. Two perpendicular forces applied to an object give a resultant force of 30 N. If the resultant has an orientation of 60°, determine, rounded to the nearest unit, the norms of the two forces.

$\|\overrightarrow{F_1}\| = \|\overrightarrow{R}\| \cos 60° = 30 \cos 60° = 15$ N

$\|\overrightarrow{F_2}\| = \|\overrightarrow{R}\| \sin 60° = 30 \sin 60° = 26$ N

3. Amelia and Ben are pulling on an object. They apply respectively forces of 100 N and 80 N, with respective orientation of 40° and 120°. Claudio claims he is able, by himself, to cause the same effect on the object.

Determine the force (magnitude and orientation) that Claudio must apply to the object.

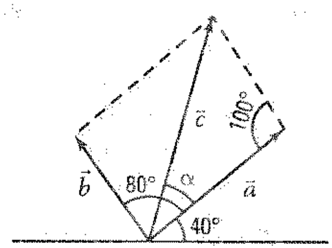
$\|\vec{c}\|^2 = 80^2 + 100^2 - 2 \times 80 \times 100 \cos 100° = 19178.4$

$\|\vec{c}\| = 138.5$; $\frac{\sin \alpha}{80} = \frac{\sin 100}{138.5} \Rightarrow \sin \alpha = 0.5689$

$\alpha = 34.7°$

Claudio applies a force of 138.7 N with orientation

$\theta_{\vec{c}} = 74.7°$.



4. Calculate the acute angle formed by the lines $l_1 : y = 2x + 1$ and $l_2 : y = \frac{1}{2}x - 3$.

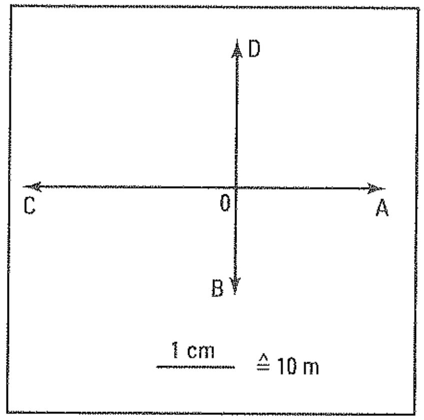
$A_1(0, 1) \in l_1$; $B_1(1, 3) \in l_1$; $A_2(0, -3) \in l_2$; $B_2(2, -2) \in l_2$; $\overrightarrow{A_1B_1} = (1, 2)$; $\overrightarrow{A_2B_2} = (2, 1)$;

$\cos\theta = \frac{\overrightarrow{A_1B_1} \cdot \overrightarrow{A_2B_2}}{\|\overrightarrow{A_1B_1}\|\|\overrightarrow{A_2B_2}\|} = \frac{4}{5} \Rightarrow \theta = 36.9°$.

6.1 Geometric vectors

Activity 1 Concept of vector

Four archaeologists Alex, Bridget, Celia and Dennis are conducting an archaeological excavation. They leave the research centre located at point 0. Alex moves 20 m eastwards and Bridget moves 15 m southwards. Each motion is represented by a directed arrow with length proportional to the distance traveled.



a) Represent, using an arrow, the following motions.
1. Celia moves 30 m westwards.
2. Dennis moves 20 m northwards.

b) Alex and Dennis travel the same distance. Explain why, however, the arrows representing Alex's and Dennis' motions are not the same.

They don't move in the same direction.

*Some observations can be described by a single number, like the number of passengers on a train, while describing the motion of the train requires **2 characteristics**, namely the **direction** of the motion and its **magnitude** (speed of the train).*

VECTOR AND SCALAR

- A number that can, by itself, describe a quantity is called a **scalar**.
  Ex.: The age, height, weight of a person are scalars.
- The two characteristics, direction and magnitude, required to describe an observation are called a **vector**.
  Ex.: The observation of a moving train, the blowing wind or the flow of a river is described by a vector indicating the direction and magnitude of the motion.

1. For each of the following observations, indicate if it can be described by a scalar or a vector.

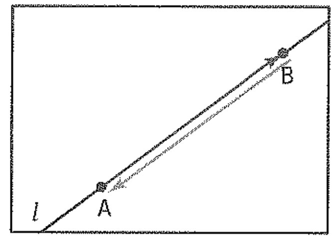
a) Temperature ***Scalar*** **b)** Speed of a plane ***Vector***

c) Volume of a solid ***Scalar*** **d)** Earth's gravity ***Vector***

e) Mass of an atom ***Scalar*** **f)** Motion of a boat ***Vector***

ACTIVITY 2 Description of a geometric vector

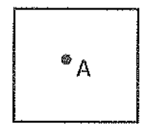
a) The arrow on the right is the geometric representation of vector AB. This vector is written $\overrightarrow{AB}$.



- A is called the **origin** and B is called the **endpoint** of the vector.
- The arrow gives the **direction** of the vector.
- The **norm** of the vector corresponds to the length of the line segment AB.

1. Draw vector $\overrightarrow{BA}$.
2. What is the origin of vector $\overrightarrow{BA}$? ***B***
3. What is the endpoint of vector $\overrightarrow{BA}$? ***A***
4. Is it true that $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have opposite directions?
   Yes
5. Do vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same norm? ***Yes***

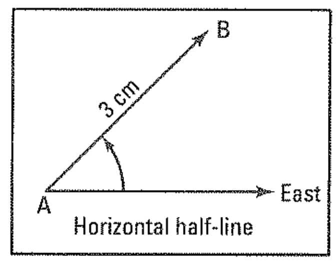
b) Consider vector $\overrightarrow{AA}$ represented on the right, having its origin equal to its endpoint.



What is the norm of vector $\overrightarrow{AA}$? ***0***

This vector, written $\vec{0}$, is called **zero vector**.

c) To describe vector $\overrightarrow{AB}$ on the right, we proceed in two steps.



1st step:
We measure the norm or length of vector $\overrightarrow{AB}$.

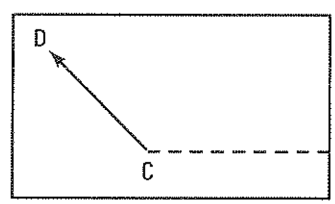
The norm of vector $\overrightarrow{AB}$ is written $\|\overrightarrow{AB}\|$. Here, we have $\|\overrightarrow{AB}\| = 3$ cm.

2nd step:
We draw the horizontal half-line oriented eastwards and passing through the origin A of the vector, then we measure the directed angle (counterclockwise direction) having initial side the horizontal half-line and terminal side the vector $\overrightarrow{AB}$.

This directed angle, written $\theta_{\overrightarrow{AB}}$, gives the orientation of vector $\overrightarrow{AB}$ and thus defines the direction of $\overrightarrow{AB}$. Here, we have $\theta_{\overrightarrow{AB}} = 45°$.

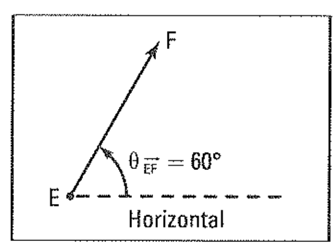
Vector $\overrightarrow{AB}$ is described by giving its norm, $\|\overrightarrow{AB}\|$, and its orientation, $\theta_{\overrightarrow{AB}}$.

1. Describe vector $\overrightarrow{CD}$ represented on the right.

   Vector $\overrightarrow{CD}$ has norm 2 cm and orientation $\theta_{\overrightarrow{CD}}$ = 135°.

2. Represent vector $\overrightarrow{EF}$ having norm 2.5 cm and orientation 60°.

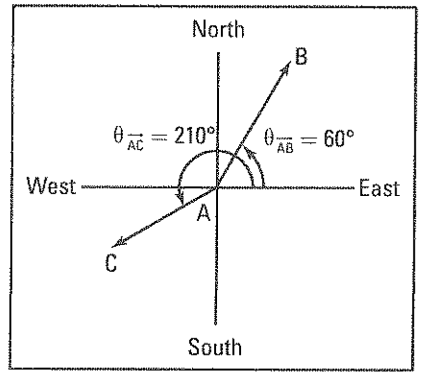
d) Vector $\overrightarrow{AB}$ on the right has a norm of 2 cm and an orientation $\theta_{\overrightarrow{AB}} = 60°$.

This vector is located 30° East of North.

It is written $\overrightarrow{AB}$: 2 cm [N 30° E].

Vector $\overrightarrow{AC}$ on the right has norm 1.5 cm and orientation $\theta_{\overrightarrow{AC}} = 210°$.

Vector $\overrightarrow{AC}$ is written $\overrightarrow{AC}$: 1.5 cm [W 30° S] or $\overrightarrow{AC}$: 1.5 cm [S 60° W].



1. Interpret the notation $\overrightarrow{AC}$: 1.5 cm [W 30° S].

   Vector $\overrightarrow{AC}$, with norm 1.5 cm, is 30° South of West.

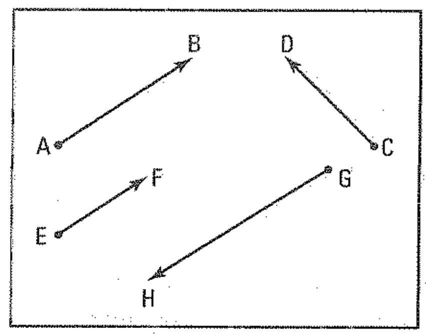
2. Interpret the notation $\overrightarrow{AC}$: 1.5 cm [S 60° W].

   Vector $\overrightarrow{AC}$, with norm 1.5 cm, is 60° West of South.

DESCRIPTION OF A GEOMETRIC VECTOR

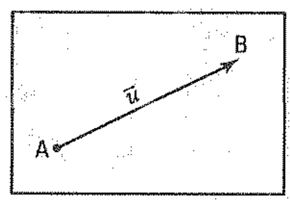
- A **geometric vector** is described by:
  - its **direction**,
  - its **norm** (or **length**).

  Ex.: On the figure on the right,
  - vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ do not have the same direction.
  - vectors $\overrightarrow{AB}$ and $\overrightarrow{EF}$ have same the direction.
  - vectors $\overrightarrow{AB}$ and $\overrightarrow{GH}$ have opposite directions.



- Vector $\overrightarrow{AB}$ with origin A and endpoint B can be written using a letter u, v, ... with an arrow above it.

  $\vec{u}$ and $\overrightarrow{AB}$ are two different notations representing the same geometric vector.

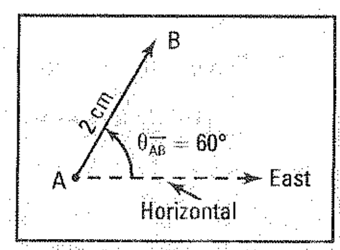


- Vector $\overrightarrow{AB}$ on the right, with norm 2 cm, has an orientation of 60°.
  - The **norm** of vector $\overrightarrow{AB}$ is written $\|\overrightarrow{AB}\|$.
  - The **orientation** of vector $\overrightarrow{AB}$ is written $\theta_{\overrightarrow{AB}}$.

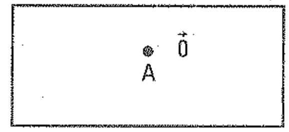
  This angle, directed in the counterclockwise direction, has for initial side the horizontal half-line directed eastwards and terminal side the vector $\overrightarrow{AB}$.

  The orientation $\theta_{\overrightarrow{AB}}$ of a vector $\overrightarrow{AB}$ gives the direction of this vector.

  We have: $0° \leq \theta_{\overrightarrow{AB}} < 360°$



- The vector $\overrightarrow{AA}$, with origin and endpoint A, is called **zero vector**. It is written $\vec{0}$.

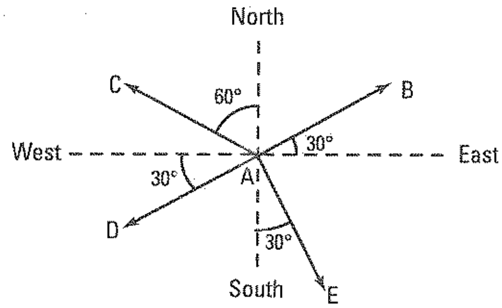


  We have: $\|\vec{0}\| = 0$. The zero vector has every direction.

- A vector $\vec{u}$ is called **unit vector** if $\|\vec{u}\| = 1$.

  Ex.: Vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ and $\overrightarrow{AE}$ on the right all have the same norm, 2 cm, but different orientations. We have:

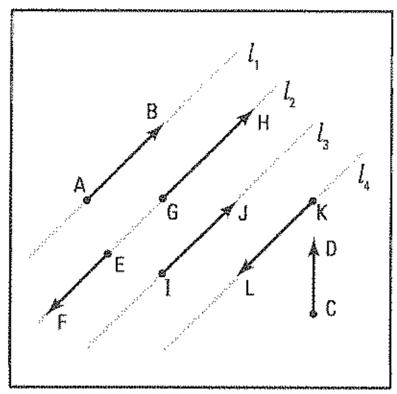
  $\theta_{\overrightarrow{AB}} = 30°, \theta_{\overrightarrow{AC}} = 150°, \theta_{\overrightarrow{AD}} = 210°$ and $\theta_{\overrightarrow{AE}} = 300°$.



- We usually describe a vector by giving its norm and a rotation angle spanning two consecutive cardinal points.

  Ex.: – Vector $\overrightarrow{AB}$ on the right, with norm 2 cm, is oriented 60° East of North. We write: 2 cm [N 60° E].
  – Vector $\overrightarrow{AC}$, with norm 2 cm, is oriented 60° West of North. $\overrightarrow{AC}$: 2 cm [N 60° W].
  – Vector $\overrightarrow{AD}$, with norm 2 cm, is oriented 30° South of West. $\overrightarrow{AD}$: 2 cm [W 30° S].
  – Vector $\overrightarrow{AE}$, with norm 2 cm, is oriented 30° East of South. $\overrightarrow{AE}$: 2 cm [S 30° E].

2. Lines l_1, l_2, l_3 and l_4 on the right are parallel.
Compare each vector on the right with vector $\overrightarrow{AB}$ according to direction and norm.



	Same direction as $\overrightarrow{AB}$	Direction opposite to $\overrightarrow{AB}$	Same norm as $\overrightarrow{AB}$
$\overrightarrow{CD}$	*No*	*No*	*No*
$\overrightarrow{EF}$	*No*	*Yes*	*No*
$\overrightarrow{GH}$	*Yes*	*No*	*No*
$\overrightarrow{IJ}$	*Yes*	*No*	*Yes*
$\overrightarrow{KL}$	*No*	*Yes*	*Yes*

3. Represent the following vectors.

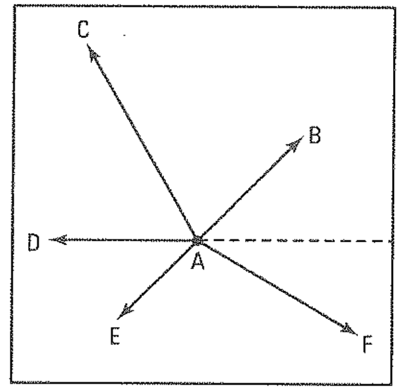
a) $\overrightarrow{AB}$ of norm 2 cm and orientation 45°.

b) $\overrightarrow{AC}$ of norm 3 cm and orientation 120°.

c) $\overrightarrow{AD}$ of norm 2 cm and orientation 180°.

d) $\overrightarrow{AE}$ of norm 1.5 cm and orientation 225°.

e) $\overrightarrow{AF}$ of norm 2.5 cm and orientation 330°.



4. Represent the following vectors.

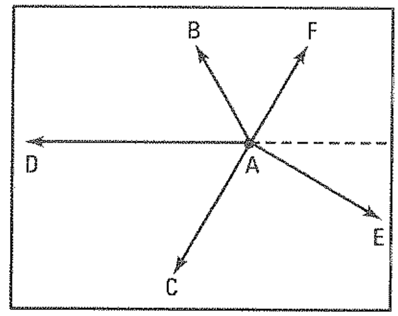
a) $\overrightarrow{AB}$: 1.5 cm [N 30° W].

b) $\overrightarrow{AC}$: 2 cm [S 30° W].

c) $\overrightarrow{AD}$: 3 cm [W].

d) $\overrightarrow{AE}$: 2 cm [E 30° S].

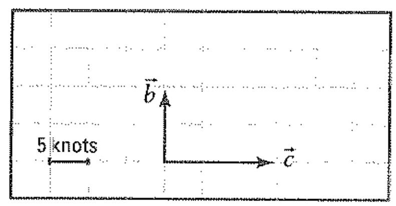
e) $\overrightarrow{AF}$: 1.5 cm [N 30° E].



5. Represent the following situations.

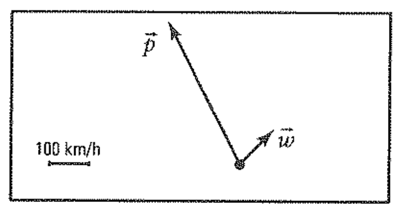
a) A boat crosses a river perpendicularly. The speed of the boat is 10 knots and the speed of the current is 15 knots.

***$\vec{b}$* : speed of boat, *$\vec{c}$* : current speed.**



b) A plane is moving at 400 km/h 30° West of North and the wind blows at 100 km/h 45° in the East of North.

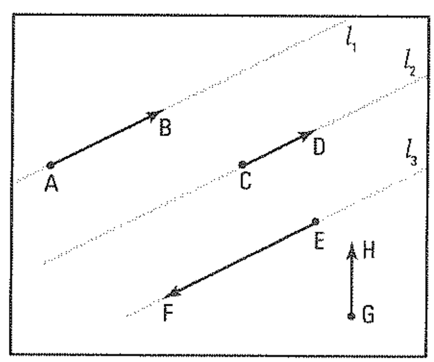
***$\vec{p}$* : speed of plane; *$\vec{w}$* : wind speed.**



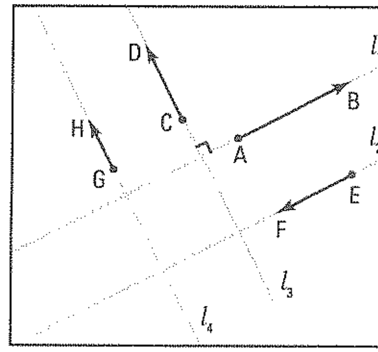
ACTIVITY 3 Comparison of vectors

a) Lines l_1, l_2 and l_3 on the right are parallel.
Two vectors are called **collinear** or **parallel** when they have the same direction or opposite directions.

1. Which vectors are collinear with vector $\overrightarrow{AB}$?
   $\overrightarrow{CD}$* and *$\overrightarrow{EF}$
2. Which vector collinear with $\overrightarrow{AB}$ has the same direction as $\overrightarrow{AB}$? ***$\overrightarrow{CD}$***
3. Which vector collinear with $\overrightarrow{AB}$ has the direction opposite to $\overrightarrow{AB}$? ***$\overrightarrow{EF}$***



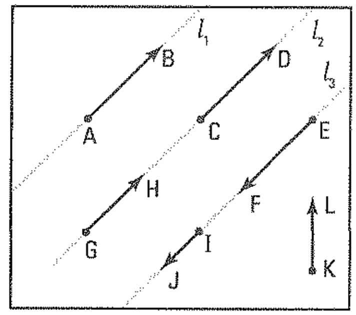
b) Consider lines l_1, l_2, l_3 and l_4 such that $l_1 /\!/ l_2$, $l_3 /\!/ l_4$ and $l_1 \perp l_3$. Two vectors are called **perpendicular** or **orthogonal** when they have perpendicular directions.



1. Find the vectors orthogonal to vector $\overrightarrow{AB}$. ***$\overrightarrow{CD}$ and $\overrightarrow{GH}$***
2. Find the vectors orthogonal to vector $\overrightarrow{CD}$. ***$\overrightarrow{AB}$ and $\overrightarrow{EF}$***
3. What can be said of vectors $\overrightarrow{AB}$ and $\overrightarrow{EF}$?

   They are collinear with opposite directions.
4. What can be said of vectors $\overrightarrow{CD}$ and $\overrightarrow{GH}$? ***They are collinear with same direction.***

c) Lines l_1, l_2 and l_3 on the right are parallel.



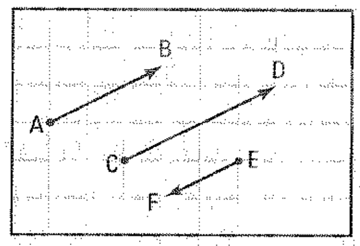
1. Two vectors are called **equal** or **equipollent** if they have same direction and same norm. Which vector, among the vectors on the right, is equal to vector $\overrightarrow{AB}$? ***$\overrightarrow{CD}$***
2. Two vectors are **opposite** if they have opposite directions and same norm. Which vector, among the vectors on the right, is opposite to vector $\overrightarrow{AB}$? ***$\overrightarrow{EF}$***

COMPARISON OF VECTORS

- Two vectors $\vec{u}$ and $\vec{v}$ are called **collinear** or **parallel** if they have the same direction or opposite directions. We write: $\vec{u} /\!/ \vec{v}$.

  Ex.: Vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$ and $\overrightarrow{EF}$ on the right are collinear.

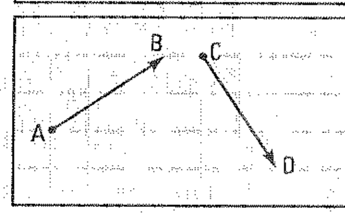
  We have: $\overrightarrow{AB} /\!/ \overrightarrow{CD} /\!/ \overrightarrow{EF}$.



- Two vectors $\vec{u}$ and $\vec{v}$ are called **orthogonal** if they have perpendicular directions. We write: $u \perp \vec{v}$.

  Ex.: Vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ on the right are orthogonal.

  We have: $\overrightarrow{AB} \perp \overrightarrow{CD}$.

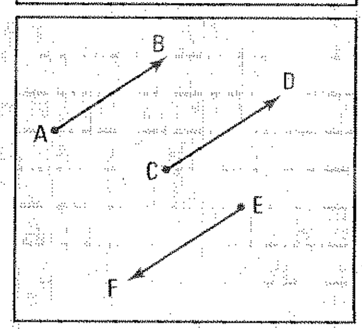


- Two vectors $\vec{u}$ and $\vec{v}$ are called **equal** or **equipollent** if they have the same direction and the same norm.

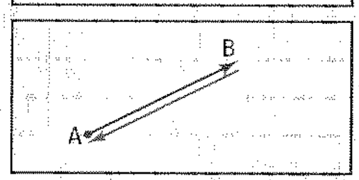
  We write: $\vec{u} = \vec{v}$.
- Two vectors $\vec{u}$ and $\vec{v}$ are called **opposite** if they have opposite directions and same norm.

  We write: $\vec{u} = -\vec{v}$ or $\vec{v} = -\vec{u}$.

  Ex.: Vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ on the right are equal and vectors $\overrightarrow{AB}$ and $\overrightarrow{EF}$ are opposite. We have: $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AB} = -\overrightarrow{EF}$



- Given two points A and B, vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are opposite. We have: $\overrightarrow{AB} = -\overrightarrow{BA}$.



6. Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are opposite. Compare $\|\overrightarrow{AB}\|$ and $\|\overrightarrow{CD}\|$. *$\|\overrightarrow{AB}\| = \|\overrightarrow{CD}\|$*

7. Consider parallelogram ABCD on the right.

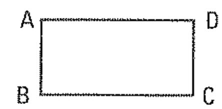
a) Complete:

1. $\overrightarrow{AB}$ = *$\overrightarrow{DC}$* 2. $\overrightarrow{AD}$ = *$\overrightarrow{BC}$*

b) What can be said of vectors

1. $\overrightarrow{AB}$ and $\overrightarrow{CD}$? *$\overrightarrow{AB} = -\overrightarrow{CD}$* 2. $\overrightarrow{AD}$ and $\overrightarrow{CB}$? *$\overrightarrow{AD} = -\overrightarrow{CB}$* 3. $\overrightarrow{AD}$ and $\overrightarrow{DA}$? *$\overrightarrow{AD} = -\overrightarrow{DA}$*

8. Consider rectangle ABCD on the right. Answer true or false.



a) $\overrightarrow{AB} \perp \overrightarrow{AD}$ *True* **b)** $\overrightarrow{AD} /\!/ \overrightarrow{BC}$ *True*

c) $\overrightarrow{AD} = \overrightarrow{BC}$ *True* **d)** $\overrightarrow{AB} = -\overrightarrow{CD}$ *True*

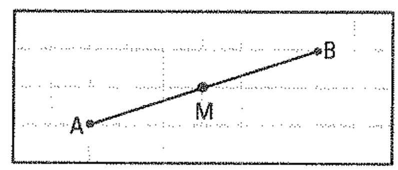
e) $\overrightarrow{AC} = \overrightarrow{BD}$ *False* **f)** $\|\overrightarrow{AC}\| = \|\overrightarrow{BD}\|$ *True*

9. **a)** If $\overrightarrow{AB}$ is a vector of length 2 cm and orientation 60°, describe $\overrightarrow{BA}$.

$\overrightarrow{BA}$ is a vector of length 2 cm and orientation 240°.

b) Let $\overrightarrow{CD}$: 3 cm [N 30° E], define $\overrightarrow{DC}$. ***$\overrightarrow{DC}$: 3 cm [S 30° W]***

10. Consider a line segment AB and its midpoint M.
What can be said of vectors

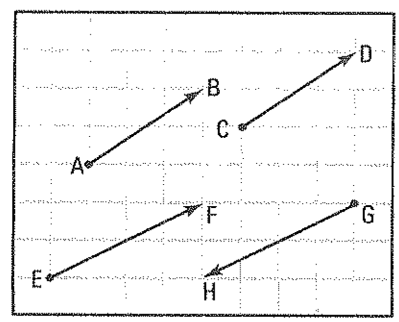


a) $\overrightarrow{AM}$ and $\overrightarrow{MB}$? ***They are equal.***

b) $\overrightarrow{MA}$ and $\overrightarrow{MB}$? ***They are opposite.***

11. **a)** Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal. Compare

1. norms $\|\overrightarrow{AB}\|$ and $\|\overrightarrow{CD}\|$. *$\|\overrightarrow{AB}\| = \|\overrightarrow{CD}\|$*
2. orientations $\theta_{\overrightarrow{AB}}$ and $\theta_{\overrightarrow{CD}}$. *$\theta_{\overrightarrow{AB}} = \theta_{\overrightarrow{CD}}$*



b) Two vectors $\overrightarrow{EF}$ and $\overrightarrow{GH}$ are opposite. Compare

1. norms $\|\overrightarrow{EF}\|$ and $\|\overrightarrow{GH}\|$. *$\|\overrightarrow{EF}\| = \|\overrightarrow{GH}\|$*
2. orientations $\theta_{\overrightarrow{EF}}$ and $\theta_{\overrightarrow{GH}}$. *$\theta_{\overrightarrow{GH}} = \theta_{\overrightarrow{EF}} + 180°$*

ACTIVITY 4 Representative of a geometric vector

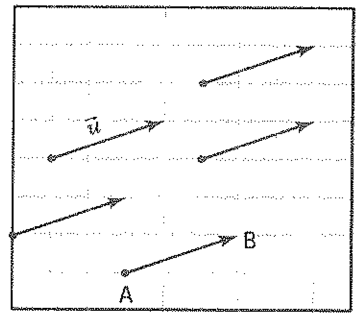
a) Consider the set $\mathbb{Q}$ of rational numbers.

1. What is the definition of a rational number? ***A number x is rational if it can be written as a fraction $\frac{a}{b}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^*$.***
2. The fraction $\frac{1}{2}$ is a representative of the rational number 0.5.
   How many representatives does the rational number 0.5 have? ***An infinite number.***
3. Complete: $0.5 = \frac{1}{2} = \frac{\mathbf{2}}{4} = \frac{\mathbf{3}}{6} = \frac{4}{\mathbf{8}} = \frac{5}{\mathbf{10}} = \ldots$
4. How many fractions are there with denominator equal to 100 representing the rational number 0.5? ***Only one, $\frac{50}{100}$.***

b) By analogy with the set $\mathbb{Q}$ of rational numbers, consider the set $\mathcal{V}$ of geometric vectors.

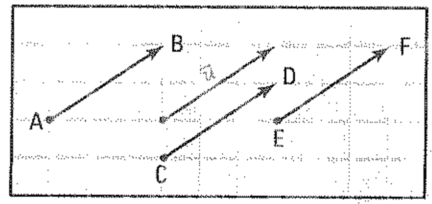
Let $\vec{u}$ be the geometric vector on the right.

1. How many arrows can represent $\vec{u}$? ***An infinite number.***
2. Draw 3 arrows representing the geometric vector $\vec{u}$.
3. Given a point A, how many arrows with origin A represent vector $\vec{u}$? ***Only one.***
4. Draw the representative of vector $\vec{u}$ having its origin at point A.

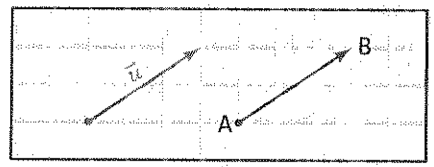


REPRESENTATIVE OF A GEOMETRIC VECTOR

- A geometric vector $\vec{u}$ has an infinite number of representatives. Ex.: $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$ are 3 representatives of vector $\vec{u}$ on the right.
- Given a geometric vector $\vec{u}$ and a point A, there exists only one arrow with origin A representing vector $\vec{u}$.
- Every arrow represents only one vector but a vector can be represented by an infinite number of arrows.



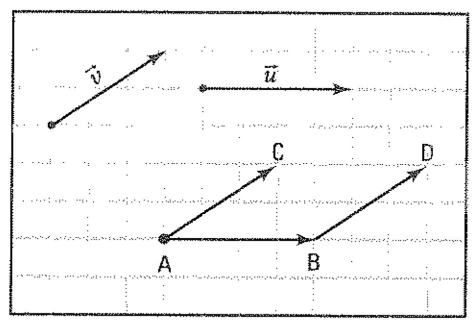
Speaking loosely, we identify a vector with the arrow representing it, in the same way we identify a rational number with the fraction representing it.



12. Consider vectors $\vec{u}$ and $\vec{v}$ on the right.

a) Starting from point A, draw

1. the vector $\overrightarrow{AB}$ representing $\vec{u}$.
2. the vector $\overrightarrow{AC}$ representing $\vec{v}$.
3. the vector $\overrightarrow{BD}$ representing $\vec{v}$.



b) What is the representative of vector $\vec{u}$ having origin C?

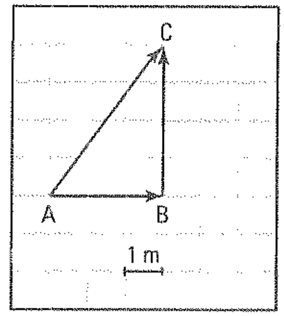
Vector $\overrightarrow{CD}$.

c) Vector $-\vec{u}$ is the opposite of vector $\vec{u}$. Find two representatives of vector $-\vec{u}$.

$\overrightarrow{BA}$ and $\overrightarrow{DC}$

6.2 Operations on geometric vectors

Activity 1 Vector addition



Starting from point A, David travels 3 m east until he reaches point B then travels 4 m north until he reaches point C.

The first motion is represented by vector $\overrightarrow{AB}$ and the second motion is represented by vector $\overrightarrow{BC}$.

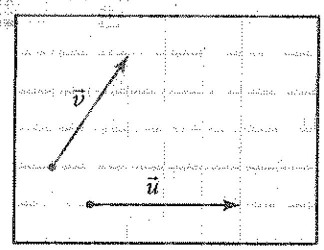
The result of these two motions is equivalent to a single motion represented by vector $\overrightarrow{AC}$.

Vector $\overrightarrow{AC}$ is called vector sum or resultant of vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. We write: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

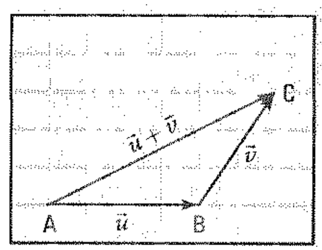
a) Find the norm of vector $\overrightarrow{AC}$. $\|\overrightarrow{AC}\| = 5\ m$

b) Find the orientation $\theta_{\overrightarrow{AC}}$ of vector $\overrightarrow{AC}$. $\theta_{\overrightarrow{AC}} = 53.1°\left(tan^{-1}\frac{4}{3}\right)$

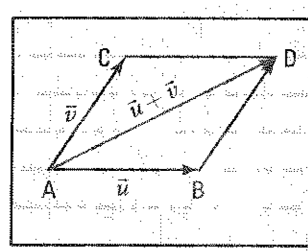
VECTOR ADDITION



- There are two methods for adding two vectors $\vec{u}$ and $\vec{v}$.
  - **Triangle method**
    Starting from any point A in the plane,
    1. we draw vector $\overrightarrow{AB}$ representing vector $\vec{u}$.
    2. we draw vector $\overrightarrow{BC}$ representing vector $\vec{v}$.
    3. we draw vector $\overrightarrow{AC}$ representing vector $\vec{u} + \vec{v}$.



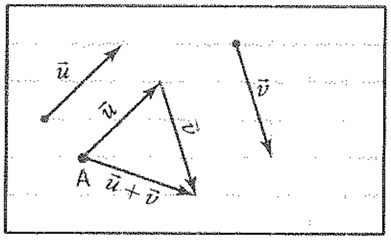
  - **Parallelogram method**
    Starting from any point A in the plane,
    1. we draw vector $\overrightarrow{AB}$ representing vector $\vec{u}$.
    2. we draw vector $\overrightarrow{AC}$ representing vector $\vec{v}$.
    3. we locate the point D such that ABDC is a parallelogram.
    4. we draw vector $\overrightarrow{AD}$ representing vector $\vec{u} + \vec{v}$.



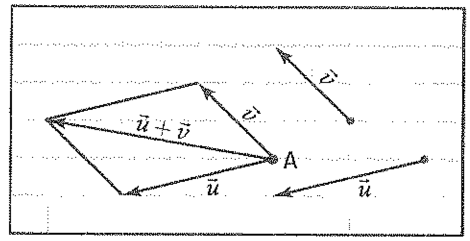
- To add two vectors, we use
  - the triangle method if the origin of one vector corresponds to the endpoint of the other.
  - the parallelogram method if both vectors have the same origin.

1. In each of the following cases, represent starting from point A the sum $\vec{u} + \vec{v}$ using the given method.

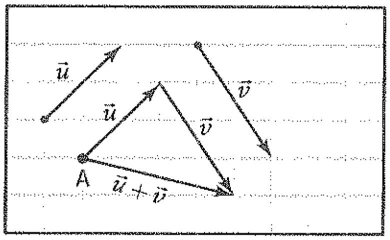
a) Triangle method.



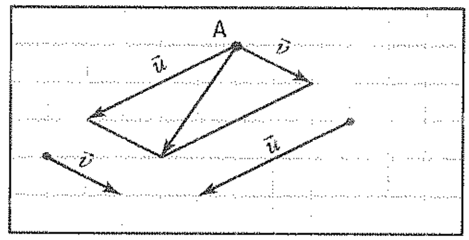
b) Parallelogram method.



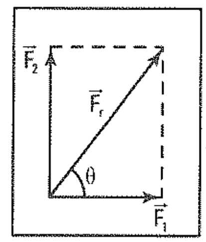
c) Triangle method.



d) Parallelogram method.



2. An object is acted upon by two perpendicular forces of 30 N [E] and 40 N [N]. Determine the resultant force $\vec{F_r}$ acting on this object.

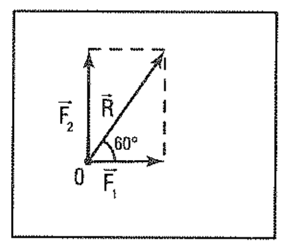


$\tan\theta = \dfrac{\|\vec{F_2}\|}{\|\vec{F_1}\|} = \dfrac{40}{30}$; $\theta = 53.1°$.

$\|\vec{F_r}\|^2 = 30^2 + 40^2 = 2500$; $\|\vec{F_r}\| = 50$ N

$\vec{F_r}$: 50 N [N 36.9° E]

3. Two perpendicular forces $\vec{F_1}$ and $\vec{F_2}$ are applied to an object. The resultant $\vec{R}$ of these two forces has norm 200 N and orientation 60°. Find the norm of each of the forces applied to the object.



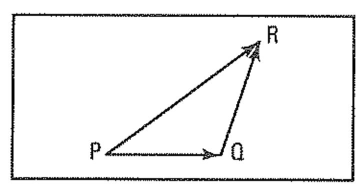
$\|\vec{F_1}\| = \|\vec{R}\| \cos 60° = 200 \cos 60° = 100$ N.

$\|\vec{F_2}\| = \|\vec{R}\| \sin 60° = 200 \sin 60° = 173.2$ N.

ACTIVITY 2 Chasles' relation

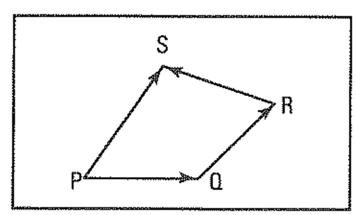
a) Consider the sum of vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ where the endpoint Q of the first vector is the origin of the second vector.

Complete: $\overrightarrow{PQ} + \overrightarrow{QR} =$ ***$\overrightarrow{PR}$***



b) Consider the sum of vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RS}$ where the endpoint of the first vector is the origin of the second one and the endpoint of the second one is the origin of the third one.

Complete: $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} =$ ***$\overrightarrow{PS}$***



c) Consider any two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ where the endpoint B of the first vector is the origin B of the second vector.

Complete: $\overrightarrow{AB} + \overrightarrow{BC} =$ ***$\overrightarrow{AC}$***

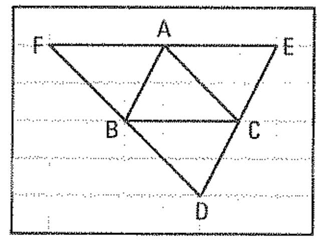
CHASLES' RELATION

For any points A, B and C, we have:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

The sum of 2 vectors with the endpoint of the first one equal to the origin of the second one is a vector whose origin is the origin of the first vector and whose endpoint is the endpoint of the second vector.

4. Consider triangle ABC on the right.



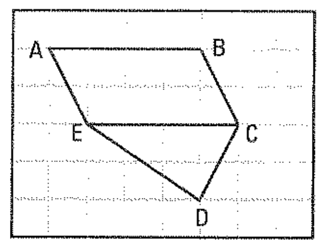
a) Locate

1. Point D knowing that $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$.
2. Point E knowing that $\overrightarrow{BE} = \overrightarrow{BA} + \overrightarrow{BC}$.
3. Point F knowing that $\overrightarrow{CF} = \overrightarrow{CA} + \overrightarrow{CB}$.

b) What can be said of triangles ABC, BCD, ACE and ABF?

They are congruent.

5. Consider parallelogram ABCE and triangle CDE.



a) Justify the steps to simplify the sum $\overrightarrow{AB} + \overrightarrow{CD}$.

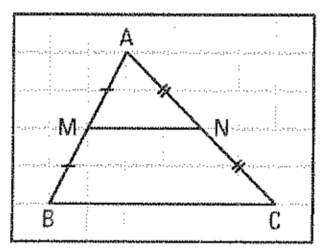
$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{EC} + \overrightarrow{CD}$ ***$\overrightarrow{AB}$ and $\overrightarrow{EC}$ are equal.***

$= \overrightarrow{ED}$ ***Chasles' relation.***

b) Simplify the sum $\overrightarrow{AE} + \overrightarrow{CD}$.

$\overrightarrow{AE} + \overrightarrow{CD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$

6. Consider triangle ABC and line segment MN joining the midpoints of sides AB and AC.



a) Explain why

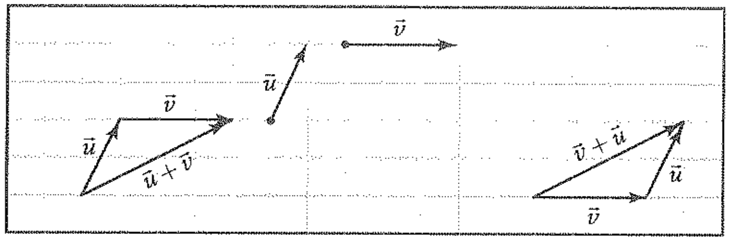
1. $\overrightarrow{BM} = \overrightarrow{MA}$. ***Vectors $\overrightarrow{BM}$ and $\overrightarrow{MA}$ have same direction and same length (M is the midpoint of $\overline{AB}$).***
2. $\overrightarrow{AN} = \overrightarrow{NC}$. ***Vectors $\overrightarrow{AN}$ and $\overrightarrow{NC}$ have same direction and same length (N is the midpoint of $\overline{AC}$).***

b) Simplify the sum $\overrightarrow{BM} + \overrightarrow{NC}$.

$\overrightarrow{BM} + \overrightarrow{NC} = \overrightarrow{MA} + \overrightarrow{AN} = \overrightarrow{MN}$

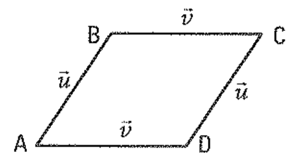
ACTIVITY 3 Properties of vector addition

a) 1. Given two vectors $\vec{u}$ and $\vec{v}$, verify that $\vec{u} + v = \vec{v} + \vec{u}$.

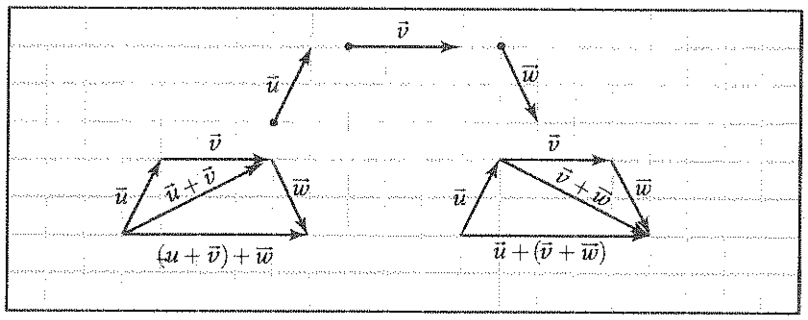


2. Refer to the parallelogram on the right to show that:

$\vec{u} + v = \vec{v} + \vec{u}$. ***$\vec{u} + \vec{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{v} + \vec{u}$***



b) 1. Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, verify that $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.



2. Use Chasles' relation to show that $(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD} = \overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CD})$.

$(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$\overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CD}) = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

c) Justify the steps showing that:

1. $\overrightarrow{AB} + \vec{0} = \overrightarrow{AB}$

$\overrightarrow{AB} + \vec{0} = \overrightarrow{AB} + \overrightarrow{BB}$ ***$\overrightarrow{BB} = \vec{0}$***

$= \overrightarrow{AB}$ ***Chasles' relation.***

2. $\vec{0} + \overrightarrow{AB} = \vec{0}$

$\vec{0} + \overrightarrow{AB} = \overrightarrow{AA} + \overrightarrow{AB}$ ***$\overrightarrow{AA} = \vec{0}$***

$= \overrightarrow{AB}$ ***Chasles' relation.***

d) 1. What can be said of vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$? ***They are opposite.***

2. What can be said of the sum $\overrightarrow{AB} + \overrightarrow{BA}$? Justify your answer.

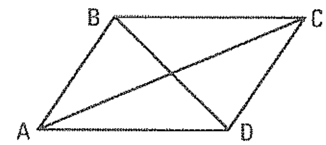
$\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA}$ Chasles' relation.

$= \vec{0}$

PROPERTIES OF VECTOR ADDITION

- The addition of vectors is **commutative**. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
- The addition of vectors is **associative**. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
- $\vec{0}$ is the **neutral element** for the addition of vectors. $\vec{u}+\vec{0}=\vec{u}$ and $\vec{0}+\vec{u}=\vec{u}$
- For any vector $\vec{u}$, there exists an **opposite vector** $-\vec{u}$ such that: $\vec{u}+(-\vec{u})=\vec{0}$ and $(-\vec{u})+\vec{u}=\vec{0}$

7. Quadrilateral ABCD on the right is a parallelogram. Simplify



a) $\overrightarrow{AB}+\overrightarrow{BC}$ ***$\overrightarrow{AC}$***

b) $\overrightarrow{AD}+\overrightarrow{DB}$ ***$\overrightarrow{AB}$***

c) $\overrightarrow{AD}+\overrightarrow{DC}+\overrightarrow{CB}$ ***$\overrightarrow{AB}$***

d) $\overrightarrow{AC}+\overrightarrow{CD}+\overrightarrow{DA}$ ***$\vec{0}$***

e) $\overrightarrow{AB}+\overrightarrow{CD}$ ***$\vec{0}$***

f) $\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{DC}$ ***$\overrightarrow{AB}$***

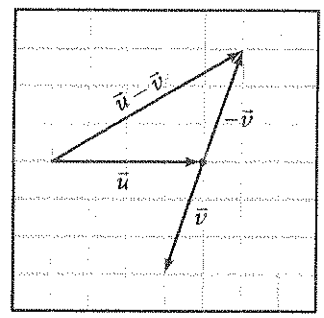
g) $\overrightarrow{AC}+\overrightarrow{BD}+\overrightarrow{CB}$ ***$\overrightarrow{AD}$***

h) $\overrightarrow{CB}+\overrightarrow{AB}+\overrightarrow{BD}+\overrightarrow{DC}$ ***$\overrightarrow{AB}$***

ACTIVITY 4 Vector subtraction

Subtracting a vector $\vec{v}$ from a vector $\vec{u}$ consists in adding vector $\vec{u}$ to the opposite of $\vec{v}$.

$$\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$$



a) Given vectors $\vec{u}$ and $\vec{v}$ on the right, represent vector $\vec{u}-\vec{v}$.

b) Justify the steps of the following subtraction.

$\overrightarrow{CD}-\overrightarrow{ED}=\overrightarrow{CD}+(-\overrightarrow{ED})$ ***$\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$***

$=\overrightarrow{CD}+\overrightarrow{DE}$ ***$\overrightarrow{DE}$ is the opposite of $\overrightarrow{ED}$.***

$=\overrightarrow{CE}$ ***Chasles' relation.***

VECTOR SUBTRACTION

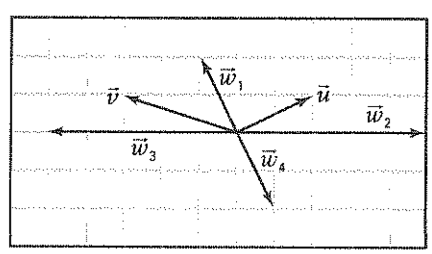
For any vectors $\vec{u}$ and $\vec{v}$, we have: $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$

Thus, $\overrightarrow{AB}-\overrightarrow{CB}=\overrightarrow{AB}+(-\overrightarrow{CB})$ Subtraction rule.

$=\overrightarrow{AB}+\overrightarrow{BC}$ $\overrightarrow{BC}$ is the opposite of $\overrightarrow{CB}$.

$=\overrightarrow{AC}$ Chasles' relation.

8. Consider vectors $\vec{u}$ and $\vec{v}$ on the right.
Construct the vectors



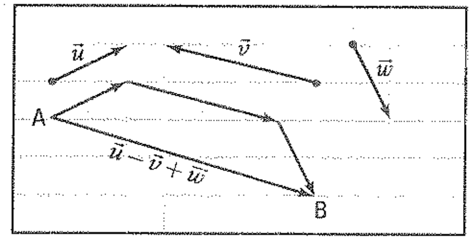
a) $\vec{w_1}=u+\vec{v}$

b) $\vec{w_2}=\vec{u}-\vec{v}$

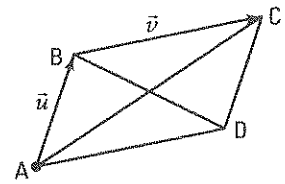
c) $\vec{w_3}=\vec{v}-\vec{u}$

d) $\vec{w_4}=-\vec{u}-\vec{v}$

9. Consider vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ and point A.
Draw vector $\overrightarrow{AB}$ knowing that $\overrightarrow{AB} = \vec{u} - \vec{v} + \vec{w}$.



10. Consider parallelogram ABCD and vectors $\vec{u}$ and $\vec{v}$ represented by $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively. Express, as a function of $\vec{u}$ and $\vec{v}$, the vector



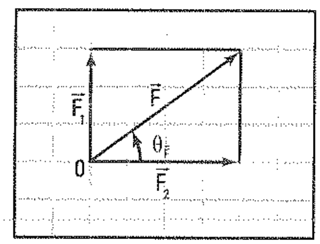
a) $\overrightarrow{AC}$ ***$\vec{u} + \vec{v}$*** b) $\overrightarrow{BD}$ ***$\vec{v} - \vec{u}$*** c) $\overrightarrow{DB}$ ***$\vec{u} - \vec{v}$***

11. Simplify the following expressions.

a) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CD}$ ***$\overrightarrow{AD}$***
b) $\overrightarrow{BC} - \overrightarrow{ED} - \overrightarrow{DC}$ ***$\overrightarrow{BE}$***
c) $\overrightarrow{BC} - \overrightarrow{BA} - \overrightarrow{DC}$ ***$\overrightarrow{AD}$***
d) $\overrightarrow{BA} - \overrightarrow{CB} + \overrightarrow{AB}$ ***$\overrightarrow{BC}$***
e) $\overrightarrow{CD} + \overrightarrow{BC} - \overrightarrow{BD}$ ***$\vec{0}$***
f) $\overrightarrow{AD} - \overrightarrow{BD} - \overrightarrow{AB}$ ***$\vec{0}$***

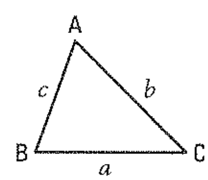
ACTIVITY 5 Calculating the length and orientation of the sum vector

a) An object located at 0 is subjected to a force $\vec{F_1}$ of 3 N oriented northwards and a force $\vec{F_2}$ of 4 N oriented eastwards.



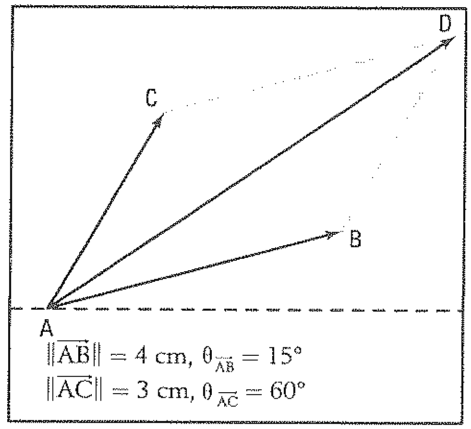
1. Draw the force $\vec{F}$ resultant of the sum of forces $\vec{F_1}$ and $\vec{F_2}$.
2. Calculate the norm of $\vec{F}$.
$\|\vec{F}\|^2 = \|\vec{F_1}\|^2 + \|\vec{F_2}\|^2 = 3^2 + 4^2 = 25 \Rightarrow \|\vec{F}\| = 5$ newtons.
3. Calculate the orientation of $\vec{F}$.
$\tan\theta_{\vec{F}} = \frac{\|\vec{F_1}\|}{\|\vec{F_2}\|} = \frac{3}{4} \Rightarrow \theta_{\vec{F}} = \tan^{-1}\frac{3}{4} = 36.9°$. $\vec{F}$ has orientation 36.9° (or vector $\vec{F}$ is at 53.1° East of North).

b) Consider triangle ABC on he right.



1. State the sine law. ***$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$***
2. State the cosine law.
$a^2 = b^2 + c^2 - 2bc \cos A$.
$b^2 = a^2 + c^2 - 2ac \cos B$.
$c^2 = a^2 + b^2 - 2ab \cos C$.

c) Consider vector $\overrightarrow{AB}$ of length 4 cm and orientation 15°, vector $\overrightarrow{AC}$ of length 3 cm and orientation 60° and vector $\overrightarrow{AD}$ representing the sum of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



1. Explain why ∠BAC measures 45°.

   m ∠BAC = $\theta_{\overrightarrow{AC}} - \theta_{\overrightarrow{AB}}$ = 60° − 15° = 45°.

2. Explain why ∠ABD measures 135°.

   ABDC is a parallelogram and consecutive angles in a parallelogram are supplementary.

3. Consider triangle ABD and use the cosine law to calculate $\|\overrightarrow{AD}\|$.

   $\|\overrightarrow{AD}\|^2 = \|\overrightarrow{AB}\|^2 + \|\overrightarrow{BD}\|^2 - 2\|\overrightarrow{AB}\|\|\overrightarrow{BD}\| \cos B$

   $\|\overrightarrow{AD}\|^2 = 4^2 + 3^2 - 2(4)(3)\cos 135° = 41.97 \Rightarrow \|\overrightarrow{AD}\| = 6.48$ cm.

4. Consider triangle ABD and use the sine law to calculate m ∠BAD.

   $\frac{\sin B}{\|\overrightarrow{AD}\|} = \frac{\sin A}{\|\overrightarrow{BD}\|}$; $\frac{\sin 135°}{6.48} = \frac{\sin A}{3} \Rightarrow \sin A = 0.3274 \Rightarrow$ m ∠BAD = 19.1°.

5. Deduce the orientation of vector $\overrightarrow{AD}$. $\theta_{\overrightarrow{AD}} = \theta_{\overrightarrow{AB}} + 19.1° = 34.1°$.

6. Describe vector $\overrightarrow{AD}$, sum of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

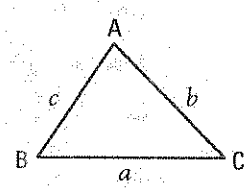
   $\overrightarrow{AD}$ has norm 6.48 cm and orientation 34.1°.

NORM AND ORIENTATION OF THE SUM VECTOR

We use the sine law and the cosine law to determine the sum vector.

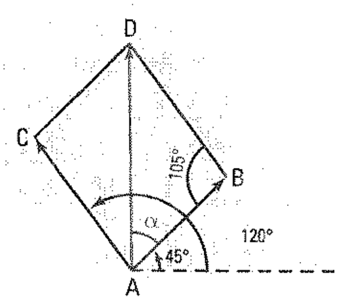
- Sine law

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



- Cosine law

  - $a^2 = b^2 + c^2 - 2bc \cos A$
  - $b^2 = a^2 + c^2 - 2ac \cos B$
  - $c^2 = a^2 + b^2 - 2ab \cos C$



Ex.: Let us determine the norm and orientation of vector $\overrightarrow{AD}$ if $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$

$\|\overrightarrow{AB}\| = 2$; $\theta_{\overrightarrow{AB}} = 45°$; $\|\overrightarrow{AC}\| = 3$; $\theta_{\overrightarrow{AC}} = 120°$ and $\theta_{\overrightarrow{AB}} = 45°$.

1. m ∠BAC = $\theta_{\overrightarrow{AC}} - \theta_{\overrightarrow{AB}}$ = 120° − 45° = 75°
2. m ∠ABD = 105° because angles BAC and ABD are two consecutive angles of the parallelogram ABDC.

3. $\|\overrightarrow{AD}\|^2 = \|\overrightarrow{AB}\|^2 + \|\overrightarrow{BD}\|^2 - 2\|\overrightarrow{AB}\| \times \|\overrightarrow{BD}\| \times \cos 105°$.

   We get: $\|\overrightarrow{AD}\|^2 = 16.10$ and so $\|\overrightarrow{AD}\| = 4.01$.
4. We apply the sine law to triangle ABD to find α.

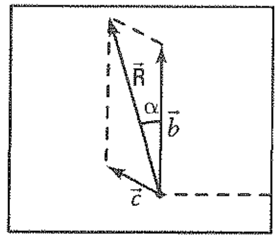
   $\frac{\sin \alpha}{\|\overrightarrow{BD}\|} = \frac{\sin 105°}{\|\overrightarrow{AD}\|} \Rightarrow \sin \alpha = 0.7226 \Rightarrow \alpha = 46.3°$.
5. We deduce $\theta_{\overrightarrow{AD}}$.

   $\theta_{\overrightarrow{AD}} = \theta_{\overrightarrow{AB}} + \alpha = 45° + 46.3° = 91.3°$

Thus, vector $\overrightarrow{AD}$ has norm 4.01 and orientation 91.3°.

12. A boat is traveling north at a speed of 25 knots. A current of 10 knots and orientation 150° acts on the boat.

a) Represent the situation.



b) Determine the actual speed of the boat and its orientation.

$\|\vec{R}\|^2 = \|\vec{b}\|^2 + \|\vec{c}\|^2 - 2\|\vec{b}\| \times \|\vec{c}\| \times \cos 120° = 975 \Rightarrow \|\vec{R}\| = 31.2$ knots.

$\frac{\sin \alpha}{10} = \frac{\sin 120°}{31.2} \Rightarrow \sin \alpha = 0.2776 \Rightarrow \alpha = 16.1° \Rightarrow \theta_{\vec{R}} = 90° + 16.1° = 106.1°$.

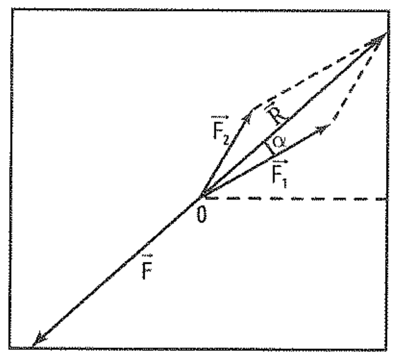
The actual speed of the boat is 31.2 knots. The boat is traveling at 16.1° West of North.

13. An object is subjected to two forces $\vec{F_1}$ and $\vec{F_2}$.

$\vec{F_1}$ has an intensity of 3 newtons and an orientation of 30°.

$\vec{F_2}$ has an intensity of 2 newtons and an orientation of 60°.

Find the force $\vec{F}$ that must be applied to the object in order to cancel the effect of forces $\vec{F_1}$ and $\vec{F_2}$.



$\vec{F} = -\vec{R}$ where $\vec{R} = \vec{F_1} + \vec{F_2}$.

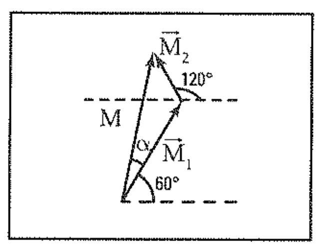
$\|\vec{R}\|^2 = 3^2 + 2^2 - 2(3)(2)\cos 150° = 23.4 \Rightarrow \|\vec{R}\| = 4.8$ N.

$\frac{\sin \alpha}{2} = \frac{\sin 150°}{4.8} \Rightarrow \sin \alpha = 0.2083 \Rightarrow \alpha = 12.0°$.

$\vec{R}$ has norm 4.8 N and orientation 42°.

$\vec{F}$ has norm 4.8 N and orientation 222°. $(\theta_{\vec{R}} + 180° = \theta_{\vec{F}})$

14. Rafael travels 2 km on foot 30° East of North then 1 km 30° West of North. Determine the length and the orientation of the resulting motion.



$\vec{M_1}$ (1st motion), $\vec{M_2}$ (2nd motion).

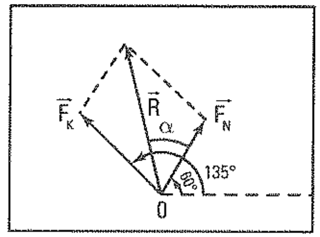
$\vec{M} = \vec{M_1} + \vec{M_2}$ (resulting motion)

$\|\vec{M}\|^2 = 1^2 + 2^2 - 2(1)(2)\cos 120° = 7 \Rightarrow \|\vec{M}\| = 2.646$ km.

$\frac{\sin \alpha}{1} = \frac{\sin 120°}{2.646} \Rightarrow \sin \alpha = 0.3273 \Rightarrow \alpha = 19.1° \Rightarrow \theta_{\vec{M}} = 60° + \alpha = 79.1°$.

Rafael travels 2.646 km with orientation 79.1°.

15. Nomi and Karen are pulling an object. The forces $\overrightarrow{F_N}$ and $\overrightarrow{F_K}$ applied to the object are, respectively, of 150 N and 200 N with orientations 60° and 135°.



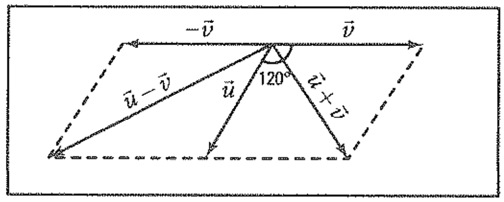
a) What is the intensity of the resultant force $\overrightarrow{R}$, sum of the two forces $\overrightarrow{F_N}$ and $\overrightarrow{F_K}$?

$\|\vec{R}\|^2 = 150^2 + 200^2 - 2(150)(200)\cos 105° = 78029 \Rightarrow \|\vec{R}\| = 279.3$ N.

b) Find the orientation of the resultant force $\overrightarrow{R}$.

$\frac{\sin\alpha}{200} = \frac{\sin 105°}{279.3} \Rightarrow \sin\alpha = 0.6916 \Rightarrow \alpha \approx 43.8° \Rightarrow \theta_{\vec{R}} = 43.8° + 60° = 103.8°$.

16. Vectors $\vec{u}$ and $\vec{v}$ on the right form an angle of 120° and have norm $\|\vec{u}\| = 3.6$ and $\|\vec{v}\| = 4$ respectively. Calculate the norm, rounded to the nearest tenth of a unit, of



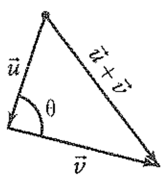
a) $\vec{u}+\vec{v}$

$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos 60° = 14.56 \Rightarrow \|\vec{u}+\vec{v}\| = 3.8$.

b) $\vec{u}-\vec{v}$ ***$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos 120° = 43.36 \Rightarrow \|\vec{u}-\vec{v}\| = 6.6$.***

17. Determine the measure, rounded to the nearest tenth of a unit, of the angle θ between $\vec{u}$ and $\vec{v}$ if

a) $\|\vec{u}\| = 3.6$; $\|\vec{v}\| = 4.1$; $\|\vec{u}+\vec{v}\| = 4.5$

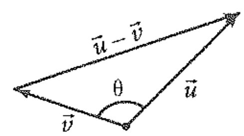


$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$

$20.25 = 29.77 - 29.52\cos\theta$

$\cos\theta = 0.3225 \Rightarrow \theta = 71.2°$

b) $\|\vec{u}\| = 4.2$; $\|\vec{v}\| = 3.2$; $\|\vec{u}-\vec{v}\| = 6.3$

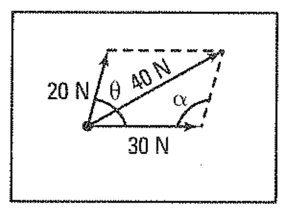


$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\| + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$

$39.69 = 27.88 - 26.88\cos\theta$

$\cos\theta = -0.4393 \Rightarrow \theta = 116.1°$

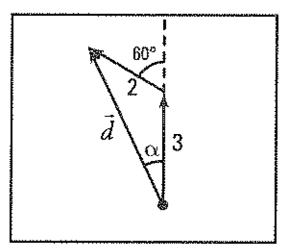
18. A force of 20 N and another force of 30 N applied to the same object yield a resultant of 40 N. What is, rounded to the nearest tenth of a unit, the angle between the two forces?



$40^2 = 30^2 + 20^2 - 2(30)(20)\cos\alpha \Rightarrow \cos\alpha = -0.25 \Rightarrow \alpha = 104.5°$

$\theta = 180° - \alpha = 75.5°$.

19. Valerie travels 3 km North on a sailboat. She then changes her route by turning 60° westwards and traveling 2 km. Determine at what distance Valerie is from her starting point and how she is oriented at the end of her trip.



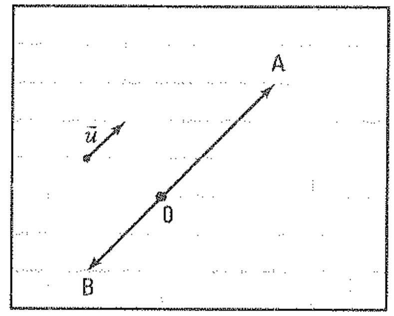
$\|\vec{d}\|^2 = 3^2 + 2^2 - 2(3)(2)\cos 120° = 19 \Rightarrow \|\vec{d}\| = 4.36$ km.

$\frac{\sin\alpha}{2} = \frac{\sin 120°}{4.36} \Rightarrow \sin\alpha = 0.3974 \Rightarrow \alpha = 23.4°$.

Valerie is 4.36 km from the starting point, 23.4° West of North.

ACTIVITY 6 Multiplication of a vector by a real number

Consider a vector $\vec{u}$ and a point 0.



a) 1. Draw the vector $\overrightarrow{0A}$ if $\overrightarrow{0A} = 3\vec{u}$. (Note: $3\vec{u} = \vec{u} + \vec{u} + \vec{u}$).

2. Compare vectors $\vec{u}$ and $3\vec{u}$ according to

1) direction; *They have the same direction.*

2) norm. $\|3\vec{u}\| = 3\|\vec{u}\|$

b) 1. Draw the vector $\overrightarrow{0B}$ if $\overrightarrow{0B} = -2\vec{u}$. (Note: $-2\vec{u} = -\vec{u} + -\vec{u}$).

2. Compare vectors $\vec{u}$ and $-2\vec{u}$ according to

1) direction; *They have opposite directions.*

2) norm. $\|-2\vec{u}\| = 2\|\vec{u}\|$

c) Let k be a real number. ($k \neq 0$).

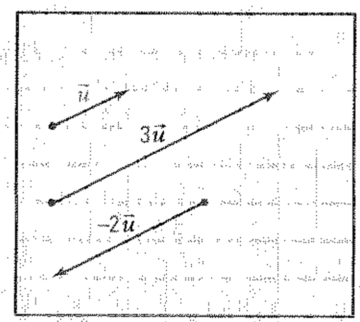
1. Complete by indicating the sign of k, vectors $\vec{u}$ and $k\vec{u}$ have

1) the same direction. $k > 0$ 2) opposite directions. $k < 0$

2. Compare $\|k\vec{u}\|$ and $\|\vec{u}\|$. $\|k\vec{u}\| = |k|\|\vec{u}\|$

d) Determine the value of the real number k if $k\vec{u} = \vec{0}$. $k = 0$

MULTIPLICATION OF A VECTOR BY A REAL NUMBER



- The product of a non-zero vector $\vec{u}$ by a real number k is a vector written $k\vec{u}$. Thus

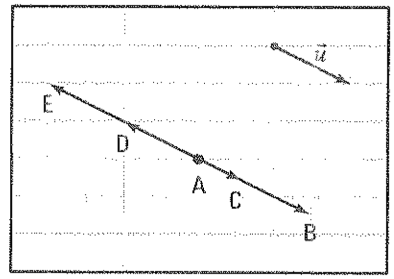
$$k \times \vec{u} = k\vec{u}$$

  - $k\vec{u}$ and $\vec{u}$ have same direction if $k > 0$,

    $k\vec{u}$ and $\vec{u}$ have opposite directions if $k < 0$.
  - $\|k\vec{u}\| = |k| \times \|\vec{u}\|$.
- Note that: $0 \times \vec{u} = \vec{0}$; $1 \times \vec{u} = \vec{u}$; $(-1) \times \vec{u} = -\vec{u}$.
- Vectors $\vec{u}$ and $k\vec{u}$ are called collinear or parallel.
- Two vectors $\vec{u}$ and $\vec{v}$ are collinear if and only if there exists a nonzero real number k such that $\vec{v} = k\vec{u}$.

$$\vec{u} /\!/ \vec{v} \Leftrightarrow \exists k \in \mathbb{R}^* : \vec{v} = k\vec{u}$$

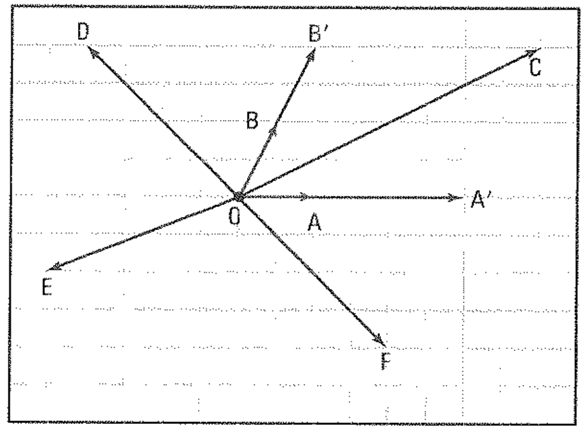
20. Consider vector $\vec{u}$ and point A on the right. Draw the following vectors.



a) $\overrightarrow{AB}$ if $\overrightarrow{AB} = \frac{3}{2}\vec{u}$. b) $\overrightarrow{AC}$ if $\overrightarrow{AC} = \frac{1}{2}\vec{u}$.

c) $\overrightarrow{AD}$ if $\overrightarrow{AD} = (-1) \times \vec{u}$. d) $\overrightarrow{AE}$ if $\overrightarrow{AE} = -2\vec{u}$.

21. Consider vectors $\overrightarrow{0A}$ and $\overrightarrow{0B}$ in the plane on the right.



a) Draw the vector

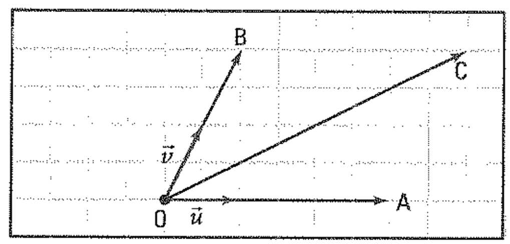
1. $\overrightarrow{0A'}$ if $\overrightarrow{0A'} = 3\,\overrightarrow{0A}$.
2. $\overrightarrow{0B'}$ if $\overrightarrow{0B'} = 2\,\overrightarrow{0B}$.
3. $\overrightarrow{0C}$ if $\overrightarrow{0C} = 3\,\overrightarrow{0A} + 2\,\overrightarrow{0B}$.

b) Draw the vector $\overrightarrow{0D}$ if $\overrightarrow{0D} = -3\,\overrightarrow{0A} + 2\,\overrightarrow{0B}$.

c) Draw the vector $\overrightarrow{0E}$ if $\overrightarrow{0E} = -2\,\overrightarrow{0A} - \overrightarrow{0B}$.

d) Draw the vector $\overrightarrow{0F}$ if $\overrightarrow{0F} = 3\,\overrightarrow{0A} - 2\,\overrightarrow{0B}$.

22. Consider 2 vectors $\vec{u}$ and $\vec{v}$ and a point 0. Locate the following points.

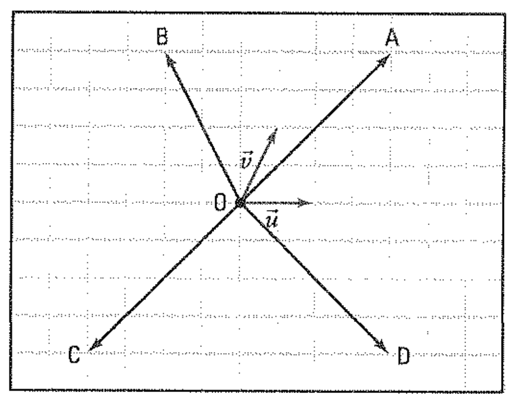


a) A if $\overrightarrow{0A} = 3\vec{u}$.

b) B if $\overrightarrow{0B} = 2\vec{v}$.

c) C if $\overrightarrow{0C} = 3\vec{u} + 2\vec{v}$.

23. Consider 2 vectors $\vec{u}$ and $\vec{v}$ and a point 0. Locate the following points.



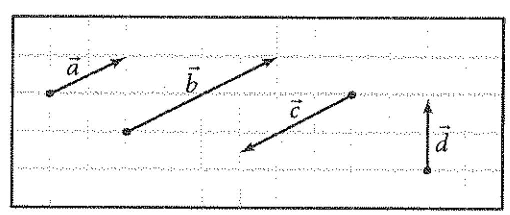
a) A if $\overrightarrow{0A} = \vec{u} + 2\vec{v}$.

b) B if $\overrightarrow{0B} = -2\vec{u} + 2\vec{v}$.

c) C if $\overrightarrow{0C} = -\vec{u} - 2\vec{v}$.

d) D if $\overrightarrow{0D} = 3\vec{u} - 2\vec{v}$.

24. Consider vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ on the right.



a) 1. Does there exist a real number k such that $\vec{b} = k\vec{a}$? ***Yes, $\vec{b} = 2\vec{a}$***

2. What can be said of vectors $\vec{a}$ and $\vec{b}$?

They are collinear.

b) Does there exist real number k such that $\vec{c} = k\vec{a}$? If so, what can we conclude?
Yes, $\vec{c} = -1.5\vec{a}$. $\vec{a}$ and $\vec{c}$ are collinear.

c) Does there exist a real number k such that $\vec{d} = k\vec{a}$? If so, what can we conclude?
No, because $\vec{a}$ and $\vec{d}$ are not parallel.

25. Consider a vector $\vec{u}$ and a real number k. Complete:

a) $k \neq 0$ and $k\vec{u} = \vec{0} \Rightarrow$ *$\vec{u} = \vec{0}$* b) $\vec{u} \neq \vec{0}$ and $k\vec{u} = \vec{0} \Rightarrow$ *$k = 0$* c) $k\vec{u} = \vec{0} \Leftrightarrow$ *$k = 0$ or $\vec{u} = \vec{0}$*

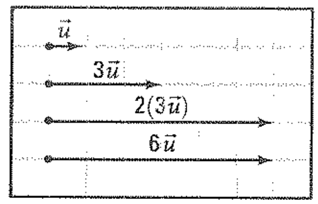
26. Let $\vec{u}$ be a vector such that $\|\vec{u}\| = 6$. Find the norm of

a) $3\vec{u}$ *18* b) $-5\vec{u}$ *30* c) $-\frac{1}{2}\vec{u}$ *3* d) $\frac{\vec{u}}{3}$ *2* e) $\frac{\vec{u}}{\|\vec{u}\|}$ *1*

ACTIVITY 7 Properties of multiplication of a vector by a real number

a) Consider vector $\vec{u}$ on the right.

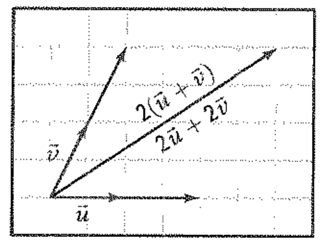
1. Verify that $2(3\vec{u}) = (2 \times 3)\vec{u}$.
2. If a and b are two real numbers, is it true that $a(b\vec{u}) = (ab)\vec{u}$? *Yes*



b) Which real number is the neutral element when we multiply a vector by a real number? Justify your answer. *1 since $1 \times \vec{u} = \vec{u}$*

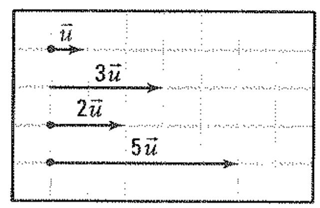
c) Consider vectors $\vec{u}$ and $\vec{v}$ on the right.

1. Compare $2(\vec{u} + \vec{v})$ and $2\vec{u} + 2\vec{v}$. *$2(\vec{u} + \vec{v}) = 2\vec{u} + 2\vec{v}$*
2. If a is any real number, is it true that $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$? *Yes*



d) Consider vector $\vec{u}$ on the right.

1. Verify that $(3 + 2)\vec{u} = 3\vec{u} + 2\vec{u}$.
2. If a and b are two real numbers, is it true that $(a + b)\vec{u} = a\vec{u} + b\vec{u}$?
Yes



PROPERTIES OF MULTIPLICATION OF A VECTOR BY A REAL NUMBER

Consider two vectors $\vec{u}$ and $\vec{v}$ and two real numbers a and b.
Multiplication of a vector by a real number has the following properties:

- Associativity: $a(b\vec{u}) = (ab)\vec{u}$
- 1 is the neutral element: $1 \times \vec{u} = \vec{u}$
- Distributivity over the sum of vectors: $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
- Distributivity over the sum of real numbers: $(a + b)\vec{u} = a\vec{u} + b\vec{u}$

27. Reduce the following expressions.

a) $2\vec{u}+6\vec{u}$ ***$8\vec{u}$*** b) $-2(5\vec{u})$ ***$-10\vec{u}$*** c) $5\vec{u}-7\vec{u}$ ***$-2\vec{u}$***

d) $3(2\vec{u}+3\vec{v})$ ***$6\vec{u}+9\vec{v}$*** e) $3(2\vec{u})-2(3\vec{u})$ ***$\vec{0}$*** f) $-(\vec{u}-2\vec{v})$ ***$-\vec{u}+2\vec{v}$***

28. Reduce the following expressions.

a) $-2\vec{u}+3\vec{v}+5\vec{u}-4\vec{v}$ ***$3\vec{u}-\vec{v}$*** b) $-(\vec{u}+2\vec{v})+2(\vec{u}-\vec{v})$ ***$\vec{u}-4\vec{v}$***

c) $3(2\vec{u}-\vec{v})-(2\vec{u}-3\vec{v})$ ***$4\vec{u}$*** d) $-2(2\vec{u}-3\vec{v})-3(\vec{u}+2\vec{v})$ ***$-7\vec{u}$***

29. Justify the steps to reduce: $2\overrightarrow{AB}-2\overrightarrow{DC}+2\overrightarrow{BC}-2\overrightarrow{BD}$.

Steps	Justifications
$2\overrightarrow{AB}-2\overrightarrow{DC}+2\overrightarrow{BC}-2\overrightarrow{BD}$	
$=2\overrightarrow{AB}+2\overrightarrow{CD}+2\overrightarrow{BC}+2\overrightarrow{DB}$	***$-\overrightarrow{PQ}=\overrightarrow{QP}$***
$=2\overrightarrow{AB}+2\overrightarrow{BC}+2\overrightarrow{CD}+2\overrightarrow{DB}$	***Addition is commutative.***
$=2\left(\overrightarrow{AB}+2\overrightarrow{BC}\right)+2\left(\overrightarrow{CD}+2\overrightarrow{DB}\right)$	***Double factorization.***
$=2\overrightarrow{AC}+2\overrightarrow{CB}$	***Chasles' relation.***
$=2\left(\overrightarrow{AC}+\overrightarrow{CB}\right)$	***Factorization.***
$=2\overrightarrow{AB}$	***Chasles' relation.***

30. Reduce the following expression. $5\overrightarrow{AC}+\overrightarrow{BC}-2\overrightarrow{AC}-4\overrightarrow{BC}+\overrightarrow{BE}-2\overrightarrow{CE}+2\overrightarrow{BE}-\overrightarrow{CE}$.

$3\overrightarrow{AC}-3\overrightarrow{BC}+3\overrightarrow{BE}-3\overrightarrow{CE}=3\overrightarrow{AC}+3\overrightarrow{CB}+3\overrightarrow{BE}+3\overrightarrow{EC}$

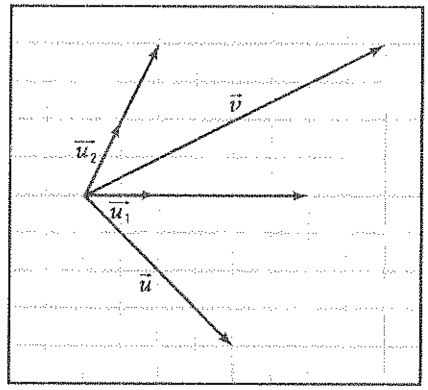
$=3\left(\overrightarrow{AC}+\overrightarrow{CB}\right)+3\left(\overrightarrow{BE}+\overrightarrow{EC}\right)=3\overrightarrow{AB}+3\overrightarrow{BC}=3\left(\overrightarrow{AB}+\overrightarrow{BC}\right)=3\overrightarrow{AC}$

6.3 Vector basis

ACTIVITY 1 Linear combination of vectors

Consider 2 vectors $\vec{u_1}$ and $\vec{u_2}$.

Any expression of the form $c_1\vec{u_1} + c_2\vec{u_2}$ where c_1 and c_2 are real numbers is called **linear combination** of vectors $\vec{u_1}$ and $\vec{u_2}$.



a) Explain why a linear combination of vectors always yields a vector. *The product of a real number with a vector is a vector. Therefore, $c_1\vec{u_1}$ and $c_2\vec{u_2}$ are vectors. The sum of 2 vectors is a vector, so $c_1\vec{u_1} + c_2\vec{u_2}$ is a vector.*

b) Represent the vector $\vec{u}$ knowing that $\vec{u} = 3\vec{u_1} - 2\vec{u_2}$.

c) Express vector $\vec{v}$ on the right as a linear combination of the vectors $\vec{u_1}$ and $\vec{u_2}$.

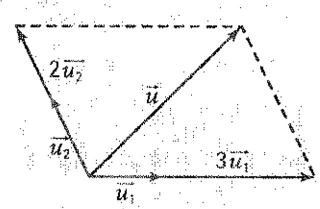
$\vec{v} = 3\vec{u_1} + 2\vec{u_2}$

LINEAR COMBINATION OF VECTORS

- Given a sequence of vectors $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}$, any expression of the form $c_1\vec{u_1} + c_2\vec{u_2} + \ldots + c_n\vec{u_n}$ where $c_i \in \mathbb{R}$ is a **linear combination** of the vectors $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}$.
- Note that a linear combination of vectors is a vector.

  Ex.: Given the vectors $\vec{u_1}$ and $\vec{u_2}$, $3\vec{u_1} + 2\vec{u_2}$ is a linear combination of the vectors $\vec{u_1}$ and $\vec{u_2}$.

  We have: $\vec{u} = 3\vec{u_1} + 2\vec{u_2}$.



- A vector $\vec{u}$ is a linear combination of the vectors $\vec{u_1}, \vec{u_2}, \ldots \vec{u_n}$ if there exist real numbers $c_1, c_2, \ldots, c_n$ such that $\vec{u} = c_1\vec{u_1} + c_2\vec{u_2} + \cdots + c_n\vec{u_n}$.

1. Consider vectors $\vec{u_1}, \vec{u_2}$ and $\vec{u_3}$ on the right. Represent the following linear combinations.

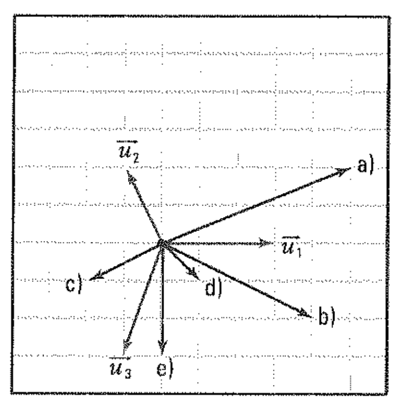
a) $2\vec{u_1} + \vec{u_2}$

b) $\vec{u_1} - \vec{u_2}$

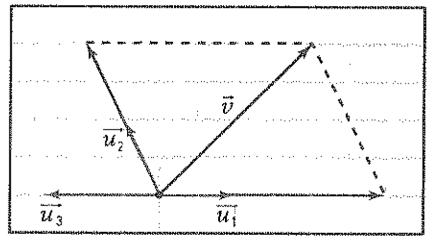
c) $\vec{u_2} + \vec{u_3}$

d) $\vec{u_1} + \vec{u_2} + \vec{u_3}$

e) $2\vec{u_1} + 3\vec{u_2} - 3\vec{u_3}$



2. Consider vectors $\vec{u_1}$, $\vec{u_2}$ and $\vec{u_3}$ on the right.

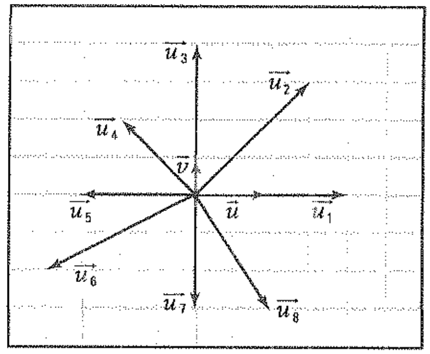


a) Express vector $\vec{v}$ as a linear combination of the vectors

1. $\vec{u_1}$ and $\vec{u_2}$ ***$3\vec{u_1} + 2\vec{u_2}$***
2. $\vec{u_2}$ and $\vec{u_3}$ ***$2\vec{u_2} - 2\vec{u_3}$***

b) Can we express vector $\vec{v}$ as a linear combination of $\vec{u_1}$ and $\vec{u_3}$? ***No***

3. Consider the vectors on the right. Express each of the following vectors as a linear combination of vectors $\vec{u}$ and $\vec{v}$.



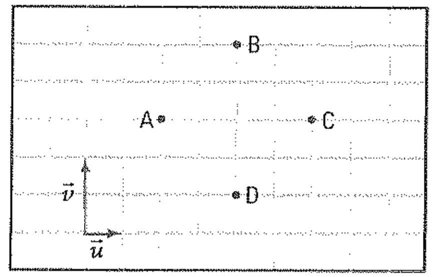
a) $\vec{u_1}$ ***$2\vec{u}$*** b) $\vec{u_2}$ ***$1.5\vec{u} + 3\vec{v}$***

c) $\vec{u_3}$ ***$4\vec{v}$*** d) $\vec{u_4}$ ***$-\vec{u} + 2\vec{v}$***

e) $\vec{u_5}$ ***$-1.5\vec{u}$*** f) $\vec{u_6}$ ***$-2\vec{u} - 2\vec{v}$***

g) $\vec{u_7}$ ***$-3\vec{v}$*** h) $\vec{u_8}$ ***$\vec{u} - 3\vec{v}$***

4. Consider the points of the right. Express each of the following vectors as a linear combination of vectors $\vec{u}$ and $\vec{v}$.



a) $\overrightarrow{AB}$ ***$2\vec{u} + \vec{v}$*** b) $\overrightarrow{BA}$ ***$-2\vec{u} - \vec{v}$***

c) $\overrightarrow{AC}$ ***$4\vec{u}$*** d) $\overrightarrow{CA}$ ***$-4\vec{u}$***

e) $\overrightarrow{AD}$ ***$2\vec{u} - \vec{v}$*** f) $\overrightarrow{DA}$ ***$-2\vec{u} + \vec{v}$***

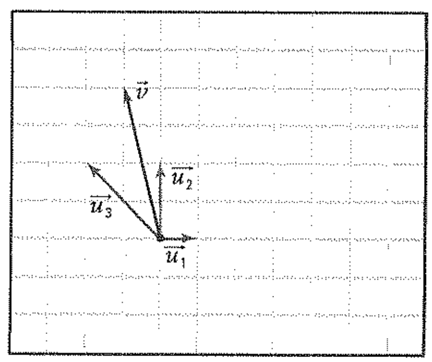
g) $\overrightarrow{BD}$ ***$-2\vec{v}$*** h) $\overrightarrow{CD}$ ***$-2\vec{u} - \vec{v}$***

ACTIVITY 2 Vector basis

Consider in a plane the vectors $\vec{u_1}$, $\vec{u_2}$ and $\vec{u_3}$ and the vector $\vec{v}$.

a) Verify that

1. $\vec{v} = \vec{u_1} + \vec{u_2} + \vec{u_3}$ 2. $\vec{v} = 5\vec{u_1} - \vec{u_2} + 3\vec{u_3}$



b) Is the expression of $\vec{v}$ as a linear combination of vectors $\vec{u_1}$, $\vec{u_2}$ and $\vec{u_3}$ unique? ***No***

c) Express vector $\vec{v}$ as a linear combination of vectors $\vec{u_1}$ and $\vec{u_2}$.
$\vec{v} = -\vec{u_1} + 2\vec{u_2}$

d) Is the expression of $\vec{v}$ as a linear combination of vectors $\vec{u_1}$ and $\vec{u_2}$ unique? ***Yes***

e) Can we say that for any vector $\vec{v}$ in that plane, there exist two unique real numbers a and b such that $\vec{v} = a\vec{u_1} + b\vec{u_2}$? ***Yes***

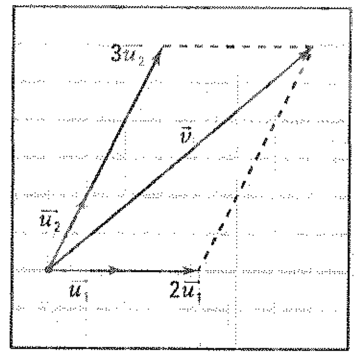
If so, we say that vectors $\vec{u_1}$ and $\vec{u_2}$ form a ***vector basis*** *of the plane.*

VECTOR BASIS

In a plane, two vectors $\vec{u_1}$ and $\vec{u_2}$ that are not parallel (not collinear) form a vector basis.

The two vectors $\vec{u_1}$ and $\vec{u_2}$ of this basis can generate any vector $\vec{v}$ in the plane, meaning that given any vector $\vec{v}$ in the plane, there exist two **unique** real numbers c_1 and c_2 such that:

$$\vec{v} = c_1\vec{u_1} + c_2\vec{u_2}$$



The real numbers c_1 and c_2 are called **components** of vector $\vec{v}$ relative to the basis $\{\vec{u_1}, \vec{u_2}\}$.

Ex.: Vector $\vec{v}$ above can be written in a unique way as a linear combination of the basis vectors $\vec{u_1}$ and $\vec{u_2}$.

We have: $\vec{v} = 2\vec{u_1} + 3\vec{u_2}$.

5. True or false?

a) Three vectors in the plane can form a vector basis. ***False***

b) Two vectors in the plane always form a vector basis. ***False***

c) Two vectors that are not parallel always form a vector basis. ***True***

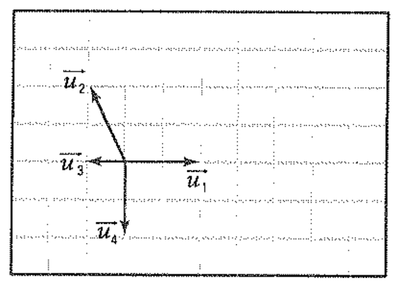
d) If $\vec{u_1}$ and $\vec{u_2}$ form a vector basis of the plane then any vector $\vec{v}$ in the plane can be written in a unique way as a linear combination of the vectors $\vec{u_1}$ and $\vec{u_2}$. ***True***

6. Consider the vectors on the right.

a) Explain why $\vec{u_1}, \vec{u_2}$ and $\vec{u_4}$ do not form a vector basis.
Three vectors cannot form a vector basis of a plane.

b) Explain why $\vec{u_1}$ and $\vec{u_3}$ do not form a vector basis.
Vectors $\vec{u_1}$ and $\vec{u_3}$ are parallel.

c) Explain why $\vec{u_1}$ and $\vec{u_2}$ form a vector basis.
$\vec{u_1}$ and $\vec{u_2}$ are 2 vectors that are not parallel.

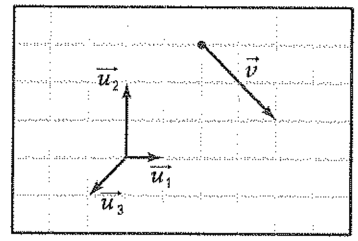


7. Consider the vectors on the right.

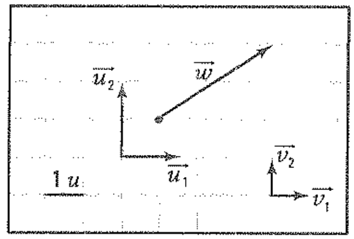
a) Express vector $\vec{v}$ as a linear combination of the vectors in the vector basis.

1. $\{\vec{u_1}, \vec{u_2}\}$ ***$\vec{v} = 2\vec{u_1} - 1\vec{u_2}$***
2. $\{\vec{u_1}, \vec{u_3}\}$ ***$\vec{v} = 4\vec{u_1} + 2\vec{u_3}$***
3. $\{\vec{u_2}, \vec{u_3}\}$ ***$\vec{v} = -2\vec{u_2} - 2\vec{u_3}$***

b) Do the components of vector $\vec{v}$ relative to a basis depend on the basis? ***Yes***



ACTIVITY 3 Orthogonal basis – Orthonormal basis



a) Explain why two orthogonal vectors in a plane form a vector basis of the plane.
Two orthogonal vectors are not parallel.

b) A basis is orthogonal when the two vectors of the basis are orthogonal. If, in addition, the vectors are unit vectors then the orthogonal basis is called orthonormal.

1. Find, in the plane on the right, the two orthogonal bases. $\{\vec{u_1}, \vec{u_2}\}$ *and* $\{\vec{v_1}, \vec{v_2}\}$
2. Which one of the two bases is orthonormal? $\{\vec{v_1}, \vec{v_2}\}$

c) Express vector $\vec{w}$ as a linear combination of the vectors of the basis.

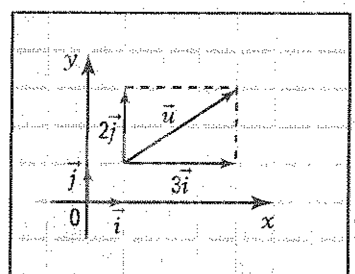
1. $\{\vec{u}_1, \vec{u}_2\}$ $\vec{w} = 2\vec{u_1} + \vec{u_2}$
2. $\{\vec{v}_1, \vec{v}_2\}$ $\vec{w} = 3\vec{v_1} + 2\vec{v_2}$

ORTHOGONAL BASIS – ORTHONORMAL BASIS

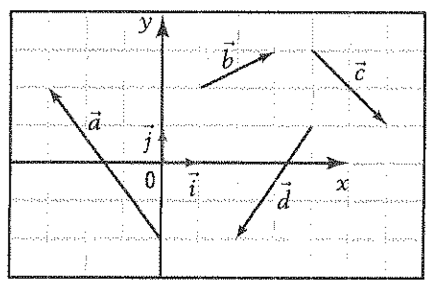
- A vector basis with orthogonal vectors is called **orthogonal**. If, in addition, the vectors are unit vectors, then the basis is called **orthonormal**.
- The Cartesian plane usually uses an orthonormal basis. We denote by $\vec{i}$ and $\vec{j}$ the vectors of an orthonormal basis.

  We have: $\vec{i} \perp \vec{j}$; $\|\vec{i}\| = 1$ and $\|\vec{j}\| = 1$.

  Ex.: Vector $\vec{u}$, in the orthonormal basis $\{\vec{i}, \vec{j}\}$ on the right, is written: $\vec{u} = 3\vec{i} + 2\vec{j}$.



8. Consider the Cartesian plane on the right.



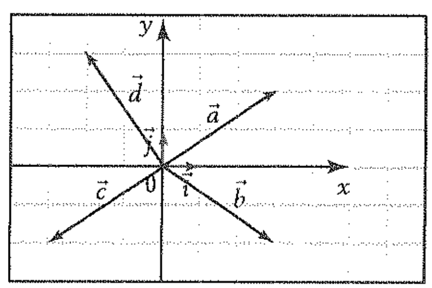
a) Express, as a linear combination of vectors $\vec{i}$ and $\vec{j}$, the vector

1. $\vec{a}$ $-3\vec{i} + 4\vec{j}$
2. $\vec{b}$ $2\vec{i} + \vec{j}$
3. $\vec{c}$ $2\vec{i} - 2\vec{j}$
4. $\vec{d}$ $-2\vec{i} - 3\vec{j}$

b) Determine

1. $\|\vec{a}\|$ 5
2. $\|\vec{b}\|$ $\sqrt{5}$
3. $\|\vec{c}\|$ $2\sqrt{2}$
4. $\|\vec{d}\|$ $\sqrt{13}$

9. Consider the Cartesian plane on the right. Represent, starting from the origin 0, the vector



a) $\vec{a} = 3\vec{i} + 2\vec{j}$.

b) $\vec{b} = 3\vec{i} - 2\vec{j}$.

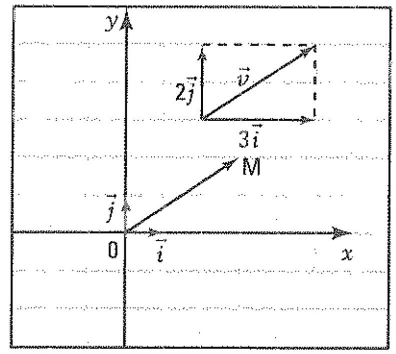
c) $\vec{c} = -3\vec{i} - 2\vec{j}$.

d) $\vec{d} = -2\vec{i} + 3\vec{j}$.

6.4 Algebraic vectors

ACTIVITY 1 Vectors in the Cartesian plane

Consider the geometric vector $\vec{v}$ located in the Cartesian plane on the right.



a) Write vector $\vec{v}$ as a linear combination of vectors $\vec{i}$ and $\vec{j}$.

$\vec{v} = 3\vec{i} + 2\vec{j}$

b) 1. Is it true that to any geometric vector $\vec{u}$ located in the Cartesian plane corresponds a unique couple (a, b) such that $\vec{u} = a\vec{i} + b\vec{j}$ where $a \in \mathbb{R}$ and $b \in \mathbb{R}$? *True*

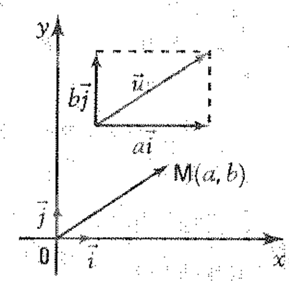
2. The couple (a, b) defines algebraically the vector $\vec{u}$. What is the couple defining vector $\vec{v}$? *(3, 2)*

c) 1. Draw vector $\overrightarrow{0M}$, representative with origin 0 of vector $\vec{v}$.

2. Verify that the coordinates of point M are equal to the components of vector $\vec{v}$.

ALGEBRAIC VECTORS

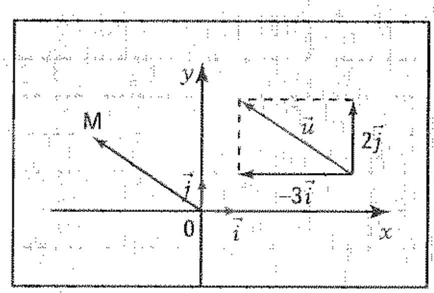
- Consider the Cartesian plane and the orthonormal basis $\{\vec{i}, \vec{j}\}$.
  An **algebraic vector** $\vec{u}$ in the Cartesian plane is defined by a couple (a, b) where $a \in \mathbb{R}$ and $b \in \mathbb{R}$. We write: $\vec{u} = (a, b)$.



- We have the equivalence:

$$\vec{u} = (a, b) \Leftrightarrow \vec{u} = a\vec{i} + b\vec{j}$$

  a is called **horizontal component** and b **vertical component.**
- If $\overrightarrow{0M}$ is the representative with origin 0 of the vector $\vec{u}$, then the coordinates (a, b) of point M correspond to the components (a, b) of vector $\vec{u}$.
  Ex.: Consider the vector $\vec{u} = (-3, 2)$.
  - We have: $\vec{u} = -3\vec{i} + 2\vec{j}$.
  - $\overrightarrow{0M}$ is the representative with origin 0 of vector $\vec{u}$.
    We have: M(–3, 2).



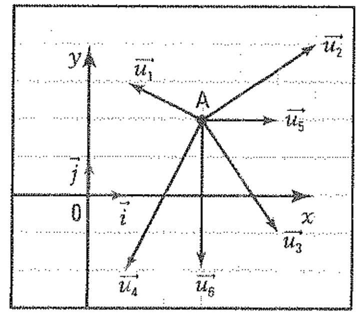
- The zero vector $\vec{0}$ is defined by $\vec{0} = (0, 0)$.
- Vectors $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$ are **equal** if and only if the corresponding components of the two vectors are equal.

$$\vec{u} = \vec{v} \Leftrightarrow a = c \text{ and } b = d$$

1. Consider, in the Cartesian plane on the right, point A(3, 2). Draw the representative with origin A of each of the following vectors.

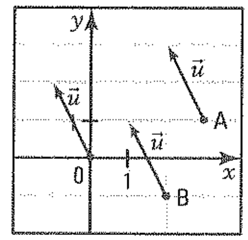
a) $\vec{u_1} = (-2, 1)$ b) $\vec{u_2} = (3, 2)$ c) $\vec{u_3} = (2, -3)$

d) $\vec{u_4} = (-2, -4)$ e) $\vec{u_5} = (2, 0)$ f) $\vec{u_6} = (0, -4)$



2. Represent vector $\vec{u}(-1, 2)$ by an arrow with origin

a) A(3, 1) b) B(2, –1) c) 0(0, 0)



3. Determine the orientation of the following vectors.

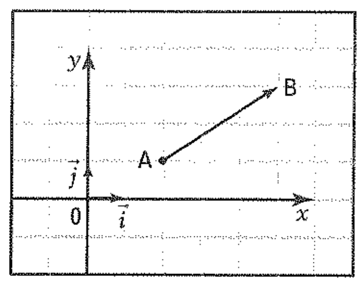
a) $\vec{a} = (3, 0)$ ***0°*** b) $\vec{b} = (0, 2)$ ***90°*** c) $\vec{c} = (2, 2)$ ***45°*** d) $\vec{d} = (-2, 0)$ ***180°***

e) $\vec{e} = (1, \sqrt{3})$ ***60°*** f) $\vec{f} = (-1, -\sqrt{3})$ ***240°*** g) $\vec{g} = (-1, -1)$ ***225°*** h) $\vec{h} = (3, 4)$ ***53.1°***

i) $\vec{i} = (-3, -4)$ ***233.1°*** j) $\vec{j} = (-3, 4)$ ***126.9°*** k) $\vec{k} = (3, -4)$ ***306.9°*** l) $\vec{l} = (3, -3)$ ***315°***

ACTIVITY 2 Calculating the components of a vector

Consider vector $\overrightarrow{AB}$ on the right with origin A(2, 1) and endpoint B(5,3).



a) 1. Write vector $\overrightarrow{AB}$ as a linear combination of vectors $\vec{i}$ and $\vec{j}$.

$\overrightarrow{AB} = 3\vec{i} + 2\vec{j}$

2. What are the components of vector $\overrightarrow{AB}$? ***(3, 2)***

b) Verify that

1. the horizontal component of vector $\overrightarrow{AB}$ is equal to the difference of the x-coordinates: $(x_B - x_A)$. ***$x_B - x_A = 5 - 2 = 3$***

2. the vertical component of vector $\overrightarrow{AB}$ is equal to the difference of the y-coordinates: $(y_B - y_A)$. ***$y_B - y_A = 3 - 1 = 2$***

CALCULATING THE COMPONENTS OF A VECTOR

- Given two points $A(x_A, y_A)$ and $B(x_B, y_B)$, the vector $\overrightarrow{AB}$ has components:

$$\overrightarrow{AB} = (x_B - x_A, y_B - y_A)$$

- Note that the horizontal component of a vector is equal to the difference between the x-coordinate of the endpoint and the x-coordinate of the origin and that the vertical component of a vector is equal to the difference between the y-coordinate of the endpoint and the y-coordinate of the origin.

Ex.: Consider the points A(2, 1), B(4, 4), C(4, –1), D(–2, –1) and E(–1, 3).

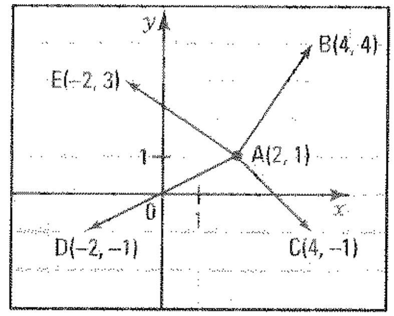
We have:

$\overrightarrow{AB} = (4 - 2, 4 - 1) = (2, 3)$

$\overrightarrow{AC} = (4 - 2, -1 - 1) = (2, -2)$

$\overrightarrow{AD} = (-2 - 2, -1 - 1) = (-4, -2)$

$\overrightarrow{AE} = (-1 - 2, 3 - 1) = (-3, 2)$



4. Consider points A(1, –2), B(–2, 5) and C(–3, –1). Find the components of

a) $\overrightarrow{AB}$ *(-3, 7)* **b)** $\overrightarrow{AC}$ *(-4, 1)* **c)** $\overrightarrow{BC}$ *(-1, -6)* **d)** $\overrightarrow{CB}$ *(1, 6)*

5. Consider A(–1, 2) and B(x, y). Find the coordinates of B if $\overrightarrow{AB} = (2, -3)$.

$\overrightarrow{AB} = (x + 1, y - 2) \Rightarrow x = 1$ and $y = -1$; B(1, -1)

6. Consider points A(0, 3), B(4, 6), C(–2, –3) and D(7, 2).

a) Find P(x, y) if $\overrightarrow{CP} = \overrightarrow{AB}$. ***$\overrightarrow{CP} = (x + 2, y + 3)$; $\overrightarrow{AB} = (4, 3) \Rightarrow$ P(2, 0)***

b) Find Q(x, y) if $\overrightarrow{QC} = \overrightarrow{CD}$. ***$\overrightarrow{QC} = (-2 - x, -3 - y)$; $\overrightarrow{CD} = (9, 5) \Rightarrow$ Q(-11, -8)***

7. Consider points A(–1, 2), B(3, 4) and C(2, –3).

a) Find the point D(x, y) if $\overrightarrow{AB} = \overrightarrow{CD}$. ***$\overrightarrow{AB} = (4, 2)$; $\overrightarrow{CD} = (x - 2, y + 3) \Rightarrow$ D(6, -1)***

b) What can be said of the quadrilateral ABDC? Justify your answer.

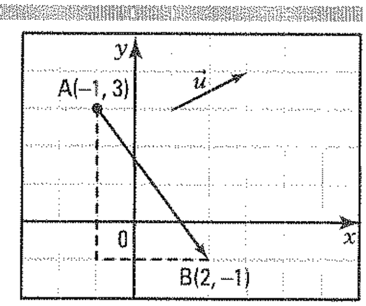
ABDC is a parallelogram since $\overrightarrow{AB} = \overrightarrow{CD}$.

c) Verify that $\overrightarrow{AC} = \overrightarrow{BD}$. ***$\overrightarrow{AC} = (3, -5)$; $\overrightarrow{BD} = (3, -5)$***

ACTIVITY 3 Norm of an algebraic vector

a) 1. Consider the vector $\vec{u} = (2, 1)$. Calculate $\|\vec{u}\|$. ***$\|\vec{u}\| = \sqrt{5}$***

2. Consider the vector $\vec{u} = (a, b)$. Establish a formula to calculate $\|\vec{u}\|$. ***$\|\vec{u}\| = \sqrt{a^2 + b^2}$***



b) Consider the vector $\overrightarrow{AB}$ such that A(–1, 3) and B(2, –1).

1. Calculate the components of $\overrightarrow{AB}$. ***$\overrightarrow{AB} = (3, -4)$***

2. Calculate $\|\overrightarrow{AB}\|$. ***$\|\overrightarrow{AB}\| = 5$***

c) Let $\overrightarrow{AB}$ be a vector such that $A(x_A, y_A)$ and $B(x_B, y_B)$. Establish a formula to calculate the norm of $\overrightarrow{AB}$. ***$\|\overrightarrow{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$***

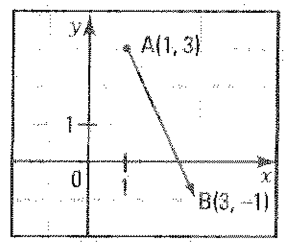
NORM OF AN ALGEBRAIC VECTOR

- If $\vec{u} = (a, b)$, then $\|\vec{u}\| = \sqrt{a^2 + b^2}$.
- If $A(x_A, y_A)$ and $B(x_B, y_B)$, then $\|\overrightarrow{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$.

Ex.: Consider A(1, 3) and B(3, −1). We have:

$\overrightarrow{AB} = (2, -4)$.

$\|\overrightarrow{AB}\| = \sqrt{(2)^2 + (-4)^2} = \sqrt{4+16} = \sqrt{20}$.



8. Consider A(−1, 2) and B(3, 4). Calculate

a) $\|\overrightarrow{AB}\|$ *$\sqrt{20}$* b) $\|\overrightarrow{BA}\|$ *$\sqrt{20}$* c) $\|\overrightarrow{AA}\|$ *0*

9. Given any three points A, B and C, answer true or false.

a) $\|\overrightarrow{AB}\| = \|\overrightarrow{BA}\|$ ***True***

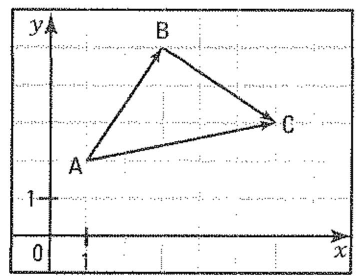
b) $\|\overrightarrow{AB} + \overrightarrow{BC}\| = \|\overrightarrow{AB}\| + \|\overrightarrow{BC}\|$ ***False***

c) $\|\overrightarrow{AB} - \overrightarrow{CB}\| = \|\overrightarrow{AC}\|$ ***True***

d) $\|\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}\| = 0$ ***True***

ACTIVITY 4 Operations between algebraic vectors

Consider points A(1, 2), B(3, 5) and C(6, 3).



a) 1. Calculate the components of

1) $\overrightarrow{AB}$ ***(2, 3)*** 2) $\overrightarrow{BC}$ ***(3, −2)*** 3) $\overrightarrow{AC}$ ***(5, 1)***

2. What is the sum vector $\overrightarrow{AB} + \overrightarrow{BC}$? ***$\overrightarrow{AC}$***

3. Verify that

1) the horizontal component of the sum vector is equal to the sum of the horizontal components. ***5 = 2 + 3***

2) the vertical component of the sum vector is equal to the sum of the vertical components. ***1 = 3 + −2***

b) If $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$, what are the components of the sum $\vec{u} + \vec{v}$? ***$\vec{u} + \vec{v} = (a + c, b + d)$***

c) 1. What is the difference vector $\overrightarrow{AC} - \overrightarrow{BC}$? ***$\overrightarrow{AB}$***

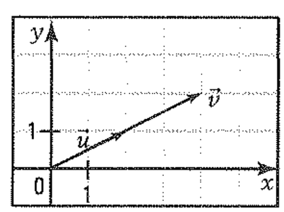
2. Verify that

1) the horizontal component of the difference vector is equal to the difference of the horizontal components. ***2 = 5 − 3***

2) the vertical component of the difference vector is equal to the difference of the vertical components. ***3 = 1 − (−2)***

d) If $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$, what are the components of the difference $\vec{u} - \vec{v}$? ***$\vec{u} - \vec{v} = (a - c, b - d)$***

e) Let $\vec{u} = (2, 1)$ be the vector represented on the right.



1. Draw the vector $\vec{v} = 2\vec{u}$.

2. What are the components of $\vec{v}$? ***(4, 2)***

f) 1. If $\vec{u} = (a, b)$, find the components of the vector $\vec{v}$ if $\vec{v} = k\vec{u}$ ($k \in \mathbb{R}$).

$\vec{v} = (ka, kb)$

2. If $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$, under what conditions are vectors $\vec{u}$ and $\vec{v}$ parallel?

If there exists a real number k such that c = ka and d = kb.

OPERATIONS BETWEEN ALGEBRAIC VECTORS

Given vectors $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$, we have:

$$\boxed{\vec{u} + \vec{v} = (a + c, b + d)}$$
$$\boxed{\vec{u} - \vec{v} = (a - c, b - d)}$$
$$\boxed{k\vec{u} = (ka, kb)} \quad (k \in \mathbb{R})$$

10. Consider the vectors $\vec{u} = (-3, 4)$, $\vec{v} = (2, -3)$ and $\vec{w} = (1, -2)$. Calculate

a) $-2\vec{u}$ ***(6, −8)***
b) $\vec{u} + \vec{v} - \vec{w}$ ***(−2, 3)***
c) $-\vec{w}$ ***(−1, 2)***
d) $\|-3\vec{u}\|$ ***15***
e) $2\vec{u} + 3\vec{v}$ ***(0, −1)***
f) $3\vec{u} - 2\vec{v} + \vec{w}$ ***(−12, 16)***
g) $\|\vec{u} + \vec{v}\|$ $\boldsymbol{\sqrt{2}}$
h) $\|\vec{u} - \vec{v} + 5\vec{w}\|$ ***3***
i) $0\vec{u}$ ***(0, 0)***

11. Consider the points A(−1, 2), B(1,3), C(2, −3) and D(3, −4). Calculate

a) $2\overrightarrow{AB}$ ***(4, 2)***
b) $\overrightarrow{AB} + \overrightarrow{AC}$ ***(5, −4)***
c) $\overrightarrow{AD} - \overrightarrow{BC}$ ***(3, 0)***
d) $3\overrightarrow{AB} - 2\overrightarrow{AC}$ ***(0, 13)***
e) $\|2\overrightarrow{AB} - 4\overrightarrow{CD}\|$ ***6***
f) $\|\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC}\|$ $\boldsymbol{2\sqrt{13}}$

12. In each of the following cases, indicate if vectors $\vec{u}$ and $\vec{v}$ are parallel. If so, justify your answer.

a) $\vec{u} = (-1, 2)$ and $\vec{v} = (3, -6)$ ***Yes,*** $\vec{v} = -3\vec{u}$
b) $\vec{u} = (2, -3)$ and $\vec{v} = (4, -5)$ ***No***
c) $\vec{u} = (6, -9)$ and $\vec{v} = (4, -6)$ ***Yes,*** $\vec{v} = \frac{2}{3}\vec{u}$
d) $\vec{u} = (0, 1)$ and $\vec{v} = (2, 0)$ ***No***
e) $\vec{u} = (2, 0)$ and $\vec{v} = (-4, 0)$ ***Yes,*** $\vec{v} = -2\vec{u}$
f) $\vec{u} = (1, 2)$ and $\vec{v} = (-1, -2)$ ***Yes,*** $\vec{v} = -\vec{u}$

13. Determine the unit vectors among the following vectors.

$\vec{u_1} = \left(-\frac{3}{5}, \frac{4}{5}\right)$, $\vec{u_2} = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\vec{u_3} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\vec{u_4} = (-1, 0)$, $\vec{u_5} = (\cos\theta, \sin\theta)$

$\vec{u_1}, \vec{u_3}, \vec{u_4}$ ***and*** $\vec{u_5}$ ***are unit vectors.***

14. a) Consider $\vec{u} = (3, 4)$, verify that $\frac{\vec{u}}{\|\vec{u}\|}$ is a unit vector.

$\frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{3}{5}, \frac{4}{5}\right); \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$

b) Let $\vec{u} = (a, b)$ be a nonzero vector. Show that $\frac{\vec{u}}{\|\vec{u}\|}$ is a unit vector.

$\|\vec{u}\| = \sqrt{a^2 + b^2}$; $\frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right)$

$\left\|\frac{\vec{u}}{\|\vec{u}\|}\right\| = \sqrt{\left(\frac{a}{\sqrt{a^2 + b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2}}\right)^2} = \sqrt{\frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2}} = \sqrt{\frac{a^2 + b^2}{a^2 + b^2}} = \sqrt{1} = 1$

15. Consider $\vec{u} = (a, b)$. Show that $\|k\vec{u}\| = |k| \cdot \|\vec{u}\|$ $(k \in \mathbb{R})$.

$\|k\vec{u}\| = \|k(a, b)\| = \|(ka, kb)\| = \sqrt{(ka)^2 + (kb)^2} = \sqrt{k^2(a^2 + b^2)}$
$= \sqrt{k^2} \cdot \sqrt{a^2 + b^2} = |k| \cdot \|\vec{u}\|$

16. Consider $\vec{v} = (3, 4)$.

a) 1. Find the unit vector $\vec{u_1}$ having the same direction as $\vec{v}$.

$\vec{u_1} = \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{3}{5}, \frac{4}{5}\right)$

2. Find the vector $\vec{v_1}$, with norm equal to 2, having the same direction as $\vec{v}$.

$\vec{v_1} = 2\vec{u_1} = \left(\frac{6}{5}, \frac{8}{5}\right)$

b) 1. Find the unit vector $\vec{u_2}$ having the direction opposite to $\vec{v}$.

$\vec{u_2} = -\vec{u_1} = \left(\frac{-3}{5}, \frac{-4}{5}\right)$

2. Find the vector $\vec{v_2}$, with norm equal to 7, having direction opposite to $\vec{v}$.

$\vec{v_2} = 7\vec{u_2} = \left(\frac{-21}{5}, \frac{-28}{5}\right)$

17. Determine the direction and the norm of each of the following vectors.

a) $\vec{a} = (-1, -1)$ *45° West of South;* $\|\vec{a}\| = \sqrt{2}$

b) $\vec{b} = (0, 2)$ *North;* $\|\vec{b}\| = 2$

c) $\vec{c} = (2, 3)$ *56.3° North of East;* $\|\vec{c}\| = \sqrt{13}$

d) $\vec{d} = (-2, 2)$ *45° West of North;* $\|\vec{d}\| = 2\sqrt{2}$

e) $\vec{e} = (1, -\sqrt{3})$ *60° South of East;* $\|\vec{e}\| = 4$

18. Find the components of the vectors opposite to the following vectors.

a) $\vec{a}(-1, 2)$ *(1, −2)* **b)** $\vec{b}(2, -3)$ *(−2, 3)* **c)** $\vec{c}(2, 3)$ *(−2, −3)*

19. In each of the following cases, find the components of vector $\vec{v}$.

a) $\vec{u} = (-1, 2), \vec{w} = (2, -3)$ and $\vec{u} + \vec{v} = \vec{w}$ $\vec{v} = (3, -5)$

b) $\vec{u} = (2, -3), \vec{w} = (3, -1)$ and $\vec{u} - \vec{v} = \vec{w}$ $\vec{v} = (-1, -2)$

c) $\vec{w} = (4, -2)$ and $\vec{w} = -2\vec{v}$ $\vec{v} = (-2, 1)$

d) $\vec{u} = (2, -1$, $\vec{w} = (2, 1)$ and $\vec{w} = 3\vec{u} + 2\vec{v}$ $\vec{v} = (-2, 2)$

e) $\vec{u} = (1, 2), \vec{w} = (-1, -2)$ and $\vec{w} = 2\vec{u} - 3\vec{v}$ $\vec{v} = (1, 2)$

20. Write each of the following vectors as a linear combination of vectors $\vec{i}$ and $\vec{j}$.

a) $\overrightarrow{AB}$ if A(−1, 2) and B(3, −1) $\overrightarrow{AB} = 4\vec{i} - 3\vec{j}$

b) $\overrightarrow{CD}$ if C(2, −3) and D(−3, 1) $\overrightarrow{CD} = -5\vec{i} + 4\vec{j}$

c) $\overrightarrow{EF}$ if E(0, 2) and F(−1, 0) $\overrightarrow{EF} = -\vec{i} - 2\vec{j}$

d) $\overrightarrow{GH}$ if G(2, −4) and H(2, 1) $\overrightarrow{GH} = 0\vec{i} + 5\vec{j}$

21. Consider vectors $\vec{u_1} = (1, 2)$ and $\vec{u_2} = (2, 4)$.

a) Do vectors $\vec{u_1}$ and $\vec{u_2}$ form a vector basis? Justify your answer.

No, because $\vec{u_1}$ *and* $\vec{u_2}$ *are collinear.*

b) Is it possible to write any vector $\vec{v}$ of the plane as linear combination of vectors $\vec{u_1}$ and $\vec{u_2}$?

No

22. Consider vectors $\vec{u_1} = (1, 2)$ and $\vec{u_2} = (-1, 1)$.

a) Do vectors $\vec{u_1}$ and $\vec{u_2}$ form a basis? Justify your answer.

Yes, because they are not parallel.

b) Is it possible to write any vector $\vec{v}$ of the plane as linear combination of vectors $\vec{u_1}$ and $\vec{u_2}$?

Yes

c) Express vector $\vec{v} = (1, 8)$ as a linear combination of vectors $\vec{u_1}$ and $\vec{u_2}$.

$\vec{v} = 3\vec{u_1} + 2\vec{u_2}$

23. Consider vectors $\vec{u_1} = (-1, 2)$ and $\vec{u_2} = (3, 2)$ in the plane.

a) Explain why vectors $\vec{u_1}$ and $\vec{u_2}$ form a vector basis of the plane.

Vectors $\vec{u}_1$ and $\vec{v}_2$ are not parallel.

b) Find the components of vector $\vec{v}$ if $\vec{v} = 3\vec{u_1} + 2\vec{u_2}$. $\vec{v} = (3, 10)$

c) Write vector $\vec{w} = (1, 6)$ as a linear combination of vectors $\vec{u_1}$ and $\vec{u_2}$.

$\vec{w} = 2\vec{u_1} + \vec{u_2}$

24. Consider, in the Cartesian plane, the basis formed by the vectors $\vec{u_1} = (2, 1)$ and $\vec{u_2} = (-1, 2)$. Express each of the following vectors as a linear combination of vectors $\vec{u_1}$ and $\vec{u_2}$.

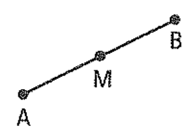
a) $\vec{u} = (4, 7)$ $\vec{u} = 3\vec{u_1} + 2\vec{u_2}$

b) $\vec{v} = (-7, 4)$ $\vec{v} = -2\vec{u_1} + 3\vec{u_2}$

c) $\vec{w} = (5, 0)$ $\vec{w} = 2\vec{u_1} - \vec{u_2}$

ACTIVITY 5 Midpoint of a line segment

A line segment AB and its midpoint M are represented on the right.



a) 1. What can be said of vectors $\overrightarrow{MA}$ and $\overrightarrow{MB}$? ***They are opposite.***

2. Complete: $\overrightarrow{MA} + \overrightarrow{MB} = \vec{0}$

b) Complete using the appropriate real number.

1. $\overrightarrow{AB} = \boxed{2}\,\overrightarrow{AM}$ 2. $\overrightarrow{AM} = \boxed{\frac{1}{2}}\,\overrightarrow{AB}$

MIDPOINT OF A LINE SEGMENT

The following propositions are equivalent.

1. M is the midpoint of AB.
2. $\overrightarrow{MA} + \overrightarrow{MB} = \vec{0}$.
3. $\overrightarrow{AM} = \overrightarrow{MB}$.
4. $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$.
5. $\overrightarrow{AB} = 2\overrightarrow{AM}$.

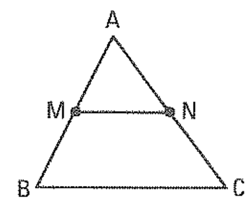
25. Let A(x_A, y_A) and B(x_B, y_B) be the endpoints of a line segment AB. Let M be the midpoint of line segment AB.

Show that the coordinates of M are: $\left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right)$.

M midpoint of $\overrightarrow{AB} \Leftrightarrow \overrightarrow{AM} = \overrightarrow{MB}$

$$\Leftrightarrow (x_M - x_A, y_M - y_A) = (x_B - x_M, y_B - y_M)$$
$$\Leftrightarrow x_M - x_A = x_B - x_M \text{ and } y_M - y_A = y_B - y_M$$
$$\Leftrightarrow 2x_M = x_A + x_B \text{ and } 2y_M = y_A + y_B$$
$$\Leftrightarrow x_M = \frac{x_A + x_B}{2} \text{ and } y_M = \frac{y_A + y_B}{2}$$

26. Let A(3, 4), B(–1, 0) and C(5, 2) be the vertices of a triangle ABC. Let M and N denote the midpoints of sides AB and AC. Verify the following theorem: The line segment joining the midpoints of 2 sides of a triangle measures half the length of the 3rd side and is parallel to the 3rd side.

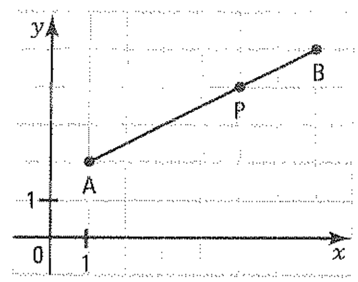


M(1, 2), N(4, 3), $\overrightarrow{MN} = (3, 1)$; $\overrightarrow{BC} = (6, 2)$. ***We have:*** $\overrightarrow{BC} = 2\overrightarrow{MN}$

$\Rightarrow \overrightarrow{BC} \,//\, \overrightarrow{MN}$ ***and*** $\|\overrightarrow{BC}\| = 2\|\overrightarrow{MN}\|$

ACTIVITY 6 Dividing point of a line segment

Consider the line segment AB on the right having endpoints A(1, 2) and B(7, 5).



a) Locate the point P such that $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB}$.

b) Determine the coordinates of P.

P(x, y); $\overrightarrow{AP} = (x - 1, y - 2)$; $\overrightarrow{AB} = (6, 3)$

$\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB} \Rightarrow (x - 1, y - 2) = \frac{2}{3}(6, 3)$

$(x - 1, y - 2) = (4, 2)$

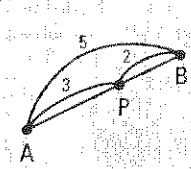
hence $x - 1 = 4$ ***and*** $y - 2 = 2$

$x = 5$ ***and*** $y = 4$

Thus, P(5, 4).

DIVIDING POINT OF A LINE SEGMENT

Point P on the right divides the line segment AB



– in a 3:2 ratio or $\frac{3}{2}$ from A,

or

– in a 2:3 ratio or $\frac{2}{3}$ from B.

We can also say that P is located at $\frac{3}{5}$ of the line segment AB from point A or at $\frac{2}{5}$ of line segment AB from B.

This situation is translated by the vector equality:

$\overrightarrow{AP} = \frac{3}{5}\overrightarrow{AB}$ or $\overrightarrow{BP} = \frac{2}{5}\overrightarrow{BA}$.

Ex.: Consider A(1, 4) and B(7, 1).

Let us find point P if $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB}$.

P(x, y); $\overrightarrow{AP} = (x - 1, y - 4)$; $\overrightarrow{AB} = (6, -3)$

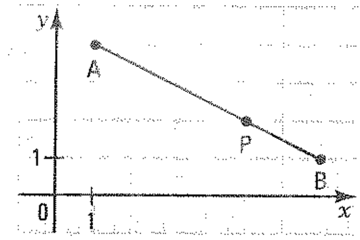
$\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB} \Rightarrow (x - 1, y - 4) = \frac{2}{3}(6, -3)$

$(x - 1, y - 4) = (4, -2)$

We deduce that $x - 1 = 4$ and $y - 4 = -2$

$x = 5$ and $y = 2$

Thus, we obtain P(5, 2).



27. Consider A(–2, 1) and B(2, 9). Find the coordinates of points P and Q if

a) $\overrightarrow{AP} = \frac{3}{4}\overrightarrow{AB}$ ***P(1, 7)***

b) $\overrightarrow{BQ} = \frac{3}{4}\overrightarrow{BA}$ ***Q(–1, 3)***

28. Consider A(–1, 5) and B(5, 2). Find the following points.

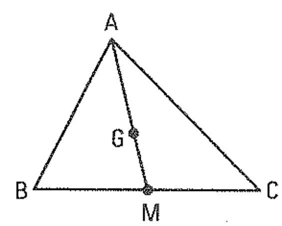
a) P if $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB}$ ***P(1, 4)***

b) Q if $\overrightarrow{AQ} = \frac{2}{3}\overrightarrow{AB}$ ***Q(3, 3)***

c) R if $\overrightarrow{AR} = \frac{1}{2}\overrightarrow{AB}$ $\boldsymbol{R\left(2, \frac{7}{2}\right)}$

d) S if $\overrightarrow{BS} = \frac{5}{6}\overrightarrow{BA}$ $\boldsymbol{S\left(0, \frac{9}{2}\right)}$

29. Consider triangle ABC on the right whose vertices are A(–5, 6), B(–2, 2) and C(4, 4).



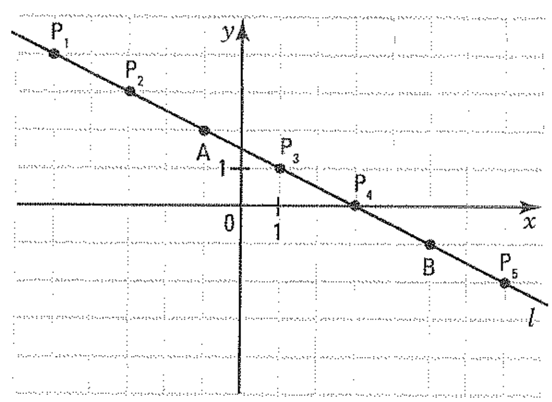
a) Determine the coordinates of point M, if $\overrightarrow{AM}$ is the median relative to side BC. ***M(1, 3)***

b) Determine the coordinates of the centre of gravity G of the triangle knowing that G divides each median in a 2:1 ratio from the corresponding vertex.

$\overrightarrow{AG} = \frac{2}{3}\overrightarrow{AM}$; $(x + 5, y - 6) = \frac{2}{3}(6, -3)$; ***G(–1, 4).***

c) Verify the following property: If G is the centre of gravity of a triangle ABC then $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$. $\overrightarrow{GA} = (4, -2)$; $\overrightarrow{GB} = (1, 2)$; $\overrightarrow{GC} = (-5, 0)$. ***We have:*** $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$

30. Consider the line l passing through points A(–1, 2) and B(5, –1).



a) Find the coordinates of points P_1, P_2, P_3, P_4 and P_5 knowing that

1. $\overrightarrow{AP_1} = -\frac{2}{3}\overrightarrow{AB}$ ***P_1(–5, 4)***

2. $\overrightarrow{BP_2} = \frac{4}{3}\overrightarrow{BA}$ ***P_2(–3, 3)***

3. $\overrightarrow{AP_3} = \frac{1}{3}\overrightarrow{AB}$ ***P_3(1, 1)***

4. $\overrightarrow{AP_4} = \frac{2}{3}\overrightarrow{AB}$ ***P_4(3, 0)***

5. $\overrightarrow{BP_5} = -\frac{1}{3}\overrightarrow{BA}$ ***P_5(7, –2)***

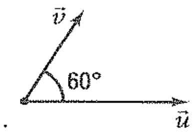
b) Locate points P_1, P_2, P_3, P_4 and P_5 on line l.

6.5 Scalar product

ACTIVITY 1 Definition of the scalar product

The scalar product of two vectors $\vec{u}$ and $\vec{v}$ written $\vec{u} \cdot \vec{v}$ is defined by:

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \times \|\vec{v}\| \times \cos\theta$ where $\theta \in [0, \pi]$ is the angle formed by vectors $\vec{u}$ and $\vec{v}$.



a) Calculate the scalar product of vectors $\vec{u}$ and $\vec{v}$ on the right if $\|\vec{u}\| = 4$ and $\|\vec{v}\| = 3$.

$\vec{u} \cdot \vec{v} = 4 \times 3 \times \cos 60° = 6.$

b) Explain why the scalar product of any two vectors $\vec{u}$ and $\vec{v}$ is a real number and not a vector.

$\|\vec{u}\| \in \mathbb{R}$, $\|\vec{v}\| \in \mathbb{R}$, $\cos\theta \in \mathbb{R}$. The product of 3 real numbers is a real number.

c) What can be said about the angle θ between two vectors if

1. $\vec{u} \cdot \vec{v} > 0$? *$\cos\theta > 0 \Rightarrow \theta$ is acute.*
2. $\vec{u} \cdot \vec{v} < 0$? *$\cos\theta < 0 \Rightarrow \theta$ is obtuse.*
3. $\vec{u} \cdot \vec{v} = 0$? *$\cos\theta = 0 \Rightarrow \theta$ is right.*

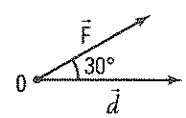
d) Complete:

1. $\vec{u} \cdot \vec{v} = 0 \Rightarrow$ *$\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\vec{u} \perp \vec{v}$*
2. $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\vec{u} \perp \vec{v} \Rightarrow \vec{u} \cdot \vec{v} =$ *0*

e) $\vec{u}$ and $\vec{v}$ being two nonzero vectors, complete:

1. $\vec{u} \cdot \vec{v} = 0 \Leftrightarrow$ *$\vec{u}$ and $\vec{v}$ are perpendicular.*
2. $\vec{u} \cdot \vec{v} \neq 0 \Leftrightarrow$ *$\vec{u}$ and $\vec{v}$ are not perpendicular.*

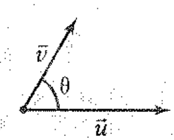
f) A force $\vec{F}$ of 10 newtons is applied to an object 0, which then moves 5 m in direction $\vec{d}$.



In physics, the work W performed by this force is defined by the scalar product $W = \vec{F} \cdot \vec{d}$ where force $\vec{F}$ is in newtons, the displacement $\vec{d}$ is in metres and work W is in joules.
Calculate the work W. *$W = 10 \times 5 \times \cos 30° = 43.3$ joules.*

SCALAR PRODUCT

- The scalar product of two vectors $\vec{u}$ and $\vec{v}$, written $\vec{u} \cdot \vec{v}$, is the real number defined by

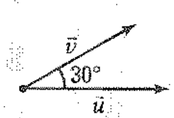


$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \times \|\vec{v}\| \times \cos\theta$$

where $\theta \in [0, \pi]$ is the angle formed by vectors $\vec{u}$ and $\vec{v}$.

- Given two nonzero vectors $\vec{u}$ and $\vec{v}$ we have the equivalence:

$$\vec{u} \cdot \vec{v} = 0 \Leftrightarrow \vec{u} \perp \vec{v}$$



Ex.: Vectors $\vec{u}$ and $\vec{v}$ form a 30° angle and have norms $\|\vec{u}\| = 4$ and $\|\vec{v}\| = 3$.
We have: $\vec{u} \cdot \vec{v} = 4 \times 3 \times \cos 30° = 6\sqrt{3}$.

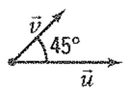
1. Calculate the following scalar products.

a) $\vec{i} \cdot \vec{j}$ ***0***   b) $\vec{i} \cdot \vec{i}$ ***1***   c) $\vec{j} \cdot \vec{j}$ ***1***

2. Let $\vec{u}$ be a vector such that $\|\vec{u}\| = 5$. Calculate $\vec{u} \cdot \vec{u}$. ***25***

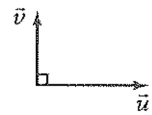
3. Calculate, in each case, $\vec{u} \cdot \vec{v}$ if

a) $\|\vec{u}\| = 3$, $\|\vec{v}\| = 2$



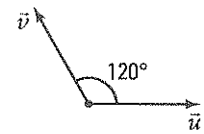
$\boldsymbol{\vec{u} \cdot \vec{v} = 3\sqrt{2}}$

b) $\|\vec{u}\| = 3$, $\|\vec{v}\| = 2$



$\boldsymbol{\vec{u} \cdot \vec{v} = 0}$

c) $\|\vec{u}\| = 4$, $\|\vec{v}\| = 3$



$\boldsymbol{\vec{u} \cdot \vec{v} = -6}$

4. Show that, for any vector $\vec{u}$, we have: $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$.

$\boldsymbol{\vec{u} \cdot \vec{u} = \|\vec{u}\| \times \|\vec{u}\| \times \cos 0° = \|\vec{u}\|^2 \Rightarrow \|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}}$

5. Show that the scalar product is commutative.

Let us show that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

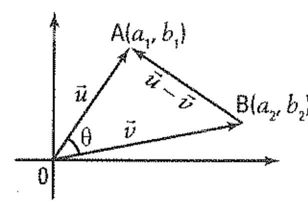
We have: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \times \|\vec{v}\| \times \cos\theta$

$= \|\vec{v}\| \times \|\vec{u}\| \times \cos\theta$ ***(Multiplication in ℝ is commutative.)***

$= \vec{v} \cdot \vec{u}$

ACTIVITY 2 Scalar product in the Cartesian plane

a) Consider vectors $\vec{u} = (a_1, b_1)$; $\vec{v} = (a_2, b_2)$ and $\vec{u} - \vec{v}$ represented by the arrows $\overrightarrow{0A}$, $\overrightarrow{0B}$ and $\overrightarrow{BA}$ respectively. Justify the steps showing that $\vec{u} \cdot \vec{v} = a_1a_2 + b_1b_2$.



Steps	Justifications
1. $\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$	***Cosine law applied to △0AB.***
2. $\|\vec{u}\|\|\vec{v}\| \cos\theta = \frac{1}{2}\left[\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2\right]$	***We isolate*** $\|\vec{u}\|\|\vec{v}\| \cos\theta$.
3. $\vec{u} \cdot \vec{v} = \frac{1}{2}\left[\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2\right]$	***Definition of the scalar product.***

4. Reduce the right hand side of the last equality to show that $\vec{u} \cdot \vec{v} = a_1a_2 + b_1b_2$.

$\vec{u} \cdot \vec{v} = \frac{1}{2}\left[(a_1^2 + b_1^2) + (a_2^2 + b_2^2) - (a_1 - a_2)^2 - (b_1 - b_2)^2\right]$

$\vec{u} \cdot \vec{v} = \frac{1}{2}\left[a_1^2 + b_1^2 + a_2^2 + b_2^2 - a_1^2 + 2a_1a_2 - a_2^2 - b_1^2 - b_2^2 + 2b_1b_2\right]$

$\vec{u} \cdot \vec{v} = \frac{1}{2}\left[2a_1a_2 + 2b_1b_2\right]$

$\vec{u} \cdot \vec{v} = a_1a_2 + b_1b_2$

b) Consider vectors $\vec{u} = (1, \sqrt{3})$ and $\vec{v} = (-1, \sqrt{3})$.

1. Calculate $\vec{u} \cdot \vec{v}$. ***2***
2. Determine, using the definition of the scalar product, the cosine of the angle θ formed by vectors $\vec{u}$ and $\vec{v}$.

   $\cos \theta = \frac{u \cdot v}{\|\vec{u}\|\|\vec{v}\|} = \frac{2}{2 \times 2} = \frac{1}{2}$
3. What is the angle θ formed by vectors $\vec{u}$ and $\vec{v}$? ***60°***

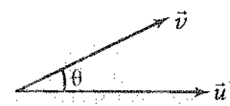
SCALAR PRODUCT IN THE CARTESIAN PLANE

- Let $\vec{u} = (a_1, b_1)$ and $\vec{v} = (a_2, b_2)$, we have:

$$\vec{u} \cdot \vec{v} = a_1a_2 + b_1b_2$$

- Let θ be the angle formed by vectors $\vec{u}$ and $\vec{v}$, we have:

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

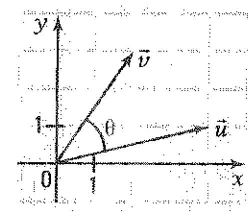


Ex.: Let $\vec{u} = (4, 1)$ and $\vec{v} = (2, 3)$. We have:

$$\vec{u} \cdot \vec{v} = 4 \times 2 + 1 \times 3 = 11$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{11}{\sqrt{17}\sqrt{13}} = \frac{11}{\sqrt{221}}$$

hence $\theta = \cos^{-1} \frac{11}{\sqrt{221}} = 42.3°$



6. Calculate the scalar product $\vec{u} \cdot \vec{v}$ in each of the following cases.

a) $\vec{u} = (2, 3)$ and $\vec{v} = (4, -1)$ ***5*** **b)** $\vec{u} = 3\vec{i} - 2\vec{j}$ and $\vec{v} = -2\vec{i} + \vec{j}$ ***-8***

7. Determine, in each case, whether vectors $\vec{u}$ and $\vec{v}$ are perpendicular.

a) $\vec{u} = (2, -3)$ and $\vec{v} = (3, 2)$ ***Yes*** **b)** $\vec{u} = 2\vec{i} + 5\vec{j}$ and $\vec{v} = 7\vec{i} - 3\vec{j}$ ***No***

8. Find, in each of the following cases, the value of a if vectors $\vec{u}$ and $\vec{v}$ are perpendicular.

a) $\vec{u} = (a + 1, 2)$ and $\vec{v} = (-3, a - 2)$

a = -7

b) $\vec{u} = (a + 2, -1)$ and $\vec{v} = (a - 2, 5)$

a = -3 or a = 3

9. Consider points A(1, 3), B(2, 5), C(2, 2) and D(4, 1).
Show that the lines AB and CD are perpendicular.

It is sufficient to show that $\overrightarrow{AB} \cdot \overrightarrow{CD} = 0$. $\overrightarrow{AB} = (1, 2)$, $\overrightarrow{CD} = (2, -1)$.

We have: $\overrightarrow{AB} \cdot \overrightarrow{CD} = 0$.

ACTIVITY 3 Properties of the scalar product

a) Consider vectors $\vec{u}=(1,2)$, $\vec{v}=(2,1)$ and $\vec{w}=(3,2)$.

1. Verify that $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$.

   $\vec{u}\cdot\vec{v}=4;\ \vec{v}\cdot\vec{u}=4$

2. Verify that $2\vec{u}\cdot 3\vec{v}=(2\times 3)\,\vec{u}\cdot\vec{v}$.

   $2\vec{u}\cdot 3\vec{v}=(2, 4)\cdot(6, 3)=24;\ (2\times 3)\,\vec{u}\cdot\vec{v}=6(4)=24$

3. Verify that $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$.

   $\vec{u}\cdot(\vec{v}+\vec{w})=(1, 2)\cdot(5, 3)=11;\ \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}=4+7=11$

b) Consider vectors $\vec{u}=(a_1,b_1)$, $\vec{v}=(a_2,b_2)$ and $\vec{w}=(a_3,b_3)$.

1. Show that $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$.

$\vec{u}\cdot\vec{v}$	$=a_1,a_2+b_2,b_2$	*Definition of the scalar product.*
	$=a_2a_1+b_2b_1$	*Multiplication in $\mathbb{R}$ is commutative.*
	$=\vec{v}\cdot\vec{u}$	*Definition of the scalar product.*

2. Show that $r\vec{u}\cdot s\vec{v}=(rs)\vec{u}\cdot\vec{v}$.

$r\vec{u}\cdot s\vec{v}$	$=(ra_1,rb_1)\cdot(sa_2,sb_2)$	*Multiplication of a vector by a real number.*
	$=rsa_1a_2+rsb_1b_2$	*Definition of the scalar product.*
	$=rs(a_1a_2+b_1b_2)$	*Factorization.*
	$=(rs)\vec{u}\cdot\vec{v}$	*Definition of the scalar product.*

3. Show that $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$.

$\vec{u}\cdot(\vec{v}+\vec{w})$	$=(a_1,b_1)\cdot(a_2+a_3,\ b_2+b_3)$	*Definition of the sum of 2 vectors.*
	$=a_1(a_2+a_3)+b_1(b_2+b_3)$	*Definition of the scalar product.*
	$=a_1a_2+a_1a_3+b_1b_2+b_1b_3$	*Multiplication is distributive over addition in $\mathbb{R}$.*
	$=a_1a_2+b_1b_2+a_1a_3+b_1b_3$	*Addition is commutative in $\mathbb{R}$.*
	$=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$	*Definition of the scalar product.*

PROPERTIES OF THE SCALAR PRODUCT

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be any three vectors, we have:

- $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$
- $r\vec{u}\cdot s\vec{v}=(rs)\vec{u}\cdot\vec{v}$ $\quad(r\in\mathbb{R}, s\in\mathbb{R})$
- $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

10. Knowing that $\vec{u}\cdot\vec{v}=3$ and $\vec{u}\cdot\vec{w}=2$, calculate

a) $(2\vec{u})\cdot\vec{v}$ ***6*** b) $\vec{u}\cdot(5\vec{v})$ ***15*** c) $(-2\vec{u})\cdot(3\vec{v})$ ***-18***

d) $\vec{u}\cdot(\vec{v}+\vec{w})$ ***5*** e) $\vec{u}\cdot(\vec{v}-\vec{w})$ ***1*** f) $2\vec{u}\cdot(3\vec{v}-2\vec{w})$ ***10***

11. Knowing that $\vec{u}\perp\vec{v}$ and that $\|\vec{u}\|=3$ and $\|\vec{v}\|=2$, calculate

a) $\vec{u}\cdot(\vec{u}+\vec{v})$ $\vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}=9+0=9$

b) $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v})$ $\vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}-\vec{v}\cdot\vec{v}=5$

c) $(2\vec{u}+3\vec{v})\cdot(3\vec{u}-2\vec{v})$ $6\vec{u}\cdot\vec{u}-4\vec{u}\cdot\vec{v}+9\vec{v}\cdot\vec{u}-6\vec{v}\cdot\vec{v}=30$

12. Consider three points A, B and C on a triangle.
Determine if triangle ABC is right. If, so indicate the right angle.

a) A(–1, 2), B(2, 3) and C(–2, 5)

$\overrightarrow{AB}=(3,1)$; $\overrightarrow{AC}=(-1,3)$; $\overrightarrow{AB}\cdot\overrightarrow{AC}=0\Rightarrow\Delta ABC$ ***is right at A.***

b) A(2, 3), B(–1, 4) and C(0, 7)

$\overrightarrow{BA}=(3,-1)$; $\overrightarrow{BC}=(1,3)$; $\overrightarrow{BA}\cdot\overrightarrow{BC}=0\Rightarrow\Delta ABC$ ***is right at C.***

c) A(1, 2), B(2, –1) and C(–1, 1)

$\overrightarrow{AB}\cdot\overrightarrow{AC}\neq0$; $\overrightarrow{BA}\cdot\overrightarrow{BC}\neq0$ ***and*** $\overrightarrow{CA}\cdot\overrightarrow{CB}\neq0\Rightarrow\Delta ABC$ ***is not a right triangle.***

13. Consider two vectors $\vec{u}$ and $\vec{v}$ such that $\|\vec{u}\|=2$ and $\|\vec{v}\|=3$. Calculate $\vec{u}\cdot\vec{v}$ if

a) $\vec{u}$ and $\vec{v}$ have the same direction. $\vec{u}\cdot\vec{v}=6$

b) $\vec{u}$ and $\vec{v}$ have opposite directions. $\vec{u}\cdot\vec{v}=-6$

14. Consider the vector $\vec{w}=(3,-4)$.

a) Find the two unit vectors $\vec{u_1}$ and $\vec{u_2}$ which are perpendicular to $\vec{w}$.

$\vec{u_1}=\left(\frac{4}{5},\frac{3}{5}\right)$ ***and*** $\vec{u_2}\left(\frac{-4}{5},\frac{-3}{5}\right)$

b) Find the two vectors $\vec{v_1}$ and $\vec{v_2}$ with norm equal to 10 and which are perpendicular to $\vec{w}$.

$\vec{v_1}=10\vec{u_1}=(8,6)$ ***and*** $\vec{v_2}=10\vec{u_2}=(-8,-6)$

15. Calculate, in each case, the angle θ formed by vectors $\vec{u}$ and $\vec{v}$.

a) $\vec{u}=(-3,2)$ and $\vec{v}=(1,4)$ $\cos\theta=0.3363;\ \theta=70.3°$

b) $\vec{u}=(2,-5)$ and $\vec{v}=(3,2)$ $\cos\theta=-0.2060;\ \theta=101.9°$

16. Calculate, in each case, the angle θ formed by vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

a) A(–1, 2), B(2, 3), C(2, –3) and D(0, –4).

$\overrightarrow{AB}=(3,1)$; $\overrightarrow{CD}=(-2,-1)$, $\cos\theta=-0.9899;\ \theta=171.9°$

b) A(2, 3), B(0, 5), C(–1, 2) and D(2, 6).

$\overrightarrow{AB}=(-2,2)$; $\overrightarrow{CD}=(3,4)$, $\cos\theta=0.1414;\ \theta=81.9°$

17. Consider a triangle ABC whose vertices are A(2, 6), B(−1, 2) and C(4, 3). Calculate the measure of each angle of triangle ABC.

$\overrightarrow{AB} = (-3, -4)$; $\overrightarrow{AC} = (2, -3)$; $\cos A = 0.3328$; $m\angle A = 70.6°$

$\overrightarrow{BA} = (3, 4)$; $\overrightarrow{BC} = (5, 1)$; $\cos B = 0.7452$; $m\angle B = 41.8°$

$\overrightarrow{CA} = (-2, 3)$; $\overrightarrow{CB} = (-5, -1)$; $\cos C = 0.3807$; $m\angle C = 67.6°$

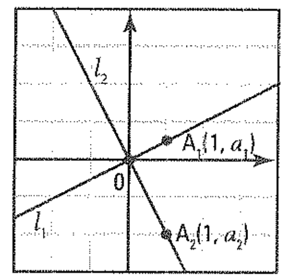
18. Calculate, rounded to the nearest unit, the acute angle θ formed by lines $l_1: y = 2x + 1$ and $l_2: y = -2x + 3$.

$A_1(0, 1) \in l_1$; $A_2(1, 3) \in l_1$; $B_1(0, 3) \in l_2$; $B_2(1, 1) \in l_2$;

$\overrightarrow{A_1A_2} = (1, 2)$; $\overrightarrow{B_1B_2} = (1, -2)$; $\cos\theta = -0.6$; $\theta = 127°$

The acute angle θ between l_1 and l_2 is equal to 53°.

19. Consider lines l_1 and l_2 with respective equations $l_1: y = a_1x\ (a_1 \neq 0)$ and $l_2: y = a_2x\ (a_2 \neq 0)$. Show the equivalence: $l_1 \perp l_2 \Leftrightarrow a_1a_2 = -1$.



$0(0, 0) \in l_1 \cap l_2$; $A_1(1, a_1) \in l_1$; $A_2(1, a_2) \in l_2$

$\overrightarrow{0A_1}$ is parallel to l_1 and $\overrightarrow{0A_2}$ is parallel to l_2.

$l_1 \perp l_2 \Leftrightarrow \overrightarrow{0A_1} \cdot \overrightarrow{0A_2} = 0 \Leftrightarrow 1 + a_1a_2 = 0 \Leftrightarrow a_1a_2 = -1$.

20. A triangle ABC has vertices A(1, 3), B(−1, 1) and C(3, −1).
Calculate the measure of angle ABC to the nearest tenth of a unit.

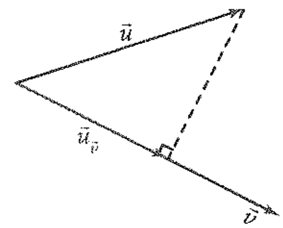
$\overrightarrow{AB} = (-2, -2)$; $\overrightarrow{AC} = (2, -4)$: $\overrightarrow{AB} \cdot \overrightarrow{AC} = 4$.

$\cos A = \dfrac{4}{2\sqrt{2} \cdot 2\sqrt{5}} = \dfrac{\sqrt{10}}{10}$; $m\angle A = 71.6°$.

6.6 Orthogonal projections

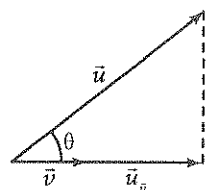
ACTIVITY 1 Orthogonal projection of a vector

a) Consider vectors $\vec{u}$ and $\vec{v}$ on the right.
Using a square, we projected vector $\vec{u}$ onto vector $\vec{v}$.
The vector we obtained, represented in blue, is the **orthogonal projection** of vector $\vec{u}$ onto vector $\vec{v}$. It is written $\vec{u}_{\vec{v}}$.

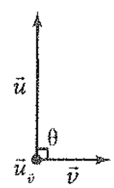


b) 1. In each of the following cases, represent the vector $\vec{u}_{\vec{v}}$, orthogonal projection of $\vec{u}$ onto $\vec{v}$. (Use a square)

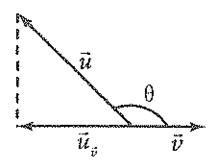
1)



2)



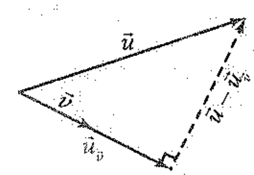
3)



2. Let θ denote the angle between vectors $\vec{u}$ and $\vec{v}$. Answer true or false.
   1) When θ is acute, vectors $\vec{u}_{\vec{v}}$ and $\vec{v}$ have the same direction. *True*
   2) When θ is obtuse, vectors $\vec{u}_{\vec{v}}$ and $\vec{v}$ have opposite directions. *True*
   3) When θ is right, vector $\vec{u}_{\vec{v}}$ is the zero vector. *True*

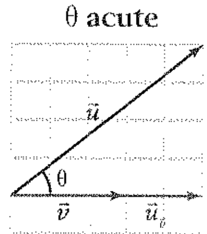
ORTHOGONAL PROJECTION OF A VECTOR

- Given vectors $\vec{u}$ and $\vec{v}$ on the right, the vector represented in blue is the orthogonal projection of vector $\vec{u}$ onto vector $\vec{v}$.
  It is written: $\vec{u}_{\vec{v}}$.



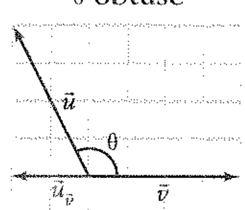
- **Properties**
  - Vectors $\vec{u}_{\vec{v}}$ and $\vec{v}$ are parallel: $\vec{u}_{\vec{v}} \parallel \vec{v}$.
  - Vector $(\vec{u} - \vec{u}_{\vec{v}})$ is perpendicular to $\vec{v}$: $(\vec{u} - \vec{u}_{\vec{v}}) \perp \vec{v}$.
- We distinguish the following 3 cases according to the angle θ between $\vec{u}$ and $\vec{v}$.

θ acute



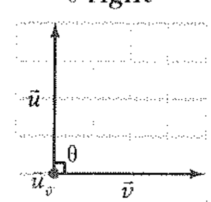
$\vec{u}_{\vec{v}}$ and $\vec{v}$ have same direction.

θ obtuse



$\vec{u}_{\vec{v}}$ and $\vec{v}$ have opposite directions.

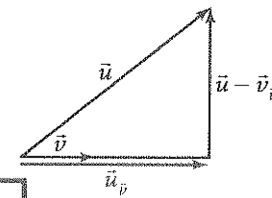
θ right



$\vec{u}_{\vec{v}} = \vec{0}$

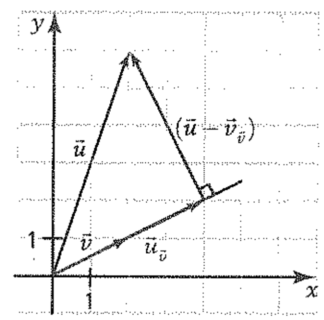
ACTIVITY 2 Formula for the orthogonal projection of $\vec{u}$ onto $\vec{v}$

a) Consider vectors $\vec{u}, \vec{v}$ and vector $\vec{u}_{\vec{v}}$, orthogonal projection of $\vec{u}$ onto $\vec{v}$.
Justify the steps showing that $\vec{u}_{\vec{v}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}$.



	Steps	Justifications
1.	$\vec{u}_{\vec{v}} \mathbin{/\!/} \vec{v}$	***Property of*** $\vec{u}_{\vec{v}}$.
2.	$\vec{u}_{\vec{v}} = k\vec{v}$	***Definition of collinear vectors*** $\vec{u}_{\vec{v}} \mathbin{/\!/} \vec{v}$.
3.	$(\vec{u} - \vec{u}_{\vec{v}}) \perp \vec{v}$	***Property of*** $\vec{u}_{\vec{v}}$.
4.	$(\vec{u} - k\vec{v}) \perp \vec{v}$	***Substituting in 3*** $(\vec{u}_{\vec{v}} = k\vec{v})$.
5.	$(\vec{u} - k\vec{v}) \cdot \vec{v} = 0$	***Property of the scalar product since*** $(\vec{u} - k\vec{v}) \perp \vec{v}$.
6.	$\vec{u} \cdot \vec{v} - k\vec{v} \cdot \vec{v} = 0$	***Distributivity of the scalar product.***
7.	$k = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$	***We isolate k.***
8.	$k = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}$	$\vec{v} \cdot \vec{v} = \|\vec{v}\|\|\vec{v}\| \cos 0° = \|\vec{v}\|^2$.
9.	$\vec{u}_{\vec{v}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}$	***Substituting in 2*** $(\vec{u}_{\vec{v}} = k\vec{v})$.

b) Consider vectors $\vec{u} = (2, 6)$ and $\vec{v} = (2, 1)$.

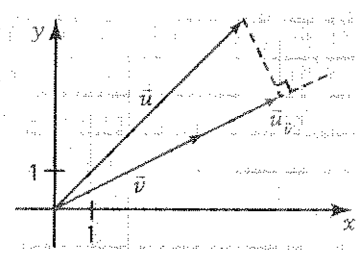


1. Show that $\vec{u}_{\vec{v}} = 2\vec{v}$.
   $\vec{u} \cdot \vec{v} = 10; \|\vec{v}\|^2 = 5; \vec{u}_{\vec{v}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = 2\vec{v}.$
2. Determine the components of $\vec{u}_{\vec{v}}$.
   $\vec{u}_{\vec{v}} = 2\vec{v} = 2(2, 1) = (4, 2)$
3. Determine the components of vector $(\vec{u} - \vec{u}_{\vec{v}})$.
   $\vec{u} - \vec{u}_{\vec{v}} = (2, 6) - (4, 2) = (-2, 4)$
4. Verify the property: $\vec{u}_{\vec{v}} \perp (\vec{u} - \vec{u}_{\vec{v}})$.
   It is sufficient to verify that $\vec{u}_{\vec{v}} \cdot (\vec{u} - \vec{u}_{\vec{v}}) = 0$.
   $(4, 2) \cdot (-2, 4) = -8 + 8 = 0.$

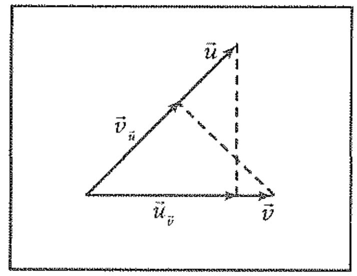
FORMULA FOR THE ORTHOGONAL PROJECTION OF $\vec{u}$ onto $\vec{v}$

$$\vec{u}_{\vec{v}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}$$

Ex.: $\vec{u} = (5, 5); \vec{v} = (4, 2)$
$\vec{u} \cdot \vec{v} = 30; \quad \|\vec{v}\|^2 = 20$
$\vec{u}_{\vec{v}} = \frac{3}{2}\vec{v}$



1. Consider vectors $\vec{u}$ and $\vec{v}$ on the right.



a) Using a square, represent

1. vector $\vec{u}_{\vec{v}}$, orthogonal projection of vector $\vec{u}$ onto $\vec{v}$.
2. vector $\vec{v}_{\vec{u}}$, orthogonal projection of vector $\vec{v}$ onto $\vec{u}$.

b) Show that $\vec{v}_{\vec{u}} = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u}$

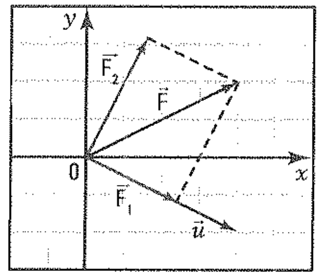
We need only to exchange the letters u and v in the formula $\vec{u}_{\vec{v}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}$

2. Consider vectors $\vec{u} = (-3, 4)$ and $\vec{v} = (2, 3)$. Find the components of

a) $\vec{u}_{\vec{v}}$ $\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{6}{13}\vec{v} = \left(\frac{12}{13}, \frac{18}{13}\right)$

b) $\vec{v}_{\vec{u}}$ $\frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{6}{25}\vec{u} = \left(\frac{-18}{25}, \frac{24}{25}\right)$

3. An object, located in 0, is moving in the direction of vector $\vec{u} = (4, -2)$ under a force $\vec{F} = (4, 2)$.



We want to express $\vec{F}$ as a sum $\vec{F}_1 + \vec{F}_2$ of two forces, one in the direction of the motion, the other perpendicular to this direction.

a) Determine the components of $\vec{F}_1$.

$\vec{F}_1 = \vec{F}_{\vec{u}} = \frac{\vec{F} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{3}{5}\vec{u} = \left(\frac{12}{5}, \frac{-6}{5}\right)$

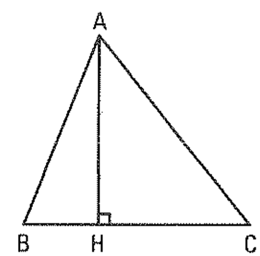
b) Determine the components of $\vec{F}_2$.

Since $\vec{F} = \vec{F}_1 + \vec{F}_2$ ***then*** $\vec{F}_2 = \vec{F} - \vec{F}_1 = (4, 2) - \left(\frac{12}{5}, \frac{-6}{5}\right) = \left(\frac{8}{5}, \frac{16}{5}\right)$

c) Verify that forces $\vec{F}_1$ and $\vec{F}_2$ are perpendicular.

$\vec{F}_1 \cdot \vec{F}_2 = \left(\frac{12}{5}, \frac{-6}{5}\right) \cdot \left(\frac{8}{5}, \frac{16}{5}\right) = 0 \Rightarrow \vec{F}_1 \perp \vec{F}_2$

4. Consider triangle ABC with vertices A(6, 3), B(1, 2) and C(4, 8). Let $\overline{AH}$ be the height from vertex A relative to the base BC.



a) What is the orthogonal projection of vector $\overrightarrow{BA}$ onto the direction of vector $\overrightarrow{BC}$?

$\overrightarrow{BA}_{\overrightarrow{BC}} = \overrightarrow{BH}$

b) Determine the components of vector $\overrightarrow{BH}$.

$\overrightarrow{BH} = \left(\frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BC}\|^2}\right)\overrightarrow{BC} = \left(\frac{(5, 1) \cdot (3, 6)}{(3, 6) \cdot (3, 6)}\right)(3, 6) = \frac{7}{15}(3, 6) = \left(\frac{7}{5}, \frac{14}{5}\right)$

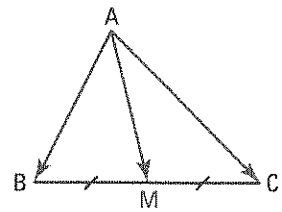
c) Deduce the coordinates of point H.

$H(x, y)$; $\overrightarrow{BH} = (x - 1, y - 2)$ ***and*** $\overrightarrow{BH} = \left(\frac{7}{5}, \frac{14}{5}\right) \Rightarrow H\left(\frac{12}{5}, \frac{24}{5}\right)$

6.7 Vectors and geometric proofs

Activity 1 Median theorem

The vector supported by the median from a vertex of a triangle is equal to half the sum of the vectors supported by the sides from that same vertex.



Hypothesis: $\overline{AM}$ is the median from vertex A of triangle $\triangle ABC$.

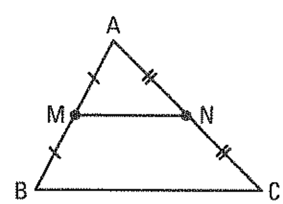
Conclusion: $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$

Justify the steps of the proof leading to this conclusion.

	Steps	Justifications
1.	$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}$ ①	*Chasles' relation.*
2.	$\overrightarrow{AM} = \overrightarrow{AC} + \overrightarrow{CM}$ ②	*Chasles' relation.*
3.	$2\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} + \overrightarrow{AC} + \overrightarrow{CM}$	*Adding corresponding sides of equalities ① and ②.*
4.	$2\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BM} + \overrightarrow{CM}$	*Commutativity of addition.*
5.	$2\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{AC}$	*$\overrightarrow{BM} + \overrightarrow{CM} = \vec{0}$ since M is the midpoint of $\overline{BC}$.*
6.	$\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$	*Dividing both sides by 2.*

Activity 2 Theorem on the midpoints of the sides of a triangle

The line segment joining the midpoints of the sides of a triangle is parallel to the 3rd side and measures half the length of the 3rd side.



Hypothesis: M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{AC}$.

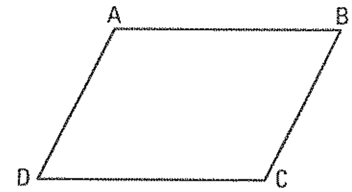
Conclusion: $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$

Justify the steps of the proof leading to this conclusion.

	Steps	Justifications
1.	$\overrightarrow{BA} = 2\overrightarrow{MA}$ ①	*M midpoint of $\overline{AB}$ (hypothesis).*
2.	$\overrightarrow{AC} = 2\overrightarrow{AN}$ ②	*N midpoint of $\overline{AC}$ (hypothesis).*
3.	$\overrightarrow{BA} + \overrightarrow{AC} = 2\overrightarrow{MA} + 2\overrightarrow{AN}$	*Adding corresponding sides of equalities ① and ②.*
4.	$\overrightarrow{BC} = 2(\overrightarrow{MA} + \overrightarrow{AN})$	*Chasles' relation and factorization.*
5.	$\overrightarrow{BC} = 2\overrightarrow{MN}$	*Chasles' relation.*
6.	$\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$	*Dividing both sides by 2.*

ACTIVITY 3 Parallelogram theorem

If a quadrilateral has two parallel congruent sides then this quadrilateral is a parallelogram.



Hypothesis: $\overrightarrow{AB} = \overrightarrow{DC}$.

Conclusion: ABCD is a parallelogram.

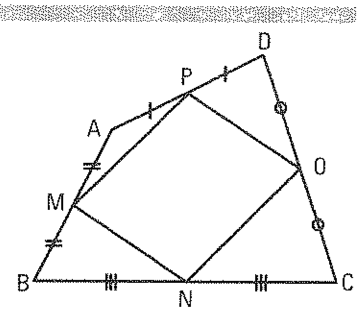
a) Justify the steps of the proof leading to this conclusion.

	Steps	Justifications
1.	$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB}$ ①	***Chasles' relation.***
2.	$\overrightarrow{DC} = \overrightarrow{DB} + \overrightarrow{BC}$ ②	***Chasles' relation.***
3.	$\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{DB} + \overrightarrow{BC}$	***$\overrightarrow{AB} = \overrightarrow{DC}$ (hypothesis) and transitivity of equalities ① and ②.***
4.	$\overrightarrow{AD} = \overrightarrow{BC}$	***We subtract $\overrightarrow{DB}$ from both sides.***
5.	AD//BC	***$\overrightarrow{AD} = \overrightarrow{BC} \Rightarrow \overrightarrow{AD}$ and $\overrightarrow{BC}$ have the same direction.***
6.	ABCD is a parallelogram.	***Opposite sides are pairwise parallel (AB//DC and AD//BC)***

b) Complete: $\overrightarrow{AB} = \overrightarrow{DC} \Leftrightarrow \overrightarrow{AD} =$ ***$\overrightarrow{BC}$***

ACTIVITY 4 Theorem on the midpoints of the sides of a quadrilateral

If we join the midpoints of adjacent sides of any quadrilateral, we obtain a parallelogram.



Hypotheses: M midpoint of $\overline{AB}$, N midpoint of $\overline{BC}$.
0 midpoint of $\overline{CD}$, P midpoint of $\overline{AD}$.

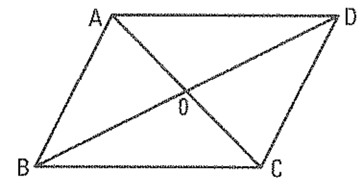
Conclusion: ABCD is a parallelogram.

Justify the steps of the proof leading to this conclusion.

	Steps	Justifications
1.	$\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AC}$ ①	***Theorem on the midpoints applied to △ABC.***
2.	$\overrightarrow{P0} = \frac{1}{2}\overrightarrow{AC}$ ②	***Theorem on the midpoints applied to △ADC.***
3.	$\overrightarrow{MN} = \overrightarrow{P0}$	***Transitivity of equalities ① and ②.***
4.	MN0P is a parallelogram.	***Parallelogram theorem.***

Activity 5 Parallelogram diagonals theorem

The diagonals of a parallelogram intersect at their midpoint.



Hypotheses: – ABCD is a parallelogram.

– 0 is the midpoint of $\overline{AC}$.

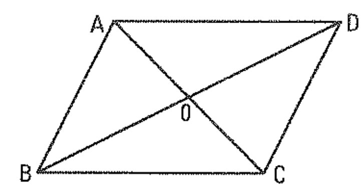
Conclusion: – 0 is the midpoint of $\overline{BD}$.

Justify the steps of the proof leading to this conclusion.

	Steps	Justifications
1.	$\overrightarrow{BC} = \overrightarrow{B0} + \overrightarrow{0C}$ ①	*Chasles' relation.*
2.	$\overrightarrow{AD} = \overrightarrow{A0} + \overrightarrow{0D}$ ②	*Chasles' relation.*
3.	$\overrightarrow{B0} + \overrightarrow{0C} = \overrightarrow{A0} + \overrightarrow{0D}$	*$\overrightarrow{BC} = \overrightarrow{AD}$ (ABCD parallelogram), transitivity of equalities ① and ②.*
4.	$\overrightarrow{B0} + \overrightarrow{A0} = \overrightarrow{A0} + \overrightarrow{0D}$ ③	*$\overrightarrow{0C} = \overrightarrow{A0}$ since 0 is the midpoint of $\overline{AC}$.*
5.	$\overrightarrow{B0} = \overrightarrow{0D}$	*We subtract $\overrightarrow{A0}$ from both sides of equality ③.*
6.	0 is the midpoint of $\overline{BD}$.	*Definition of the midpoint of a line segment.*

Activity 6 Quadrilateral diagonals theorem

If the diagonals of a quadrilateral intersect at their midpoint, then this quadrilateral is a parallelogram.



Hypotheses: – 0 is the midpoint of $\overline{AC}$.

– 0 is the midpoint of $\overline{BD}$.

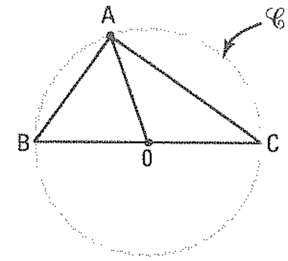
Conclusion: – ABCD is a parallelogram.

Justify the steps of the proof leading to this conclusion.

	Steps	Justifications
1.	$\overrightarrow{AD} = \overrightarrow{A0} + \overrightarrow{0D}$	*Chasles' relation.*
2.	$\overrightarrow{AD} = \overrightarrow{0C} + \overrightarrow{B0}$	*$\overrightarrow{A0} = \overrightarrow{0C}$ (0 midpoint of $\overline{AC}$) and $\overrightarrow{0D} = \overrightarrow{B0}$ (0 midpoint of $\overline{BD}$)*
3.	$\overrightarrow{AD} = \overrightarrow{B0} + \overrightarrow{0C}$	*Commutativity of addition.*
4.	$\overrightarrow{AD} = \overrightarrow{BC}$	*Chasles' relation.*
5.	ABCD is a parallelogram.	*Parallelogram theorem ($\overrightarrow{AD} = \overrightarrow{BC} \Leftrightarrow$ ADCB parallelogram).*

ACTIVITY 7 Theorem on the angle inscribed in a half-circle

Any inscribed angle intercepting a half-circle is right.



Hypotheses: – $\mathscr{C}$ is a circle of radius r centred at 0.

– $\angle BAC$ is an inscribed angle.

– $\overline{BC}$ is a diameter.

Conclusion: $\angle BAC$ is right.

a) Justify the steps of the proof leading to this conclusion.

	Steps	Justifications
1.	$\overrightarrow{AB} \cdot \overrightarrow{AC} = (\overrightarrow{A0} + \overrightarrow{0B}) \cdot (\overrightarrow{A0} + \overrightarrow{0C})$	***Chasles' relation.***
2.	$= \overrightarrow{A0} \cdot \overrightarrow{A0} + \overrightarrow{A0} \cdot \overrightarrow{0C} + \overrightarrow{0B} \cdot \overrightarrow{A0} + \overrightarrow{0B} \cdot \overrightarrow{0C}$	***Distributivity of scalar product over addition.***
3.	$= r^2 + \overrightarrow{A0} \cdot (\overrightarrow{0C} + \overrightarrow{0B}) - r^2$	***Calculating the scalar products; factorization.***
4.	$= r^2 + \overrightarrow{A0} \cdot \vec{0} - r^2$	***$\overrightarrow{0C}$ and $\overrightarrow{0B}$ are opposite.***
5.	$= 0$	$\forall \vec{u}: \vec{u} \cdot \vec{0} = 0$.
6.	$\angle BAC$ is right.	$\vec{u} \cdot \vec{v} = 0 \Leftrightarrow \vec{u} \perp \vec{v}$ ***($\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$).***

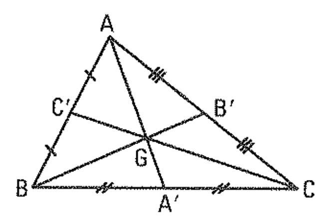
b) Explain why

1. $\overrightarrow{A0} \cdot \overrightarrow{A0} = r^2$ ***$\overrightarrow{A0} \cdot \overrightarrow{A0} = \|\overrightarrow{A0}\| \times \|\overrightarrow{A0}\| \times \cos 0° = r \times r \times 1 = r^2$***

2. $\overrightarrow{0B} \cdot \overrightarrow{0C} = -r^2$ ***$\overrightarrow{0B} \cdot \overrightarrow{0C} = \|\overrightarrow{0B}\| \times \|\overrightarrow{0C}\| \times \cos 180° = r \times r \times (-1) = -r^2$***

ACTIVITY 8 Theorem on the centre of gravity of a triangle

If G is the centre of gravity of a triangle ABC, then $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$.



Hypotheses: – $\overline{AA'}$ is a median of triangle ABC.

– G is the centre of gravity.

Conclusion: $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$

Justify the steps of the proof leading to this conclusion.

Hint: Use the following property of the centre of gravity:
The centre of gravity G divides each median in a 2:1 ratio from each vertex, in other words $\overrightarrow{AG} = 2\overrightarrow{GA'}$, $\overrightarrow{BG} = 2\overrightarrow{GB'}$ and $\overrightarrow{CG} = 2\overrightarrow{GC'}$.

	Steps	Justifications
1.	$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{GA} + (\overrightarrow{GB} + \overrightarrow{GC})$	***Associativity of addition.***
2.	$= \overrightarrow{GA} + 2\overrightarrow{GA'}$	***Median theorem. $\overline{GA'}$ is a median in $\triangle GBC$.***
3.	$= \overrightarrow{GA} + \overrightarrow{AG}$	***Property of centre of gravity.***
4.	$= \vec{0}$	***Sum of 2 opposite vectors.***

Evaluation 6

1. Complete:

a) $\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \overrightarrow{AC} =$ ***$\overrightarrow{BD}$***

b) $\overrightarrow{PQ} = \overrightarrow{RS} \Leftrightarrow$ PQSR is ***a parallelogram***

c) $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD} \Leftrightarrow$ ABDC is ***a parallelogram***

d) $\overrightarrow{AB} + \overrightarrow{BA} =$ ***$\vec{0}$***

e) $\vec{v} = k\vec{u}$ and $k \neq 0 \Rightarrow \vec{u}$ ***//*** $\vec{v}$

2. True or false?

a) $\vec{u}$ and $\vec{v}$ form a vector basis of the plane if and only if $\vec{u}$ and $\vec{v}$ are not parallel. ***True***

b) Two perpendicular vectors in a plane always form a vector basis of this plane. ***True***

c) $\{\vec{u}, \vec{v}\}$ is an orthonormal basis if and only if $\vec{u} \perp \vec{v}$ and $\|\vec{u}\| = \|\vec{v}\| = 1$. ***True***

d) M is the midpoint of the line segment AB if and only if: $\overrightarrow{AM} + \overrightarrow{BM} = \vec{0}$. ***False***

e) The scalar product of two vectors is negative if and only if the angle θ between the vectors is obtuse. ***True***

3. Determine the norm and orientation of the following vectors.

a) $\vec{u} = (-2, 2\sqrt{3})$ ***$\|\vec{u}\| = 4$; $\theta_{\vec{u}} = 120°$***

b) $\vec{u} = (\sqrt{2}, \sqrt{2})$ ***$\|\vec{u}\| = 2$; $\theta_{\vec{u}} = 45°$***

c) $\vec{u} = (-3, -4)$ ***$\|\vec{u}\| = 5$; $\theta_{\vec{u}} = 233.1°$***

d) $\vec{u} = (\sqrt{3}, -1)$ ***$\|\vec{u}\| = 2$; $\theta_{\vec{u}} = 330°$***

4. Simplify $3\overrightarrow{BC} - 2\overrightarrow{AC} + \overrightarrow{CB} + 3\overrightarrow{AD} - 2\overrightarrow{CD} + \overrightarrow{DA}$.

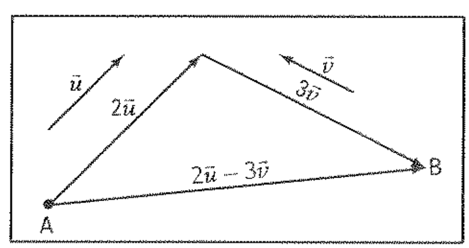
$= 3\overrightarrow{BC} + 2\overrightarrow{CA} + \overrightarrow{CB} + 3\overrightarrow{AD} + 2\overrightarrow{DC} + \overrightarrow{DA}$

$= 2\overrightarrow{BC} + 2\overrightarrow{CA} + 2\overrightarrow{AD} + 2\overrightarrow{DC}$

$= 2(\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AD} + \overrightarrow{DC})$

$= 2\overrightarrow{BC}$

5. Consider vectors $\vec{u}$ and $\vec{v}$ on the right. Starting from point A, draw the vector $\overrightarrow{AB}$ representative of $2\vec{u} - 3\vec{v}$.



6. Consider A(–2, 1), B(1, 2) and C(2, –1). Calculate

a) $3\overrightarrow{AB} - 2\overrightarrow{AC}$ ***(1, 7)***

b) $\|\overrightarrow{AB} + \overrightarrow{AC}\|$ ***$5\sqrt{2}$***

c) $\overrightarrow{AB} \cdot \overrightarrow{AC}$ ***10***

d) $(\overrightarrow{AB} + \overrightarrow{AC}) \cdot (\overrightarrow{AB} - \overrightarrow{AC})$ ***–10***

e) $\overrightarrow{AB}_{\overrightarrow{AC}}$ ***(2, –1)***

7. Consider A(–1, 2) and B(2, 6).

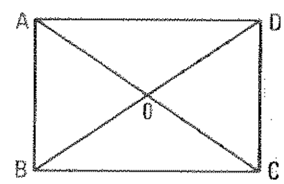
a) Find the two unit vectors $\vec{u}_1$ and $\vec{u}_2$ parallel to $\overrightarrow{AB}$.

$\overrightarrow{AB} = (3, 4)$; $\|\overrightarrow{AB}\| = 5$; $\vec{u}_1 = \left(\frac{3}{4}, \frac{4}{5}\right)$ and $\vec{u}_2 = \left(\frac{-3}{5}, \frac{-4}{5}\right)$

b) Find the two unit vectors $\vec{v}_1$ and $\vec{v}_2$ orthogonal to $\overrightarrow{AB}$.

$\vec{v}_1 = \left(\frac{4}{5}, \frac{-3}{5}\right)$ and $\vec{v}_2 = \left(\frac{-4}{5}, \frac{3}{5}\right)$

8. Consider the rectangle on the right. Complete



a) $\overrightarrow{AB} \cdot \overrightarrow{AD} =$ ***0***

b) $2\overrightarrow{A0} =$ ***$\overrightarrow{AC}$***

c) $\overrightarrow{BA} + \overrightarrow{BC} =$ ***$\overrightarrow{BD}$***

d) $\overrightarrow{AD} - \overrightarrow{BA} =$ ***$\overrightarrow{AC}$***

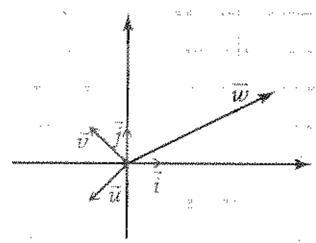
e) $\overrightarrow{AB} =$ ***$\overrightarrow{DC}$***

f) $\overrightarrow{AD} + \overrightarrow{CB} =$ ***$\vec{0}$***

g) $\overrightarrow{0A} + \overrightarrow{DC} =$ ***$\overrightarrow{0B}$***

h) $\overrightarrow{0A} + \overrightarrow{0B} + \overrightarrow{0C} + \overrightarrow{0D} =$ ***$\vec{0}$***

9. Consider the orthonormal bases $\{\vec{i}, \vec{j}\}$ and $\{\vec{u}, \vec{v}\}$ as well as the vector $\vec{w}$.



a) Express the following vectors as a linear combination of vectors $\vec{i}$ and $\vec{j}$.

1. $\vec{u}$ ***$-\vec{i} - \vec{j}$*** 2. $\vec{v}$ ***$-\vec{i} + \vec{j}$*** 3. $\vec{w}$ ***$4\vec{i} + 2\vec{j}$***

b) Express vectors $\vec{w}$ as a linear combination of vectors $\vec{u}$ and $\vec{v}$.

$\vec{w} = -3\vec{u} - \vec{v}$

c) Verify the linear combination found in b) using the linear combinations found in a).

$\vec{w} = -3\vec{u} - \vec{v} = -3(-\vec{i} - \vec{j}) - (-\vec{i} + \vec{j}) = 3\vec{i} + 3\vec{j} + \vec{i} - \vec{j} = 4\vec{i} + 2\vec{j}$

10. Consider points A(–1, 2), B(2, 1) and C(1, –3).

a) 1. Find point D knowing that $\overrightarrow{AB} = \overrightarrow{CD}$.

$\overrightarrow{AB} = (3, -1)$; $\overrightarrow{CD} = (x - 1, y + 3)$; D(4, –4)

2. Find point E knowing that $\overrightarrow{AE} = 3\overrightarrow{AB} - 2\overrightarrow{AC}$.

$3\overrightarrow{AB} - 2\overrightarrow{AC} = 3(3, -1) - 2(2, -5) = (5, 7)$; $\overrightarrow{AE} = (x + 1, y - 2)$; E(4, 9)

b) What type of quadrilateral is quadrilateral ABDC? Justify your answer.

ABDC is a parallelogram. $\overrightarrow{AB} = \overrightarrow{CD} \Rightarrow$ ABDC has two opposite sides that are parallel and congruent $\Rightarrow$ ABDC is a parallelogram according to the parallelogram theorem.

11. Calculate the acute angle θ formed by the lines $l_1 : y = \frac{3}{2}x + 1$ and $l_2 : y = -\frac{3}{2}x - 1$.

$A_1(0, 1) \in l_1$; $B_1(2, 4) \in l_1$; $A_2(0, -1) \in l_2$; $B_2(2, -4) \in l_2$

$\overrightarrow{A_1B_1} = (2, 3)$; $\overrightarrow{A_2B_2} = (2, -3)$; $\cos\theta = \dfrac{\overrightarrow{A_1B_1} \cdot \overrightarrow{A_2B_2}}{\|\overrightarrow{A_1B_1}\| \|\overrightarrow{A_2B_2}\|} = \dfrac{-5}{13}$

$\theta = 180° - 112.6° = 67.4°$.

12. Let $\vec{u}$ and $\vec{v}$ be two nonzero vectors. Indicate, for each statement, if it is true or false.

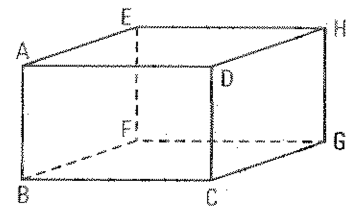
a) $(k\vec{u}) \cdot (k\vec{v}) = k(\vec{u} \cdot \vec{v})$ *False*

b) $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ *True*

c) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ *True*

d) $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$ *True*

13. Consider the rectangular-based right prism on the right. Simplify.



a) $\overrightarrow{AB} + \overrightarrow{ED}$ *$\overrightarrow{EC}$*

b) $\overrightarrow{AH} - \overrightarrow{FE}$ *$\overrightarrow{AG}$*

c) $\overrightarrow{AD} + \overrightarrow{CF} + \overrightarrow{GC}$ *$\vec{0}$*

d) $\overrightarrow{AG} - \overrightarrow{EF} - \overrightarrow{BG}$ *$\vec{0}$*

14. Consider the line l passing through points A(2, −3) and B(−4, 6).

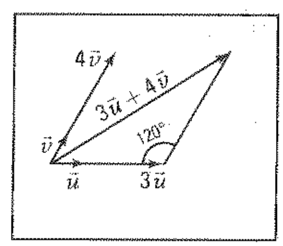
a) Find the coordinates of point P dividing the line segment AB in a 2:1 ratio from A.

$\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB}$; $(x - 2, y + 3) = \frac{2}{3}(-6, 9) = (-4, 6) \Rightarrow P(-2, 3)$

b) Find the coordinates of point Q if $\overrightarrow{AQ} = \frac{-2}{3}\overrightarrow{AB}$.

$(x - 2, y + 3) = \frac{-2}{3}(-6, 9) = (4, -6) \Rightarrow Q(6, -9)$

15. Two unit vectors $\vec{u}$ and $\vec{v}$ form a 60° angle. What is the norm, rounded to the nearest tenth of a unit, of vector $3\vec{u} + 4\vec{v}$?



$\|3\vec{u} + 4\vec{v}\|^2 = \|3\vec{u}\|^2 + \|4\vec{v}\|^2 - 2\|3\vec{u}\|\|4\vec{v}\| \cos 120°$.

$= 9 + 16 - 24 \cos 120° = 37$.

$\|3\vec{u} + 4\vec{v}\| = \sqrt{37} \approx 6.1$ u.

16. Consider vectors $\vec{u} = (1, 2)$ and $\vec{v} = (-2, 1)$.

a) 1. Show that $\vec{u}$ and $\vec{v}$ form a basis.

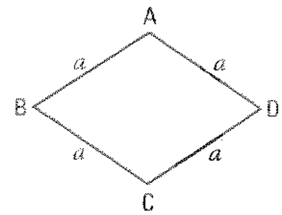
$\vec{u}$ and $\vec{v}$ are not parallel since $\frac{1}{-2} \neq \frac{2}{1}$. Therefore they form a basis.

2. Show that the basis $\{\vec{u}, \vec{v}\}$ is orthogonal. *$\vec{u} \cdot \vec{v} = 0 \Rightarrow \vec{u} \perp \vec{v}$*

b) Express vector $\vec{w} = (-1, 8)$ as a linear combination of vectors $\vec{u}$ and $\vec{v}$.

$\vec{w} = 3\vec{u} + 2\vec{v}$

17. Show the following property: the diagonals of a diamond are perpendicular.



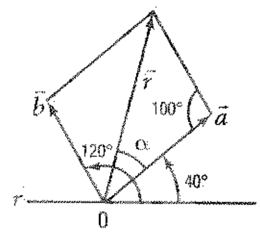
$\overrightarrow{AC} \cdot \overrightarrow{BD} = (\overrightarrow{AB} + \overrightarrow{BC}) \cdot (\overrightarrow{BC} - \overrightarrow{CD})$

$= (\overrightarrow{AB} + \overrightarrow{BC}) \cdot (\overrightarrow{BC} - \overrightarrow{AB})$

$= \overrightarrow{AB} \cdot \overrightarrow{BC} - \overrightarrow{AB} \cdot \overrightarrow{AB} + \overrightarrow{BC} \cdot \overrightarrow{BC} - \overrightarrow{BC} \cdot \overrightarrow{AB} = -a^2 + a^2 = 0$

Since $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$ then $\overrightarrow{AC} \perp \overrightarrow{BD}$

18. Alan and Ben are pulling on an object located at point 0. They apply respectively forces of 80 N and 40 N, with orientations 40° and 120°. Find the resultant $\vec{r}$ (magnitude and orientation) of these two forces.

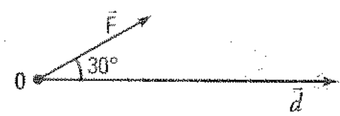


$\|\vec{r}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\| \cos 100° = 9111.35 \Rightarrow \|\vec{r}\| = 95.45$

$\frac{\sin \alpha}{\|\vec{b}\|} = \frac{\sin 100°}{\|\vec{r}\|} \Rightarrow \sin \alpha = 0.4127 \Rightarrow \alpha = 24.4° \Rightarrow \theta_{\vec{r}} = \alpha + 40° = 64.4°$

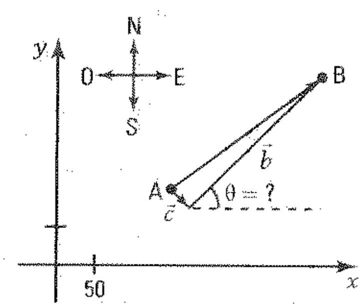
The resultant $\vec{r}$ has an intensity of 95.45 N and an orientation of 64.4°.

19. When a 100 N force with orientation 30° is applied to an object located at 0, the object moves horizontally over a distance of 20 m. What is the work, rounded to the nearest unit, performed by this force? The work, expressed in joules, is defined as the scalar product of the force vector with the displacement vector.



$\vec{F} \cdot \vec{d} = 100 \times 20 \times \cos 30° = 1000\sqrt{3} = 1732$ ***joules.***

20. A boat leaves port A and must go to port B. In the Cartesian plane on the right, measured in kilometres, points A(150, 100) and B(350, 250) represent the ports. During the crossing, a current represented by vector $\vec{c} = (15, -10)$ acts on the boat. The captain orients the boat so as to cancel the effect of the current. What is, to the nearest degree, the measure of the angle, with respect to the eastward direction, at which the captain must orient the boat to reach the port located in B?



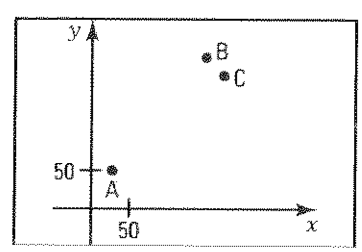
$\overrightarrow{AB} = (200, 150)$; $\vec{c} = (15, -10)$; $\vec{b} = (x, y)$

$\vec{b} + \vec{c} = \overrightarrow{AB} \Rightarrow x + 15 = 200$ *and* $y - 10 = 150 \Rightarrow \vec{b} = (185, 160)$

$\Rightarrow \tan \theta = \frac{160}{185}$; $\theta = 41°$.

The captain must orient the boat 41° North of East.

21. In a Cartesian plane, measured in km, town A is located at point (25, 50) and town B at point (150, 200). When there is no wind, the flight from town A to town B lasts one hour. If there is wind and the pilot does not take it into account, the plane will end up, after one hour of flight, at point C(160, 190). Find the speed of the wind as well as its orientation.



$\vec{v} = (10, -10)$; $\|\vec{v}\| = 14.14$.

$\tan \theta = \frac{-10}{10} = -1 \Rightarrow \theta = 315°$. ***The wind has a speed of 14.14 km/h and an orientation of 315°.***